REAL ANALYSIS QUALIFYING EXAM

SPRING 1992

Problem 1 Let (X, Σ, μ) be a measure space and $\{f_n\}$ a sequence in $L^1(\,d\mu)$ which converges a.e. to $f \in L^1(\,d\mu)$. Prove: $f_n \to f$ in $L^1(\,d\mu)$ iff $\int |f_n|\, d\mu \to \int |f|\, d\mu$.
Hint: Apply Fatou's lemma to $|f| + |f_n| - |f - f_n|$.

Problem 2 Let $\{f_n\}$ be a sequence of Lebesgue-measurable real-valued functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 |f_n(x)|\, dx = 0$$

Prove: there exists a subsequence of $\{f_n\}$ such that $\{f_{n_i}(x)\}$ converges to 0 for a.e. x.

Problem 3 Prove that Lebesgue measure λ on $\mathbb{R}$ is translation-invariant: if A is a Lebesgue-measurable subset of $\mathbb{R}$, then for each $u \in \mathbb{R}$, $u + A$ is also Lebesgue-measurable and $\lambda(u + A) = \lambda(A)$.

Problem 4 A function $f : \mathbb{R} \to \mathbb{R}$ is said to be **lower semi-continuous** provided

$$f(x) \le \liminf_{n\to\infty} f(x_n)$$

whenever $\lim_n x_n = x$. Show that every lower semi-continuous function is Borel measurable.

?. **Problem 5** Show that the function φ defined by

$$\varphi(p) = \int_0^\infty x^p e^{-x}\, dx \quad (p \ge 0)$$

is well-defined and differentiable on $(0, \infty)$.

Topics for the Graduate Exam in Real Analysis

Most of the following topics are normally covered in the course Math 525a.

This is a one hour exam.

Measures: Sigma-rings, sigma fields. Set functions and measures. Construction of measures over Euclidean n-space. Variation of signed measures. Hahn decomposition theorem. Absolute continuity. Mutually singular measures. Product measures. Regular measures. Measurable functions.

Integration: Definition and basic properties of integrable functions over an abstract measure space. The Riemann integral and its relation to the Lebesgue integral. Lebesgue's dominated convergence theorem and related results. Radon-Nikodym theorem. Fubini's theorem.

Convergence: Almost everywhere convergence, uniform convergence, almost uniform convergence, convergence in measure and in mean. Egoroff's theorem. Lusin's theorem.

Differentiation: Almost everywhere differentiable functions. Termwise differentiation and relations to limits of sequences. Bounded variation. Fundamental theorem of calculus.

Metric spaces: Topological properties, compactness, completeness, continuity of functions, contractive mapping theorem. Baire category theorem.

References:

W. Rudin, Real and Complex Analysis
Munroe, Introduction to Measure and Integration
E. Hewitt and K. Stromberg, Real and Abstract Analysis
P. Halmos, Measure Theory
R.P. Boas, A Primer of Real Functions
F. Riesz and B. Sz-Nagy, Functional Analysis
N. Dunford and J. Schwartz, Linear Operators

Real Analysis Qual

Spring 1992

① $f_n \in L^1(\mu)$; $f_n \to f$ a.e.

($\Rightarrow$) assume $f_n \to f$ in $L^1(\mu)$. Then $\int |f_n - f| d\mu \to 0$.

So then $\left| \int |f_n| - |f| d\mu \right| \le \int \big| |f_n| - |f| \big| d\mu \le \int |f_n - f| d\mu \to 0$

So $\int |f_n| \to \int |f|$

($\Leftarrow$) Suppose $\int |f_n| \to \int |f|$.

Then by fatou $\int |f| \le \liminf \int |f_n|$. so:

$2\int |f| - \limsup_n \int |f_n - f| d\mu$ (since $\limsup (a+b) \le \limsup a + \limsup b$)

$\ge \limsup_n \left[\int |f| + |f_n| - |f_n - f| \right]$

$\ge \liminf_n \left[\int |f| + |f_n| - |f_n - f| \right]$ since $\int |f_n| \to \int |f|$ & by fatou.

$\ge 2\int |f| - \int \liminf |f_n - f|$

$= 2\int |f|$

$\Rightarrow \limsup \int |f_n - f| d\mu \le 0$

now $\int |f_n - f| d\mu \ge 0 \Rightarrow \liminf \int |f_n - f| d\mu \ge 0$

$\Rightarrow \lim_n \int |f_n - f| d\mu = 0$. ▫

(2) $f_n : \mathbb{R} \to [0,1]$ Lebesgue m'able $\forall n$ $\ni$ $\lim_{n\to\infty} \int_0^1 |f_n(x)|\, dx = 0$

Then $\forall \epsilon > 0\ \exists N \ni \forall n \geq N$ $\int_0^1 |f_n(x)| dx < \epsilon$

Now $\int_0^1 |f_n(x)| dx \leq \max_{x\in[0,1]} |f_n(x)| < \epsilon$

since f_n is m'able then $\{x ; f_n(x) = \infty\}$ has measure zero, hence we can consider such a maximum existing for a.e. $x \in [0,1]$.

$\Rightarrow |f_n(x)| < \epsilon$ a.e. $x \in [0,1]$ for $\forall n \geq N$.

Thus let $\epsilon = \frac{1}{m}$ for $m \in \mathbb{N}$.

Then for each m $\exists N_m \ni |f_{N_m}(x)| \leq \frac{1}{m}$ a.e. x

Thus $|f_{N_m}(x)| \to 0$ for a.e. x.

$\square$

(3) Prove the Lebesgue measure is translation invariant.

First suppose A is an open interval. Then for any $x \in \mathbb{R}$ we have:

$x + A = x + (a,b) = \underbrace{(a+x,\ b+x)}_{\text{since again is open set.}} \in \mathcal{M}$ for any A open.

Now $\lambda(x+A) = \lambda(a+x, b+x) = b+x - a - x = b - a = \lambda(A)$

Since any open set is the union of open sets this holds for all open sets & hence for complements. Thus it holds for $\mathcal{B}_{\mathbb{R}}$ since it is the σ-algebra generated by open sets in $\mathbb{R}$. Now given $A \in \mathcal{L}$, then $A = B \cup N$ where B is Borel & N is Lebesgue-null.

Since $A + x = (B+x) \cup (N+x)$ where $B+x \in \mathcal{B}_{\mathbb{R}}$ & $\lambda(N+x) = 0$ then $A + x \in \mathcal{L}$. Clearly $\lambda(A+x) = \lambda((B+x) \sqcup (N+x) \setminus (B+x))$

$= \lambda(B+x) + \lambda(N+x \setminus B+x) = \underset{\text{by above}}{\lambda(B)} + \underset{\text{by completeness of } \lambda.}{0} = \lambda(B) + \lambda(N \setminus B) = \lambda(A)$

④ $f: \mathbb{R} \to \mathbb{R}$ w/ $f(x_0) \le \liminf\limits_{n\to\infty} f(x_n)$ $\forall x_n \to x_0$

so $f(x_0) \le \inf\{ \liminf\limits_{n\to\infty} f(x_n) ; x_n \to x_0 \}$.

Hence $f(x_0) \le \liminf\limits_{x \to x_0} f(x)$.

So then $\liminf\limits_{x\to x_0} f(x) = \lim\limits_{\delta \to 0} \left(\inf\limits_{0<|x-x_0|<\delta} f(x) \right) \ge f(x_0)$

$\Rightarrow \forall \epsilon > 0 \ \exists \delta \ni \left| \liminf\limits_{x\to x_0} f(x) - \inf\limits_{|x-x_0|<\delta} f(x) \right| < \epsilon$

So then $\liminf\limits_{x\to x_0} f(x) - \epsilon \le \inf\limits_{|x-x_0|<\delta} f(x)$

hence for any $x \in B_\delta(x_0)$ we have:

$$f(x_0) - \epsilon \le \liminf_{x\to x_0} f(x) - \epsilon \le f(x)$$

So then to show f is Borel m'able we must show $f^{-1}((a,\infty)) \in \mathcal{B}_{\mathbb{R}}$ $\forall a \in \mathbb{R}$

so then fix $a \in \mathbb{R}$ and consider $\{x : f(x) > a\} = E$.

Then if $x_0 \in E$ we have that $f(x_0) > a$.

So then $f(x) \ge f(x_0) - \epsilon > a - \epsilon$ $\forall x \in (x_0 - \delta, x_0 + \delta)$

since ϵ is arbitrary $\Rightarrow$ $f(x) > a$ $\forall x \in B_\delta(x_0)$.

So $f(B_\delta(x_0)) \subseteq (a, \infty)$. Hence $B_\delta(x_0) \subseteq E$ so E is open.

But open sets are Borel m'able by definition so $E \in \mathcal{B}_{\mathbb{R}}$. Thus f is Borel m'able.

∎

③ $\varphi(p) = \int_0^\infty x^p e^{-x} dx \quad p \geq 0$

<u>Well defined</u>: $|\varphi(p)| < \infty \;\; \forall p \geq 0 \iff |\int_0^\infty x^p e^{-x} dx| < \infty \;\; \forall p \geq 0$.

it suffices to show $x^p e^{-x} \in L^1 \;\; \forall p \in (0,\infty)$

Now $\forall p \in (0,\infty) \; \exists a \ni \forall x > a, \;\; x^p \leq e^{x/2}$

Thus $\int_0^\infty x^p e^{-x} dx = \int_0^a x^p e^{-x} dx + \int_a^\infty x^p e^{-x} dx$

$\leq a^p \int_0^a e^{-x} dx + \int_a^\infty e^{-x/2} dx < \infty$

So $x^p e^{-x} \in L^1$ & hence $|\varphi(p)| < \infty$ & so well defined

<u>Differentiable</u>:

Now we know that for $\varphi(p) = \int_a^b f(p,x) dx$ w/ $f(p,x)$ integrable $\forall p$

and where $\frac{\partial}{\partial p} f(p,x) \; \exists \;\; \forall x$ & $|\frac{\partial}{\partial p} f(p,x)| \leq g(x) \in L^1$

$\Rightarrow \varphi(p)$ diff'l.

So by well definedness $\Rightarrow f(p,x) = x^p e^{-x}$ integrable $\forall p$.

Clearly $\frac{\partial f}{\partial p} = x^p e^{-x} \ln x$ exists $\forall x \in [0,\infty)$

& $|x^p e^{-x} \ln x| \leq$

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1993

Problem 1 Define $D_r = \{z \in \mathbb{C} : |z| < r\}$, the open r-disk. Let $M > 0$ and $f_n : D_1 \to D_M$ for $n = 1, 2, \ldots$ be a sequence of analytic functions. Prove there is a subsequence which converges uniformly on $D_{1/2}$.

Problem 2 Prove or find a counterexample: Let D be a countable dense subset of $(0,1)$ and let G be an open subset of $\mathbb{R}$ such that $G \supset D$, then $G \supset (0,1)$.

Problem 3 Let f be a non-constant meromorphic function which is doubly periodic (i.e. has two periods linearly independent over the reals). Prove that f has at least one singularity.

Problem 4 How many roots of the equation $f(z) = 0$ lie in the right half-plane, where

$$f(z) = z^4 + \sqrt{2}z^3 + 2z^2 - 5z + 2$$

Hint: consider the image of the imaginary axis.

Problem 5 Show that a function $f : (a,b) \to \mathbb{R}$ which is absolutely continuous is both uniformly continuous and of bounded variation.

Problem 6 Show that $\frac{\sin x}{x} \in L^2(\mathbb{R}^+)$ and evaluate its L^2 norm.

Problem 7 Suppose f is a non-negative function which is Lebesgue integrable on $[0,1]$, and $\{r_n : n = 1, 2, \ldots\}$ is an enumeration of the rational numbers in $[0,1]$. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} f(|x - r_n|)$$

converges for a.e. $x \in [0,1]$.

REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

SPRING 1994

Problem 1 Evaluate $\int_0^\infty \frac{\log x}{1+x^2}\, dx$

Problem 2 Show that $[0,1]$ cannot be written as the countably infinite union of disjoint nonempty closed intervals.

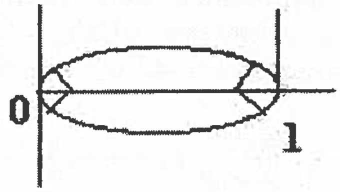
Problem 3 Let $f : D \to \mathbb{C}$ be analytic such that $\Re f(z) > 0$ for all z. Prove

$$|f(z)| \quad |f(0)|\frac{1+|z|}{1 \quad |z|}$$

Problem 4 Let $f : [1,+\infty) \to [0,+\infty)$ be Lebesgue measurable. Prove:

$$\int_1^\infty \frac{f(x)^2}{x^2} < +\infty \quad \Rightarrow \quad \int_1^\infty \frac{f(x)}{x^2}\, dx < +\infty$$

Problem 5 Map the region between the circular arcs (in the figure below) conformally to the unit disk. (Note that the top and bottom acrs intersect at right angles.)



Problem 6 Let $([0,1], \mathcal{A}, \mu)$ denote the Lebesgue measure space on $[0,1]$. Give examples to show that for $f : [0,1] \to \mathbb{R}$ the condition "f is continuous a.e." neither implies, nor is imlied by, the condition "there exists a continuous function $g : [0,1] \to \mathbb{R}$ such that $f = g$ a.e."

Problem 7 An entire function is said to have **finite order** if there exists $c > 0$ such that $|f(z)| \quad \exp(|z|^c)$ for all $|z|$ sufficiently large; the **order of** f is the infimum of all such $c > 0$. Prove that the following function is entire and has order $1/2$:

$$f(z) = \prod_{k=1}^{\infty} 1 \quad \frac{z}{k^2}$$

Problem 8 Let $\{f_n\}$ be a sequence of measurable functions on some measure space $(X, \mathcal{A}, \mu)$ with $\mu(X) < \infty$. We say the sequence is **uniformly integrable** if

$$\lim_{R\to\infty} \sum_n \int_{|f_n| \geq R} |f_n|\, d\mu = 0$$

(a) Show that if there exists $g \in L^1(\mu)$ such that $|f_n(x)| \leq g(x)$ for all x, n there the $\{f_n\}$ are uniformly integrable.

(b) Prove that if $f_n \to f$ pointwise and the $\{f_n\}$ are uniformly integrable then $f \in L^1(\mu)$ and

$$\lim_n \int f_n\, d\mu = \int f\, d\mu$$

Fall 1993

Real Analysis Qual

② D countable dense subset of $(0,1)$ & G open subset of $\mathbb{R}$ $\ni$ $D \subseteq G$.

Then let $D = \{d_i\}$. Then for each d_i $\exists$ $\delta_i > 0$ $\ni$ $B_{\delta_i}(d_i) \subseteq G$.

so $D \subseteq \cup B_{\delta_i}(d_i) \subseteq G$.

Now if $(0,1) \not\subseteq \cup B_{\delta_i}(d_i)$ $\exists$ d_i, d_{i+1} $\ni$ $B_{\delta_i}(d_i) \cap B_{\delta_{i+1}}(d_{i+1}) = \emptyset$

$\Rightarrow F = (d_i + \delta_i, d_{i+1} - \delta_{i+1}) \cap (\cup B_{\delta_i}(d_i)) = \emptyset$

But $d_i, d_{i+1} \in D \subseteq (0,1)$ so $(d_i + \delta_i, d_{i+1} - \delta_{i+1}) \subseteq (0,1)$

Hence for any $x \in F$ $\exists$ $\epsilon > 0$ $\ni$ $B_\epsilon(x) \subseteq F \subseteq (0,1)$.

But then $B_\epsilon(x) \cap (\cup B_{\delta_i}(d_i)) = \emptyset \Rightarrow B_\epsilon(x) \cap D = \emptyset$ $\Rightarrow \Leftarrow$

since D is dense in $(0,1)$.

Thus $(0,1) \subseteq \cup B_{\delta_i}(d_i) \subseteq G$. ◻

⑤ $f: (a,b) \to \mathbb{R}$ is absolutely continuous $\Rightarrow$ f is uniformly cont's.
$f \in BV$.

<u>uniformly continuous</u>: We know $\forall \epsilon > 0$ $\exists \delta$ $\ni$ for any disjoint $\bigsqcup^n (a_i, b_i)$
$\ni$ $\sum^n b_i - a_i < \delta$ $\Rightarrow$ $\sum^n |F(b_i) - F(a_i)| < \epsilon$.

Hence given any $x, y \in (a,b)$ & $n=1$ we have that $|x-y| < \delta$
$\Rightarrow |F(x) - F(y)| < \epsilon$ $\forall x, y \in (a,b)$. so F is uniformly cont's.

<u>BV</u>: We want $T_F(b) - T_F(a) = \sup \{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| \; a = x_1 \le \dots \le x_n = b \}$
to be finite.

Now given any partition $\{x_i\}_{i=1}^n$ we can refine it into partitions of length at most δ.

Hence let $\{y_{ij}\}_{j=1}^{m_i}$ be a subdivision of $[x_i, x_{i+1}]$. $x_i = y_{i1}$

$x_{i+1} = y_{im_i}$

Then $|F(x_{i+1}) - F(x_i)| = |\sum_{j=1}^{m_i} F(y_{i+1}) - F(y_i)| \le \sum^{m_i} |F(y_{i+1}) - F(y_i)| < m_i \cdot \epsilon$

since by construction $|y_i - y_{i-1}| < \delta$.

So then $\sum^{n} |F(x_i) - F(x_{i-1})| \le n \cdot m_i \cdot \epsilon < M \cdot \epsilon < \infty$.

Since ϵ is arbitrary & thus holds for all partitions we obtain

$T_F(b) - T_F(a) < \infty$ as desired.

(6) Show $\frac{\sin x}{x} \in L^2(\mathbb{R}^+)$ & find its L^2 norm.

want $\int_0^\infty \left|\frac{\sin x}{x}\right|^2 dx < \infty$. $\Leftrightarrow$ $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx < \infty$

$u = \sin^2 x \qquad v' = \frac{1}{x^2}$

$u' = 2\sin\cos x \qquad v = -\frac{1}{x}$

$$\Rightarrow \int \frac{\sin^2 x}{x^2}\, dx = -\frac{\sin^2 x}{x} + \int \frac{2\sin x \cos x\, dx}{x}$$

$$= -\frac{\sin^2 x}{x} + \int \frac{\sin 2x}{x}\, dx$$

since $\sin 2x = 2\sin x\cos x$

by $i\sin n\theta + \cos n\theta = (i\sin\theta + \cos\theta)^n$

$$= -\frac{\sin^2 x}{x} + 4\int \frac{\sin x}{x}\, dx$$

Now $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$ (by use of Residue's & $\int \frac{e^{iz}}{z}\, dz$)

also $\lim_{x\to\infty} \frac{\sin^2 x}{x} \le \lim_{x\to\infty} \frac{1}{x} = 0$ & $\lim_{x\to 0} \frac{\sin^2 x}{x} = \lim_{x\to 0} 2\sin x\cos x = 0$

so $$\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = 4\left(\frac{\pi}{2}\right) = 2\pi$$

hence $\left\|\frac{\sin x}{x}\right\|_2 = \sqrt{2\pi}$.

D

(7) f-nonnegative $(f \geq 0)$ Lebesgue integrable on $[0,1]$

$\left(\int_0^1 |f| dx < \infty\right)$ & $\{r_n\}_{n=1}^{\infty} = \mathbb{Q}$

WANT $\quad \sum_{n=1}^{\infty} \frac{1}{2^n} f(|x-r_n|) < \infty$ a.e. $x \in [0,1]$.

Consider: $\quad \int_0^1 \sum_{n=1}^{\infty} \frac{1}{2^n} f(|x-r_n|) dx = \int_0^1 \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{2^i} f|x-r_i|\right) dx$

Since $\int_0^1 |f| dx < \infty \Rightarrow |f(x)| < \infty$ a.e. $x \in [0,1]$.

Hence $|f(x)| < M$ a.e. $x \in [0,1]$ and some $M \in \mathbb{R}$.

so $\quad \left|\sum^{n} \frac{1}{2^i} f(|x-r_i|)\right| \leq \sum^{n} \frac{1}{2^i} \cdot M \leq M \cdot \sum^{\infty} \frac{1}{2^i} = 2M \ \forall n$.

now since $\sum^{n} \frac{1}{2^i} f(|x-r_i|) \to \sum^{\infty} \frac{1}{2^i} f(|x-r_i|)$ & $\int_0^1 2M dx < \infty$

Then by LDCT (Note $\sum^{n} \frac{1}{2^i} f(|x-r_i|) \in L^1([0,1])$ by (*) below):

$$\int_0^1 \lim_{n\to\infty} \sum^{n} \frac{1}{2^i} f(|x-r_i|) dx = \sum_{i=1}^{\infty} \left[\frac{1}{2^i} \int_0^1 f(|x-r_i|) dx\right]$$

$$\text{Now} \quad \int_0^1 f(|x-r_i|) dx = \int_{-r_i}^{1-r_i} f(|t|) dt = \int_{-r_i}^{0} f(|t|) dt + \int_0^{1-r_i} f(|t|) dt$$

$$= \int_0^{r_i} f(t) dt + \int_0^{1-r_i} f(t) dt < 2\int_0^1 |f| dt < \infty$$

Hence (*) $\int_0^1 f(|x-r_i|) dx < N$ for some $N \in \mathbb{R}$ & $\forall i$.

$$\Rightarrow \int_0^1 \sum^{\infty} \frac{1}{2^i} f(|x-r_i|) dx \leq N \cdot \sum^{\infty} \frac{1}{2^i} = 2N < \infty$$

so then $\sum^{\infty} \frac{1}{2^i} f(|x-r_i|) < \infty$ a.e. $x \in [0,1]$.

∎

Real Analysis Qual

Spring 1994.

(2) Suppose $[0,1] = \bigsqcup_{i=1}^{\infty} [a_i, b_i]$ w/ $a_i < b_i$ so $[a_i, b_i] \neq \emptyset$ $\forall i$.

Since they are disjoint & countable we can order them so WLOG assume $0 \leq \ldots a_i < b_i < a_{i+1} < b_{i+1} \ldots \leq 1$. Note that $b_i < a_{i+1}$ since by assumption all the closed intervals are disjoint.

Hence for any i $(b_i, a_{i+1}) \neq \emptyset$ (since $b_i < a_{i+1}$) and $(b_i, a_{i+1}) \cap \left(\bigsqcup [a_i, b_i]\right) = \emptyset$ (since we assumed the intervals were ordered).

But $[a_i, b_i] \subseteq [0,1]$ & since $[0,1]$ is connected it follows $(b_i, a_{i+1}) \subseteq [0,1]$.

But we said $[0,1] = \bigsqcup^{\infty} [a_i, b_i]$ $\Rightarrow\Leftarrow$. Thus $[0,1]$ cannot be written as a countable disjoint union of closed intervals. ∎

(4) $f : [1, +\infty) \to [0, \infty)$

Want: $\displaystyle\int_1^\infty \frac{f^2(x)}{x^2}\,dx < \infty \Rightarrow \int_1^\infty \frac{f(x)}{x^2}\,dx < \infty$.

The proof directly follows from Holder's inequality for $p = q = 2$.

Explicitly: for any $a, b \in \mathbb{R}^+ \Rightarrow 0 \leq (a-b)^2 = a^2 + b^2 - 2ab$
$$\Rightarrow ab \leq \frac{a^2}{2} + \frac{b^2}{2}$$

Thus $\displaystyle\left|\frac{f(x)}{x^2}\right| = \left|\frac{f(x)}{x} \cdot \frac{1}{x}\right| \leq \frac{1}{2}\left(\frac{f(x)}{x}\right)^2 + \frac{1}{2}\left(\frac{1}{x}\right)^2$

Hence $\displaystyle\int_1^\infty \frac{f(x)}{x^2}\,dx \leq \int_1^\infty \frac{|f(x)|}{x^2}\,dx \leq \frac{1}{2}\int_1^\infty \frac{f^2(x)}{x^2}\,dx + \frac{1}{2}\int_1^\infty \frac{1}{x^2}\,dx < \infty$

since $f(x)/x$ & $\frac{1}{x} \in L^2([1,\infty))$. ∎

⑥ $([0,1], \mathcal{A}, \mu)$ Lebesgue measure space on $[0,1]$

a) $f: [0,1] \to \mathbb{R}$ continuous a.e. $\not\Rightarrow$ $\exists\, g: [0,1] \to \mathbb{R}$ conts. w/ $f = g$ a.e.

Consider $f(x) = \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ continuous everywhere except at $x=0$.

Then if $\exists\, g = f$ a.e. continuous since $[0,1]$ is compact, g must be bounded. But $\lim_{x\to 0} g(x) = \lim_{x \to 0} f(x) = \infty$. $\Rightarrow \Leftarrow$.

Thus such a g cannot exist.

b) $\exists\, g$ continuous $\ni$ $f = g$ a.e. $\not\Rightarrow$ f is continuous a.e.

Consider $f(x) = \begin{cases} 0 & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,1] \\ 1 & x \in \mathbb{Q} \cap [0,1]. \end{cases}$

Then f is discontinuous everywhere but $g \equiv 0$ is continuous and $g = f$ a.e. since $m(\mathbb{Q}) = 0$.

$\square$

⑧ $\{f_n\}$ m'able w/ $\mu(x) < \infty$

a) Suppose $\exists\, g \in L^1(\mu) \ni |f_n(x)| \leq g(x)\ \forall x, n. \Rightarrow f_n$ is unif. integ.

Since $g \in L^1$ and $\mu(x) < \infty$ then $\int_X |g(x)|dx < \infty \Rightarrow |g(x)| < \infty$ a.e. $x \in X$

But then $\exists M \ni |g(x)| \leq M$ a.e. x.

Thus let $R > M$, so $m(\{x ; g(x) \geq R\}) = 0 \Rightarrow \int_{|g(x)| \geq R} |g(x)|dx = 0$

Then since $|f_n(x)| \leq g(x) \leq |g(x)|\ \forall x$ we have:

$$\{x ; |f_n(x)| \geq R\} \subseteq \{x ; |g(x)| \geq R\} \text{ \& hence:}$$

$$\sum_{n=1}^{\infty} \int_{|f_n| \geq R} |f_n(x)|dx \leq \sum_{n=1}^{\infty} \int_{|g(x)| \geq R} |g(x)|\, dx = 0 \qquad \forall n.$$

Thus $\lim_{R\to\infty} \sum_{n=1}^{\infty} \int_{|f_n| \geq R} |f_n(x)|dx = 0$.

b) Suppose $f_n \to f$ pointwise & $\{f_n\}$ are uniformly integrable

$\Rightarrow f \in L^1$ & $\int f_n \to \int f$.

Since $\sum_{n=1}^{\infty} \int_{|f_n| \geq R} |f_n(x)| d\mu \geq \int_{|f_n| \geq R} |f_n(x)| d\mu$ for any n then

$\forall \epsilon\ \exists R \ni \int |f_n(x)| d\mu = \int_{|f_n| \geq R} |f_n(x)| d\mu + \int_{|f_n| < R} |f_n(x)| d\mu < \epsilon + R\cdot\mu(x) < \infty\ \forall n.$

Thus $|f_n(x)| \leq M$ a.e. $x\ \forall n$. Since $\int M d\mu = \mu(x)\cdot M < \infty$ and $f_n \to f$ then by L.D.C.T. we have $f \in L^1$ & $\int f_n \to \int f$. $\square$