Spring 2012 Real Analysis Exam

Answer all four questions. Partial credit will be awarded, but in the event that you cannot fully solve a problem state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f and g be real integrable functions on a σ-finite measure space $(X, \mathcal{M}, \mu)$, and for $t \in \mathbb{R}$ let
$$F_t = \{x \in E : f(x) > t\} \quad \text{and} \quad G_t = \{x \in E : g(x) > t\}.$$
Show that
$$\int_X |f - g| d\mu = \int_{-\infty}^{\infty} \mu\left((F_t \backslash G_t) \cup (G_t \backslash F_t)\right) dt.$$

2. Show that
$$\int_\pi^\infty \frac{dx}{x^2 (\sin^2 x)^{1/3}}$$
is finite.

3. A collection of functions $\{f_\alpha\}_{\alpha \in \mathcal{A}} \subset L^1(\mu)$ on the measure space $(X, \mathcal{M}, \mu)$ is said to be *uniformly integrable* if
$$\lim_{M \to \infty} \sup_{\alpha \in \mathcal{A}} \int_{\{x : |f_\alpha(x)| > M\}} |f_\alpha| = 0.$$

a. Prove that if $f \in L^1$ then $\{f\}$ is uniformly integrable.

b. Prove that if $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{f_\beta\}_{\beta \in \mathcal{B}}$ are two collections of uniformly integrable functions then $\{f_\gamma\}_{\gamma \in A \cup B}$ is uniformly integrable.

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(1) f & g real-valued & integrable on a σ-finite space X.

Set $F_t = \{x \in X : f(x) > t\}$ &

$G_t = \{x \in X : g(x) > t\}$.

Show $\int_X |f-g| \, d\mu = \int_{\mathbb{R}} \mu((F_t - G_t) \cup (G_t - F_t)) \, dt$.

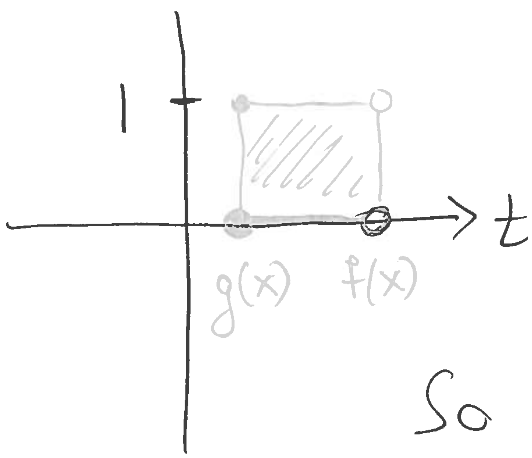
Observe $\mu((F_t - G_t) \cup (G_t - F_t))$

$= \int_{\mathbb{R}} \chi_{F_t - G_t}(x) \, d\mu + \int_{\mathbb{R}} \chi_{G_t - F_t}(x) \, d\mu$. By Tonelli,

$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{F_t - G_t}(x) + \chi_{G_t - F_t}(x) \, d\mu \, dt =$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{F_t - G_t}(x)\, dt\, d\mu + \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{G_t - F_t}(x)\, dt\, d\mu.$$

Observe $\chi_{F_t - G_t}(x) \overset{\text{think fixed } x}{=} \begin{cases} 1, & x \in F_t - G_t \\ 0, & \text{otherwise} \end{cases}$



$= \begin{cases} 1, & f(x) > t \;\&\; g(x) \le t \\ 0, & \text{otherwise} \end{cases}$

So $\displaystyle\int_{\mathbb{R}} \chi_{F_t - G_t}(x)\, dt = \begin{cases} f(x) - g(x) & \text{on } \{x : f(x) \overset{\text{or}}{\geq} g(x)\} \text{ and} \\ 0 & \text{elsewhere.} \end{cases}$

Hence, in summary, $\displaystyle\int_{\mathbb{R}} \mu(G_t \Delta F_t)\, dt$

$$= \int_{\{x : f(x) > g(x)\}} f(x) - g(x)\, dt + \int_{\{x : g(x) \geq f(x)\}} g(x) - f(x) = \int_{\mathbb{R}} |f - g|\, dt.$$

$\square$

(2) Show $\displaystyle\int_{\pi}^{\infty} \frac{dx}{x^2(\sin^2 x)^{\frac{1}{3}}}$ is finite.

The singularities occur at $x = n\pi$, $n \geq 1$.

We aim to show $\displaystyle\sum_{n=1}^{\infty} \int_{n\pi - c}^{n\pi + c} \frac{dx}{x^2 \sin^{2/3}(x)} < \infty$

for c sufficiently small.

Let $1 < \alpha < \frac{3}{2}$. Then for $c \ll 1$,

$$|x - n\pi| < c \implies |\sin x| \geq |x - n\pi|^{\alpha}.$$

So
$$\int_{n\pi-c}^{n\pi+c} \frac{dx}{x^2 \sin^{2/3}(x)} \leq \frac{1}{(n\pi - c)^2} \int_{n\pi-c}^{n\pi+c} \frac{dx}{\sin^{2/3}(x)}$$

$$\leq \frac{1}{(n\pi - c)^2} \int_{n\pi-c}^{n\pi+c} \frac{dx}{[x - n\pi]^{\frac{2}{3}\alpha}}.$$

Since $1 - \frac{2}{3}\alpha > 0$, we have

$$\int_{n\pi-c}^{n\pi+c} \frac{dx}{(x - n\pi)^{\frac{2}{3}\alpha}} = 2\int_0^c \frac{dx}{x^{2/3\alpha}} = \frac{2}{1 - \frac{2}{3}\alpha}\left(c^{1-\frac{2}{3}\alpha} - 0\right).$$

we gather

$$\sum_{n=1}^{\infty} \int_{n\pi - c}^{n\pi + c} \frac{dx}{x^2 \sin^{2/3}(x)} \leq \frac{2C^{1-\frac{2}{3}\alpha}}{1-\frac{2}{3}\alpha} \sum_{n=1}^{\infty} \frac{1}{(n\pi - c)^2} < \infty .$$

It remains to show $\displaystyle\sum_{n=1}^{\infty} \int_{n\pi + c}^{(n+1)\pi - c} \frac{dx}{x^2 \sin^{2/3}(x)} < \infty .$

Let M be s.t. $\sin x \geqslant M$ on $[c, \pi - c]$.

Then

$$\sum_{n=1}^{\infty} \int_{n\pi + c}^{(n+1)\pi - c} \frac{dx}{x^2 \sin^{2/3}(x)} \leq \frac{1}{M^{2/3}} \int_{\pi + c}^{\infty} \frac{dx}{x^2} < \infty .$$

We conclude $\displaystyle\int_{\pi}^{\infty} \frac{dx}{x^2 \sin^{2/3}(x)} < \infty .$ $\square$

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(3) $\{f_\alpha\} \subset L^1(X)$ are

uniformly integrable if

$$\lim_{M \to \infty} \sup_{\alpha} \int_{\{x : |f_\alpha(x)| > M\}} |f_\alpha| = 0 .$$

(a) If $f \in L^1$, then $\{f\}$ is uniformly integrable.

$$\int |f| = \sum_{n=0}^{\infty} \int_{\{x : n \le |f(x)| < n+1\}} |f| < \infty$$

$$\Rightarrow \text{tail} \sum_{n=M}^{\infty} \int_{\{x : n \le |f(x)| < n+1} |f|$$

$$= \int_{\{x : |f(x)| \ge M\}} |f| \longrightarrow 0 \text{ as } M \to \infty .$$

$\square$

(b) If $\{f_\alpha\}_{\alpha \in A}$ and $\{f_\beta\}_{\beta \in B}$, then $\{f_\gamma\}_{\gamma \in A \cup B}$ is uniformly integrable.

For a fixed M,

$$\sup_{\gamma \in A \cup B} \int_{\{x : |f_\gamma(x)| > M\}} |f_\gamma| \le \max \left\{ \sup_{\alpha \in A} \int_{\{x : |f_\alpha(x)| > M\}} |f_\alpha| , \sup_{\beta \in B} \int_{\{x : |f_\beta(x)| > M\}} |f_\beta| \right\}$$

$\longrightarrow 0$ as $M \longrightarrow \infty$. $\square$

(c) If $\mu(X) < \infty$ and $\{f_\alpha\}$ is uniformly integrable, then

$$\sup_\alpha \int |f_\alpha| < \infty .$$

This is not necessarily true if $\mu(X) = \infty$.

Know $\lim_{M \to 0} \sup_\alpha \int_{\{x : |f_\alpha(x)| > M\}} |f_\alpha| = 0$.

Fix $\varepsilon > 0$. Choose M so large s.t.

$$\int_{\{x : |f_\alpha(x)| > M\}} |f_\alpha| \leq \sup_\alpha \int_{\{x : |f_\alpha(x)| > M\}} |f_\alpha| < \varepsilon .$$

Also, $\sup_\alpha \int_{\{x : |f_\alpha(x)| \leq M\}} |f_\alpha| \leq M \mu(X)$.

Combining these, $\sup_\alpha \int |f| < \infty$.

Observe $f_n = \begin{cases} 1 & \text{on } [-n,0] \\ 0 & \text{elsewhere} \end{cases}$ on $\mathbb{R}$

is in L^1, $\sup_n \int_{\{x : |f_n(x)| > 1\}} |f_n| = \sup_n 0 = 0$.

And $\sup_n \int |f_n| = \infty$. $\square$

(d) $\mu(X) < \infty$. $\{f_n\} \subset L^1(\mu)$ s.t.

$f_n \to f_0$ a.e. and $\int |f_n| \to \int |f_0|$.

Show $\{f_n\}$ is uniformly integrable.

NOTE! We must assume $\int |f_0| < \infty$.

Indeed, if $\int |f_0| = \infty$, then, assuming $\{f_n\}$ is uniformly integrable,

$$\int |f_n| \leq M\mu(X) + \sup_n \int_{\{x : |f_n(x)| > M\}} |f_n|$$

$$\longrightarrow \leq M\mu(X) + \varepsilon \quad \text{for } M \gg 1.$$

Hence, $\sup_n \int |f_n| < \infty$, violating $\int |f_n| \to \infty$.

~~So we must assume $\int |f_0| < \infty$.~~

~~HOWEVER! Assuming this, we will prove the result WITHOUT assuming $\mu(X) < \infty$.~~

~~So assume $\int |f_0| < \infty$. Fix $\varepsilon > 0$~~

Choose M' s.t. $\int_{\{x:|f_0(x)|>M\}} |f_0| < \varepsilon$, for all $M \geq M'$.

First we show

$$\lim_{n \to \infty} \int_{\{x:\,|f_n(x)|>M\}} |f_n| = \int_{\{x:\,|f_0(x)|>M\}} |f_0|,$$

except possibly on a small set, where M is fixed.

Let $g_n = |f_n| \chi_{\{x:\,|f_n(x)|>M\}}$ and

$$g_0 = |f_0| \chi_{\{x:\,|f_0(x)|>M\}}.$$

$g_n \to g_0$ pointwise a.e., except possibly on a small set. Indeed, since $f_n(x) \to f_0(x)$ a.e., if $|f_0(x)| > M$, then eventually $|f_n(x)| > M$, so $g_n(x) \to g_0(x)$ where $|f_0(x)| > M$. However, if $|f_0(x)| \leq M$, it is possible (when $|f_0(x)| = M$) that

$|f_n(x)| > M$ for arbitrarily large n, in which case $g_n(x) \not\to g_0(x)$. So we must exclude these points, which is where the aforementioned 'small sets' come in.

Define $E_M = \left\{ x : \begin{array}{l} |f_0(x)| = M \text{ and} \\ |f_n(x)| > M \text{ for} \\ \text{arbitrarily large } n \end{array} \right\}$.

~~Clearly the E_M are disjoint. Then since $\mu(X) < \infty$, there is M so large s.t. $\sum_{M' \geq M} \mu(E_{M'}) < \varepsilon$, where $\varepsilon > 0$ is fixed. Set $E = \bigcup_{M' \geq M} E_{M'}$.~~

Our argument from above shows $g_n \to g_0$ a.e. on $X - E_M$

On this set $X - E_M$, we then have $g_n \leq |f_n|$, $g_n \to g_0$ a.e., $|f_n| \to |f_0|$ a.e., $g_n, |f_n|, g_0, |f_0| \in L^1(X - E_M)$. We want to use the generalized DCT, which means we also need

$$\int_{X-E_M} |f_n| \to \int_{X-E_M} |f_0| \quad (*)$$

Why is this true?

This also follows from the generalized DCT.

Since $f_n, f_0 \in L^1$ and $f_n \to f_0$ a.e. and $\int |f_n| \to \int |f_0|$ we have $|f_n - f_0| \to 0$, $|f_n - f_0| \le |f_n| + |f_0| \in L^1$, $|f_n| + |f_0| \to |f_0| + |f_0|$ a.e. and $\int |f_n| + |f_0| \to 2\int |f_0|$. By the generalized DCT, $\int |f_n - f_0| \to 0$. Consequently, $\int_F |f_n \to f_0| \to 0$, whence $\left| \int_F |f_n| - \int_F |f_0| \right| \le \int |f_n - f_0| \to 0$, whence $\int_F |f_n| \to \int_F |f_0|$; take $F = X - E_M$. (*) follows.

By the generalized DCT again, we conclude

$$\lim_{n\to\infty} \int_{\{x : |f_n(x)| > M\} - E_M} |f_n| = \int_{\{x : |f_0(x)| > M\} - E_M} |f_0| \qquad (\star)$$

($\star\star$) And on the bad set E_M, by above, i.e. ($F = E_M$),

$$\boxed{\lim_{n\to\infty} \int_{E_M} |f_n| = \int_{E_M} |f_0|.}$$

~~Hence,~~

$$\cancel{\int_{\{x : |f_n(x)| > M\} \cap E} |f_n| \le \int_E |f_n| \le \int_E |f_0| + \varepsilon \le M\mu(E) + \varepsilon \le M\varepsilon + \varepsilon}$$

~~(n sufficiently large)~~

Observe now that

$$\int_{E_M} |f_0| \leq \int_{\{x:|f_0(x)|=M\}} |f_0| \leq \int_{\{x:|f_0(x)|\geq M\}} |f_0| \longrightarrow 0 \quad \text{as } M \to \infty.$$

Referring to our choice of M' at the beginning of the solution, we may assume when M is so large that

$$\int_{\{x:|f_0(x)|>M\}} |f_0| < \varepsilon \quad \& \quad \int_{E_M} |f_0| < \varepsilon.$$

Fix such an M. Then by ($\star$) & ($\star\star$), we may take n sufficiently large s.t.

$$\int_{\{x:|f_n(x)|>M\}-E_M} |f_n| \leq \int_{\{x:|f_0(x)|>M\}-E_M} |f_0| + \varepsilon$$

$$\leq \int_{\{x:|f_0(x)|>M\}} |f_0| + \varepsilon$$

$$\leq 2\varepsilon$$

and

$$\int_{E_M} |f_n| \leq \int_{E_M} |f_0| + \varepsilon \leq 2\varepsilon.$$ Therefore,

$$\int_{\{x:|f_n(x)|>M\}} |f_n| = \int_{\{x:|f_n(x)|>M\}-E_M} |f_n| + \int_{\{x:|f_n(x)|>M\}\cap E_M} |f_n| \leq 2\varepsilon + \int_{E_M} |f_n| \leq 4\varepsilon.$$

This is true for a single value of M

but for n sufficiently large. Thus we have

proved $$\sup_{n \geq N} \int_{\{x : |f_n(x)| > M\}} |f_n| \leq 4\varepsilon.$$

But $$\sup_{n \geq N} \int_{\{x : |f_n(x)| > M\}} |f_n|$$ decreases w/ M.

Consequently, $$\lim_{M \to 0} \sup_{n \geq N} \int_{\{x : |f_n(x)| > M\}} |f_n| = 0.$$

By part (b), we may add in those f_n for $n < N$ for free. We conclude $\{f_n\}$ is uniformly integrable

$\square$

(4) $\mathbb{M}$ collection of <u>finite</u> measures on a measure space $(X, \mathcal{M})$.

(a) Show $d(\nu, \lambda) \equiv 2 \sup_{E \in \mathcal{M}} |\nu(E) - \lambda(E)|$ is a metric on $\mathbb{M}$.

$$2 \sup_{E \in \mathcal{M}} |\nu(E) - \lambda(E)| = 0 \Rightarrow \nu(E) = \lambda(E) \quad \forall E \in \mathcal{M}$$
$$\Rightarrow \nu = \lambda.$$

$d(\nu, \lambda) = d(\lambda, \nu)$ by definition.

$$|\nu(E) - \lambda(E)| \leq |\nu(E) - \mu(E)| + |\mu(E) - \lambda(E)|$$
$$\leq \sup_{F \in \mathcal{M}} |\nu(F) - \mu(F)| + \sup_{F \in \mathcal{M}} |\mu(F) - \lambda(F)|$$

$$\Rightarrow d(\nu, \lambda) \leq d(\nu, \mu) + d(\mu, \lambda). \qquad \square$$

For any finite μ s.t. $\nu, \lambda \ll \mu$ & $\nu(X) = \nu(\lambda) = 1$, set $p = \frac{d\nu}{d\mu}$ and $q = \frac{d\lambda}{d\mu}$. Prove

$$d(\nu, \lambda) = \int |p - q| = 2\left(1 - \int \min(p, q)\right).$$

Hint: $\nu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c)$

since $\nu(E) + \nu(E^c) = 1 = \lambda(E) + \lambda(E^c)$.

Observe $\int |p-q| = \int p-q + \int q-p$

$E = \{x : p(x) \geq q(x)\}$ $\quad E^c = \{x : q(x) > p(x)\}$

$= \nu(E) - \lambda(E) + \lambda(E^c) - \nu(E^c)$.

And $2(1 - \int \min(p,q)) = 2 - 2\left(\int_{E^c} p + \int_E q\right)$

$= 2 - 2\nu(E^c) - 2\lambda(E)$

$\stackrel{?}{=} \nu(E) - \lambda(E) + \lambda(E^c) - \nu(E^c)$

$\iff 2 = \nu(E) + \lambda(E) + \lambda(E^c) + \nu(E^c) = 1+1$ ✓

For any F, we have $2|\nu(F) - \lambda(F)| \stackrel{\text{Hint}}{=}$

$|\nu(F) - \lambda(F)| + |\lambda(F^c) - \nu(F^c)| = \left|\int_F p-q\right| + \left|\int_{F^c} q-p\right|$

$= \left|\int_{E\cap F} p-q - \int_{E^c\cap F} q-p\right| + \left|\int_{E^c\cap F^c} q-p - \int_{E\cap F^c} p-q\right| \leq \int_{E\cap F} p-q + \int_{E^c\cap F^c} q-p$

$\leq \int_E p-q + \int_{E^c} q-p = \int |p-q|$. $\therefore d(\nu,\lambda) \leq \int |p-q|$.

And $\int |p-q|$ is achieved:

Taking $F = E$ as previously, it is clear that the two inequalities become equalities.

$\square$