REAL ANALYSIS GRADUATE EXAM

Spring 2015

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Consider the sequence

$$f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right), \qquad n = 1, 2, \ldots .$$

Evaluate

$$\lim_n \int_0^\infty f_n(x)\, dx,$$

being careful to justify your answer.

2. Suppose that $f\colon [0, \infty) \to \mathbb{R}$ is Lebesgue integrable.
(i) Show that there exists a sequence $x_n \to \infty$ such that $f(x_n) \to 0$.
(ii) Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or a counterexample.
(iii) Suppose additionally that f is differentiable and $f'(x) \to 0$ as $x \to \infty$. Is it true that $f(x)$ must converge to 0 as $x \to \infty$? Give a proof or a counterexample.

3. Define $f_n(x) = ae^{-nax} - be^{-nbx}$ where $0 < a < b$.
(i) Show that

$$\sum_{n=1}^\infty \int_0^\infty f_n(x)\, dx = 0$$

and

$$\int_0^\infty \sum_{n=1}^\infty f_n(x)\, dx = \log(b/a).$$

(ii) What can you deduce about the value of

$$\int_0^\infty \sum_{n=1}^\infty |f_n(x)|\, dx?$$

4. Assume that f is integrable on $[0, 1]$ with respect to the Lebesgue measure m, and let $F(x) = \int_0^x f(t)\, dt$. Assume that $\phi\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz, i.e., there exists a constant $C \geq 0$ such that

$$|\phi(x_1) - \phi(x_2)| \leq C|x_1 - x_2|, \qquad x_1, x_2 \in \mathbb{R}.$$

Prove that there exists a function g which is integrable on $[0, 1]$ such that $\phi(F(x)) = \int_0^x g(t)\, dt$ for $x \in [0, 1]$.

(1)

$$f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right)$$

Evaluate $\lim\limits_{n} \int_0^\infty f_n \, dx$.

real
spring 2015

$$\int_0^\infty |f_n| = \int_0^\infty \frac{dx}{\left(1 + \frac{x}{n}\right)^n} = n \int_1^\infty \frac{du}{u^n} < \infty \quad \text{for } n > 2.$$

so $\int_0^\infty f_n$ makes sense.

Also, $f_n(x) \longrightarrow e^{-x}$ pointwise.

For $x > 0$,

$$\left|1 + \frac{x}{n}\right|^n \geq 1 + x + \frac{\cancel{n}(n-1)}{2\cancel{n}} \frac{x^2}{\cancel{n^2}} \qquad \frac{n-1}{2n} \geq \frac{1}{2} \cdot \frac{1}{2} \quad (n \geq 2)$$

$$\Rightarrow \quad \frac{1}{\left|1 + \frac{x}{n}\right|^n} \leq \frac{1}{1 + x + \frac{n-1}{2n} x^2}$$

$$\leq \frac{1}{1 + x + \frac{1}{4} x^2} = \frac{4}{4 + 4x + x^2}$$

which is in $L^1([0,\infty))$.

so by the DCT,

$$\lim_{n \to \infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \cos\left(\frac{x}{n}\right) dx = \int_0^\infty e^{-x} \, dx = 1.$$

$\blacksquare$

(2) $f: [0,\infty) \to \mathbb{R}$ Lebesgue integrable.

(i) Show there is a sequence x_n s.t. $f(x_n) \to 0$.

solution

If $\{x : |f(x)| \leq \varepsilon\}$ had measure 0,

then $\int |f| \geq \varepsilon m[0,\infty] = \infty$. So $m(\{x : |f(x)| \leq \varepsilon\}) > 0$

$\Rightarrow$ it is nonempty $\Longrightarrow$ pick $x_n \in \{x : |f(x)| \leq \frac{1}{n}\}$.

$\Rightarrow$ $f(x_n) \to 0$. $\square$

(ii) Is it true that $f(x)$ must converge to 0 as $x \to \infty$?

solution

No. $$f(x) = \begin{cases} 1, & x \in \mathbb{N} \\ 0, & \text{otherwise} \end{cases} \qquad \int f = 0.$$

(iii) Suppose f is differentiable and $f'(x) \to 0$ as $x \to \infty$. Is it true $f(x) \to 0$ as $x \to \infty$?

solution Yes. Assume $|f(x_n)| \geq 2\varepsilon$ for $x_n \leq x_n + 1 \leq x_{n+1} \leq x_{n+1} + 1 \leq \cdots$

Take x_n large enough s.t. $|f'(x_n)| \leq \frac{\varepsilon}{2}$.

Then on $[x_n, x_n+1]$, $\left| f(x) - f(x_n) \right| = \left| \int_{x_n}^{x} f' \right|$

$$\le \int_{x_n}^{x_n+1} |f'| \le \frac{\varepsilon}{2} \cdot 1.$$ Which means on

$[x_n, x_n+1]$, $|f(x)| \geq \varepsilon$. Consequently,

$$\int |f| \geq \sum \int_{x_n}^{x_n+1} |f| \geq \sum^{\infty} \varepsilon = \infty. \Rightarrow\Leftarrow.$$

(3) $f_n(x) = a e^{-nax} - b e^{-nbx}$, $0 < a < b$.

(i) Show $\displaystyle\sum_{n=1}^{\infty} \int_0^{\infty} f_n(x)\,dx = 0$

and $\displaystyle\int_0^{\infty} \sum_{n=1}^{\infty} f_n(x)\,dx = \log(b/a)$.

---

e^{-ax} and $e^{-bx} < 1$ for $x > 0$.

$$\int_0^{\infty} \sum_{n=1}^{\infty} a e^{-nax} - b e^{-nbx}$$

$|x|<1 \Rightarrow \displaystyle\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1 = \frac{1-(1-x)}{1-x} = \frac{x}{1-x}$.

$$= \int_0^{\infty} \frac{a e^{-ax}}{1-e^{-ax}} - \frac{b e^{-bx}}{1-e^{-bx}} = \cancel{\ln\left(1-e^{-ax}\right) - \ln\left(1-e^{-bx}\right)\Big|_0^{\infty}}$$

$$= \int_0^{\infty} \frac{a e^{-ax}\left(1-e^{-bx}\right) - b e^{-bx}\left(1-e^{-ax}\right)}{\left(1-e^{-ax}\right)\left(1-e^{-bx}\right)}\,dx$$

(ii) $\displaystyle\int_0^{\infty} \sum_{n=1}^{\infty} |f_n(x)|\,dx = \infty$. For otherwise, by Fubini's theorem, the two values $\sum\int$ & $\int\sum$ would coincide.

(4) f integrable on $[0,1]$, w.r.t. m.

$F(X) = \int_0^X f(t)\,dt$. $\phi : \mathbb{R} \to \mathbb{R}$ Lipschitz.

Prove there is g integrable on $[0,1]$ s.t.

$$\phi(F(X)) = \int_0^X g(t)\,dt \quad \text{for } X \in [0,1].$$

$F(0)=0$. Since $|\phi(0)-\phi(\varepsilon)| \le C|\varepsilon|$, as $\varepsilon \to 0$ we find $\phi(0)=0$. ϕ is also absolutely continuous. So ϕ is in BV. Since $\phi(0)=0$, ϕ is in NBV. Moreover, since $F(0)=0$, $\phi(F)$ is in NBV. So we obtain a measure $\mu_{\phi(F)} \ll m$ (since $\phi(F)$ absolutely continuous) s.t. $\mu_{\phi(F)}((-\infty, X]) = \phi(F(X))$. Let g be $\frac{d\mu_{\phi(F)}}{dm}$. Then by finiteness $g \in L^1$. And $\int_0^x g\,dm =$ [illegible] $= d\mu_{\phi(F)}([0,X]) = \phi(F(X))$ (since there is no mass on $[-\infty,0]$).