REAL ANALYSIS GRADUATE EXAM

Spring 2013

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose that $\{f_n\}$ is a sequence of of real valued continuously differentiable functions on $[0,1]$ such that

$$\lim_{n\to\infty}\int_0^1 |f_n(x)|\,dx = 0 \text{ and } \lim_{n\to\infty}\int_0^1 |f_n'(x)|\,dx = 0.$$

Show that $\{f_n\}$ converges to 0 uniformly on $[0,1]$.

2. Investigate the convergence of $\sum_{n=0}^{\infty} a_n$, where

$$a_n = \int_0^1 \frac{x^n}{1-x}\sin(\pi x)\,dx.$$

3. Let $(X,\mathcal{M},\mu)$ be a measure space, $f_n, f \in L^1(\mu)$. Show that $\int_X |f_n - f|\,d\mu \to 0$ as $n\to\infty$ if and only if

$$\sup_{A\in\mathcal{M}}\left|\int_A f_n d\mu - \int_A f d\mu\right| \to 0$$

as $n\to\infty$.

4. Let μ and ν be σ-finite positive measures, $\mu \geq \nu$ and assume that $\nu << \mu - \nu$ (ν is absolutely continuous with respect to $\mu - \nu$).

Prove that

$$\mu\left(\{\frac{d\nu}{d\mu} = 1\}\right) = 0.$$

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(1) $\{f_n\}$ real-valued, continuously differentiable on $[0,1]$

s.t. $\lim_{n\to\infty} \int_0^1 |f_n| = 0$ and $\lim_{n\to\infty} \int_0^1 |f_n'| = 0$.

Show $f_n \to 0$ uniformly on $[0,1]$.

Fix $\varepsilon > 0$. For any $x, y \in [0,1]$, we have

$$|f_n(x) - f_n(y)| = \left| \int_x^y f_n' \right| \leq \int_x^y |f_n'|$$

$$\leq \int |f_n'| \leq \varepsilon$$

for n sufficiently large.

Also, $m(\{x : |f_n(x)| \geq \varepsilon\}) \leq \varepsilon$ for n sufficiently large, for otherwise $\int |f_n| \geq \varepsilon^2$ for arbitrarily large n, violating $\int |f_n| \longrightarrow 0$.

Taking ε small, $n \gg 1$, let $|f_n(x_n)| \leq \varepsilon$.

By above, for all $x \in [0,1]$, $|f_n(x)| \leq |f_n(x_n)| + \varepsilon \leq 2\varepsilon$. Since n is arbitrary, $f_n \to 0$ uniformly on $[0,1]$. $\square$

(2) Investigate the convergence of $\sum_{n=0}^{\infty} a_n$ where $a_n = \int_0^1 \frac{x^n}{1-x}\sin(\pi x)\,dx$.

By Tonelli's theorem, since $\frac{x^n}{1-x}\sin(\pi x) \geq 0$ for $n \geq 1$, $x \in [0,1]$, we have

$$\sum_{n=0}^{\infty} \int_0^1 \frac{x^n}{1-x}\sin(\pi x)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{x^n}{1-x}\sin(\pi x)\,dx$$

$$= \int_0^1 \frac{\sin(\pi x)}{1-x} \sum_{n=0}^{\infty} x^n \underset{x<1}{=} \int_0^1 \frac{\sin \pi x}{(1-x)^2}. \quad \cancel{\leq \int_0^1 \frac{dx}{(1-x)^2}}$$

$\cancel{= \int_0^1 \frac{dx}{x^2}}$ And $\int_0^1 \frac{\sin \pi x}{(1-x)^2} = \infty$.

Indeed, it equals $\int_0^1 \frac{\sin \pi(1-x)}{x^2} = \pi^2 \int_0^1 \frac{\sin \pi x}{(\pi x)^2} = \pi \int_0^{\pi} \frac{\sin x}{x^2}$;

we show $\int_0^{\pi} \frac{\sin x}{x^2} = \infty$. Assume it were finite.

Since $\frac{\sin x}{x} \to 1$ from below, for $\varepsilon > 0$

choose a small $\delta > 0$ s.t. $\frac{\sin x}{x} \geq 1 - \varepsilon$ for $0 < x < \delta$.

Then $\int_0^\delta \frac{\sin x}{x^2} \geq (1-\varepsilon) \int_0^\delta \frac{dx}{x} = \infty$. $\Rightarrow\!\Leftarrow$.

We conclude $\sum_{n=0}^{\infty} a_n = \infty$. $\square$

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(3) $f_n, f \in L^1(\mu, X)$. Show

$$\int |f_n - f| \to 0 \quad \text{iff} \quad \sup_{E \in \mathcal{M}} \left| \int_E f_n - \int_E f \right| \to 0.$$

Fix $\varepsilon > 0$. For sufficiently large n,

$$\left| \int_E f_n - \int_E f \right| \leq \int |f_n - f| \leq \varepsilon$$

$$\Rightarrow \sup_{E \in \mathcal{M}} \left| \int_E f_n - \int_E f \right| \leq \varepsilon. \quad ✓$$

For the other direction, let $E_n = \{x : f_n(x) \geq f(x)\}$ and $F_n = \{x : f_n(x) < f(x)\}$.

Then $\displaystyle\int |f-f_n| = \int_{E_n} f_n - f + \int_{F_n} f - f_n$

$$= \int_{E_n} f_n - f + \left| \int_{F_n} f_n - \int_{F_n} f \right|$$

$$\leq 2 \sup_{E \in \mathcal{M}} \left| \int_E f_n - \int_E f \right| \longrightarrow 0. \qquad \square$$

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(4) μ, ν σ-finite positive measures. $\mu \geq \nu$ and $\nu \ll \mu - \nu$.

Prove $\mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = 0$.

$$\nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = \int_{\left\{\frac{d\nu}{d\mu}=1\right\}} 1 \, d\mu = \mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right).$$

Assume $\mu < \infty \Rightarrow \nu < \infty$. Then, subtracting,

$$\mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) - \nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = 0 \xRightarrow{\nu \ll \mu - \nu} \nu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right) = 0$$

$$\| $$

$$\mu\left(\left\{\frac{d\nu}{d\mu} = 1\right\}\right). \checkmark$$

Extending to μ σ-finite is trivial. $X = \bigcup X_j$ disjoint $\mu(X_j) < \infty$. On X_j, $\int_E \frac{d\nu}{d\mu}\Big|_{X_j} d\mu_j = \int_E \frac{d\nu}{d\mu} d\mu = \nu(E) = \nu_j(E) \Rightarrow \frac{d\nu}{d\mu}\Big|_{X_j} = \frac{d\nu_j}{d\mu_j}$
$\Rightarrow \{x : \frac{d\nu}{d\mu} = 1\} = \bigcup_j \{x : \frac{d\nu_j}{d\mu_j} = 1\}$. Then since $\mu_j(\{x : \frac{d\nu_j}{d\mu_j} = 1\}) = 0 \Rightarrow \mu(\{x : \frac{d\nu}{d\mu} = 1\}) = \sum 0 = 0$. $\square$