REAL ANALYSIS GRADUATE EXAM

Fall 2015

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Prove that for almost all $x \in [0,1]$, there are at most finitely many rational numbers with reduced form p/q such that $q \geq 2$ and $|x - p/q| < 1/(q \log q)^2$. (Hint: Consider intervals of length $2/(q \log q)^2$ centered at rational points p/q.)

2. Suppose that the real-valued function $f(x)$ is nondecreasing on the interval $[0,1]$. Prove that there exists a sequence of continuous functions $f_n(x)$ such that $f_n \to f$ pointwise on this interval.

3. Let (X, μ) be a finite measure space. Assume that a sequence of integrable functions f_n satisfies $f_n \to f$ in measure, where f is measurable. Assume that f_n satisfies the following property: For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) \leq \delta \implies \int_E |f_n| \, d\mu \leq \epsilon.$$

Prove that f is integrable and that

$$\lim_n \int_X |f_n - f| \, d\mu = 0.$$

4. Consider the following two statements about a function $f\colon [0,1] \to \mathbb{R}$:
(i) f is continuous almost everywhere
(ii) f is equal to a continuous function g almost everywhere.
Does (i) imply (ii)? Prove or give a counterexample. Does (ii) imply (i)? Prove or give a counterexample.

Real, Fall 2015

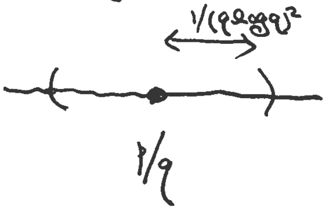
(1) For almost all $x \in [0,1]$ there are at most finitely many rational numbers w/ reduced form p/q s.t. $q \geq 2$ and $\left| x - \frac{p}{q} \right| < \frac{1}{(q \log q)^2}$.

Hint: consider intervals of length $\frac{2}{(q \log q)^2}$ centered at rational points p/q.

For each $x \in [0,1]$, define $Q(x) \in \mathbb{R}^{\geq 0}$ to be the # of $p/q^{\neq 0,1,\, q \geq 2}$ s.t. $|x - p/q| < \frac{1}{(q \log q)^2}$ & let $\mathbb{Q}(x)$ be the set of such p/q. We want to show $\int_0^1 Q(x)\,dx < \infty$, since this implies $Q(x) < \infty$ a.e. we have

$$\int_0^1 Q(x)\,dx = \int_0^1 \sum_{\substack{p/q \in \mathbb{Q} \\ q \geq 2}} \chi_{\mathbb{Q}(x)}\,dx$$

$$\overset{\text{onelli}}{=} \sum_{\substack{p/q \in \mathbb{Q} \\ q \geq 2}} \int_0^1 \chi_{\mathbb{Q}(x)}\,dx \leq \sum_{\substack{p/q \in \mathbb{Q} \\ q \geq 2}} \frac{2}{q^2 \log^2 q} \underset{p=1,\dots,q-1}{=} \sum_{q=2}^{\infty} \frac{2q-1}{q} \frac{1}{q \log^2 q}$$



$$< \infty \quad \text{since} \quad \sum_{n=2}^{\infty} \frac{1}{n \log^p n} \begin{cases} \text{converges } p > 1 \\ \text{diverges } p \leq 1 \end{cases}.$$

$\square$

(2) f real-valued, nondecreasing on $[0,1]$.

Show there exists a sequence of continuous functions f_n on $[0,1]$ s.t. $f_n \to f$ ptwise.

Index the set of countable discontinuities by $\{X_n\}_{n\in\mathbb{N}}$, where $X_n \equiv 0$ eventually if there are only finitely many discontinuities. Set $a_n \geq 0$ to be the length of the gap at the jump discontinuity X_n, where $a_n \equiv 0$ eventually if there are only finitely many discontinuities.

Define $g = f - \sum_{n\in R\subset\mathbb{N}} a_n \chi_{[X_n,1]} - \sum_{n\in L\subset\mathbb{N}} a_n \chi_{(X_n,1]}$

where $n \in R$ if $f(X_n) = f(X_n+)$ and $n \in L$ if $f(X_n) = f(X_n-)$. Then g is continuous. For simplicity, write $g = f - \sum_{n\in\mathbb{N}} f_n$ (note $\sum_{n\in\mathbb{N}} f_n$ is well-defined, being the sum of positive functions) $\sum_{n\in\mathbb{N}} f_n$ is also real-valued, since $\sum_{n\in\mathbb{N}} f_n(X) \leq \max(f(X+), f(X-)) - f(0)$.

To show g is continuous, we show $g(X-) = g(X+)$ (g is increasin
If $X \notin \{X_n\}$, it suffices to show $\sum f_n(X+) = \sum f_n(X-)$.

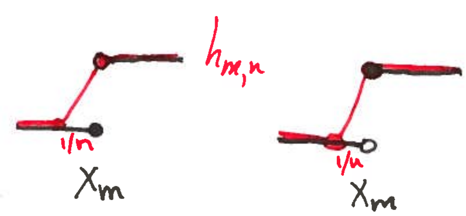
For this x, let $L_x = \{n : x_n < x\}$ and $R_x = \{n : x_n > x\}$. we show $\sum f_n(x-) = \sum f_n(x+) = \sum_{n \in L_x} a_n$. Evaluating $\sum f_n$ on a sequence $y_m \downarrow x$ we find $\sum f_n(x+) \leq \sum_{n \in L_x} a_n$ and since $\sum f_n(x) \geq \sum_{n \in L_x} a_n$ we have $\sum f_n(x+) = \sum_{n \in L_x} a_n$. A symmetric procedure shows $\sum f_n(x-) = \sum_{n \in L_x} a_n$. So g is continuous at x.

Next, let $x = x_m$. It is clear that

$$g(x-) = f(x-) - \sum_{x_n < x_m} a_n \quad \text{and} \quad g(x+) = f(x+) - \sum_{x_n \leq x_m} a_n \, ;$$

since $a_m = f(x+) - f(x-)$, g is continuous at x. So g is continuous.

Define
$$h_{m,n}(x) = \begin{cases} a_m, & x_m < x \leq 1 \\ a_m - \dfrac{a_m(x_m - t)}{1/n}, & 0 \leq x_m - t \leq \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$



And define $H_n = \sum_{m=1}^{n} h_{m,n}$.

Finally, define $g_n = g + H_n$.

It is clear that g_n is continuous and $g_n \to f$ pointwise. $\square$

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(3) (X, μ) finite. $f_n \in L^1$ s.t. for each $\varepsilon > 0$ there exists $\delta > 0$ s.t. $\mu(E) < \delta \Rightarrow \int_E |f_n| < \varepsilon$ $(\forall n)$.
Let $f_n \to f$ in measure. show $f \in L^1$ and $f_n \to f$ in L^1.

Lemma If $f_n \to f$ a.e. and $\int_F |f_n| \leq M$ $(\forall n)$, then $\int_F |f| \leq M$.

pf (lemma) $\int_F |f| = \int_F \liminf |f_n| \leq \liminf \int_F |f_n| \leq M$. ✓

Since $f_n \to f$ in measure, there is a subsequence $f_{n_k} \to f$ a.e. Fix $\varepsilon > 0$. By uniform integrability choose $\delta > 0$ s.t. $\int_F |f_{n_k}| \leq \varepsilon$ when $m(F) \leq \delta$ $(\forall k)$

By Egoroff's theorem, we may choose E s.t. $f_{n_k} \to f$ uniformly on E and $m(E^c) \leq \delta$. By the lemma, $\int_{E^c} |f| \leq \varepsilon$. Choosing k sufficiently large, we have $\int |f| \leq \int_E |f - f_{n_k}| + \int_{E^c} |f - f_{n_k}| + \int |f_{n_k}|$

$\leq \varepsilon m(X) + \int_{E^c} |f| + \int_{E^c} |f_{n_k}| + \int |f_{n_k}|$

$\leq \varepsilon m(X) + \varepsilon + \varepsilon + \int |f_{n_k}| < \infty$. $\therefore f \in L^1$! ✓

Lastly, we want to show $\lim_{n \to \infty} |f - f_n| = 0$.

Fix $\varepsilon > 0$ and choose $\delta > 0$ s.t. $\int_E |f_n| \leq \varepsilon$ when $m(E) \leq \delta$ $(\forall n)$

Choose N s.t. $\forall n \geq N$, $m\left(\{x : |f_n(x) - f(x)| \geq \varepsilon\}\right) \leq \delta$.

Set $E = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$. Then $\forall n \geq N$,

$$\int |f - f_n| = \int_{\{x : |f_n(x) - f(x)| \leq \varepsilon\}} |f - f_n| + \int_E |f - f_n|$$

$$\leq \varepsilon m(X) + \int_E |f| + \int_E |f_n|$$

$$\leq \varepsilon m(X) + \varepsilon + \varepsilon,$$

where we have once again employed the lemma.

$\therefore \int |f_n - f| \to 0$. $\square$

Let $f : [0,1] \to \mathbb{R}$.

(4) (a) T/F: f continuous a.e. $\Longrightarrow$

$\exists g : [0,1] \to \mathbb{R}$ continuous s.t.

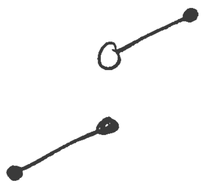
$f = g$ a.e.

(b) T/F: $\exists g : [0,1] \to \mathbb{R}$ continuous s.t.

$f = g$ a.e. $\Longrightarrow$

f is continuous a.e.

(a) F. Let $f = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ x+1, & \frac{1}{2} < x \le 1. \end{cases}$



Then f is continuous a.e.

Assume there were g as in statement. Since $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ has positive measure for all $\varepsilon > 0$, we must have $g(x-) = f(x-) = \frac{1}{2}$ and $g(x+) = f(x+) = 3/2$, contradicting g is continuous.

(b) F. Let $f(x) = \begin{cases} 0, & x \in [0,1] - \mathbb{Q} \\ 1, & x \in \mathbb{Q}. \end{cases}$

Then $f = 0$ a.e. But f is continuous nowhere.

$\square$