REAL ANALYSIS GRADUATE EXAM

Fall 2012

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let m be the Lebesgue measure on $X = [0,1]$. If

$$m(\limsup_{n\to\infty} A_n) = 1, m(\liminf_{n\to\infty} B_n) = 1,$$

prove that $m\left(\limsup_{n\to\infty}(A_n \cap B_n)\right) = 1$, where

$$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k, \liminf_{n\to\infty} B_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k.$$

2. Assume $f : X \to [0,\infty)$ is measurable. Find

$$\lim_n \int_X n \log\left[1 + \frac{f(x)}{n}\right] d\mu.$$

3. Let $f \in L^1(m)$. For $k = 1, 2, \ldots$ let f_k be the step function defined by

$$\begin{aligned} f_k(x) &= k \int_{j/k}^{(j+1)/k} f(t)dt \\ \text{for } \frac{j}{k} &< x \le \frac{j+1}{k}, j = 0, \pm 1, \ldots. \end{aligned}$$

Show that f_k converges to f in L^1 as $k \to \infty$.

4.If E is Borel set in $\mathbb{R}^n$ the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r\to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))},$$

whenever the limit exists [Here m denotes the Lebesgue measure and $B(x,r)$ is the open ball with center at x and radius r.]

(a) Show that $D_E(x) = 0$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.
(b) For $\alpha \in (0,1)$ find an example of E and x such that $D_E(x) = \alpha$.
(c) Find an example of E and x such that $D_E(x)$ does not exist.

c. Show that if $\mu(X) < \infty$ and $\{f_\alpha\}_{\alpha\in\mathcal{A}} \subset L^1(\mu)$ is uniformly integrable then

$$\sup_{\alpha\in\mathcal{A}} \int |f| d\mu < \infty.$$

Give an example to show that the conclusion fails without the condition $\mu(X) < \infty$.

d. Again let $\mu(X) < \infty$ and suppose $\{f_n\}_{n=0}^\infty \subset L^1(\mu)$ such that $f_n \to f_0$ a.e. and $\int |f_n| d\mu \to \int |f_0| d\mu$. Prove that $\{f_n\}_{n=0}^\infty$ is uniformly integrable. Hint: Consider some ϕ_M, a continuous, bounded function on $[0,\infty)$, equal to 0 on $[M,\infty)$, for which $|t|\mathbf{1}\{|t| > M\} \le |t| - \phi_M(|t|)$.

4. Let $\mathbb{M}$ be the collection of all finite measures on the measure space $(X, \mathcal{M})$.

a. Show that

$$d(\nu,\lambda) = 2 \sup_{E\in\mathcal{M}} |\nu(E) - \lambda(E)|$$

defines a metric on $\mathbb{M}$.

b. For any $\mu \in \mathbb{M}$ that dominates measures ν and λ in $\mathbb{M}$ with $\nu(X) = \lambda(X) = 1$, let

$$p = \frac{d\nu}{d\mu} \quad \text{and} \quad q = \frac{d\lambda}{d\mu}.$$

Prove

$$d(\nu,\lambda) = \int |p(x) - q(x)|\, d\mu = 2\left(1 - \int (\min\{p(x), q(x)\})\, d\mu\right).$$

Hint: notice that $\mu(E) - \lambda(E) = \lambda(E^c) - \nu(E^c)$.

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(1) m Lebesgue on $[0,1]$.

If $m(\limsup_n A_n) = 1$ and $m(\liminf_n B_n) = 1$,

show $m(\limsup_n (A_n \cap B_n)) = 1$.

Recall $\limsup_n A_n = \bigcap_k \bigcup_{n \geq k} A_n \ni x$

iff x is in infinitely many A_n. And

$\liminf_n B_n = \bigcup_K \bigcap_{n \geq k} B_n \ni x$ iff x is in

all but finitely many B_n.

We want to use

$$\underbrace{m\left(\limsup_n A_n\right)}_{1} + \underbrace{m\left(\liminf_n B_n\right)}_{1} = \underbrace{m\left(\limsup_n A_n \cup \liminf_n B_n\right)}_{\substack{\geq 1 \text{ and } \leq 1 = \text{measure } [0,1] \\ \Rightarrow = 1.}}$$

$$+ \, m\left(\limsup_n A_n \cap \liminf_n B_n\right).$$

So we hope $m\left(\limsup_n (A_n \cap B_n)\right) \geq$ (*)

$$m\left(\limsup_n A_n \cap \liminf_n B_n\right) = 2-1 = 1.$$

Indeed, $x \in \limsup_n A_n \cap \liminf_n B_n$

$\Longrightarrow$ $x \in A_n$ for infinitely many n
& $x \in B_n$ for all but finitely many n

$\Longrightarrow$ $x \in A_n \cap B_n$ for infinitely many n

$\Longrightarrow$ $x \in \limsup_n A_n \cap B_n$

$$\therefore \limsup_n A_n \cap \liminf_n B_n \subset \limsup_n A_n \cap B_n$$

and we've done since (*) follows. $\square$

(2) $f : X \to [0, \infty)$.

Evaluate $\displaystyle\lim_{n\to\infty} \int_X n \log\left(1 + \frac{f(x)}{n}\right) d\mu$.

Since $\log\left(1 + \frac{f(x)}{n}\right) \geq 0$ we want to use the MCT.

We show first that $\displaystyle\lim_{n\to\infty} n \log\left(1 + \frac{f(x)}{n}\right)$ converges to $f(x)$ everywhere.

If $f(x) = 0$, this is clearly true.

Let $f(x) \neq 0$. Then $n \log\left(1 + \frac{f(x)}{n}\right)$

$$= f(x) \frac{\log\left(1 + \frac{f(x)}{n}\right)}{\frac{f(x)}{n}} \xrightarrow[n\to\infty]{} f(x) \frac{d}{dx}\Big|_{x=1} \log x$$

$= f(x) \cdot 1$ as claimed.

Next we need to show

$n \log\left(1 + \frac{f(x)}{n}\right)$ increases w/ n for all X.

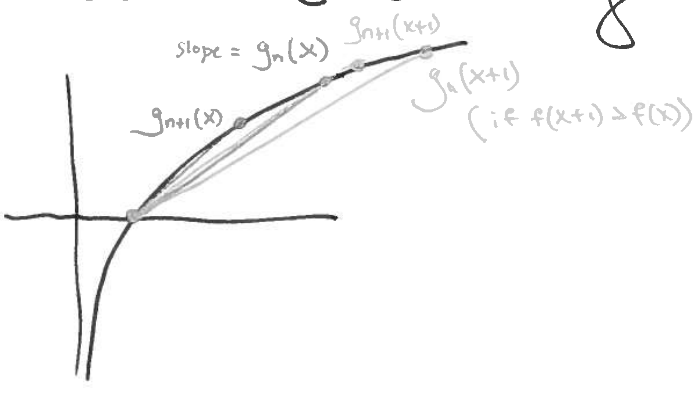
$\Vert f(x) \neq 0$

(again, we need only consider where $f(x) \neq 0$ since trivially true otherwise.)

$$f(x) \frac{\log\left(1+\frac{f(x)}{n}\right)}{\frac{f(x)}{n}} \equiv f(x)\, g_n(x).$$

To show $g_n(x)$ increases, since log is increasing it suffices to show the second derivative of log is negative at $X=1$

which is of course true.



Clearly, the slopes $g_n(X)$ increase as $\frac{f(x)}{n}$ decreases, i.e. as n increases.

This proves the $g_n(X)$ are increasing $\forall x$ w/n.

$$\therefore \text{By the MCT, } \lim_{n\to\infty} \int_X n \log\left(1+\frac{f(x)}{n}\right) d\mu = \int_X f(x)\, d\mu. \quad \square$$

(3) $f \in L^1(m)$. For $k=1,2,\ldots$

set $f_k(x) = k\int_{j/k}^{(j+1)/k} f$ (w.d. since $f \in L^1$)

where $x \in \left(\frac{j}{k}, \frac{j+1}{k}\right]$, $j = 0, \pm 1, \ldots$.

Show $f_k \to f$ in L^1.

First we show the result for $f = \chi_{[a,b]}$. Fix k.
For those j s.t. $\left(\frac{j}{k}, \frac{j+1}{k}\right] \subset [a,b]$ we have $f_k(x) = k\int_{j/k}^{(j+1)/k} \chi_{[a,b]} = 1$ $\left(x \in \left(\frac{j}{k}, \frac{j+1}{k}\right]\right)$.

Hence for such x, $|f(x) - f_k(x)| = 0$. Examining what happens at the endpoints, let $a \in \left(\frac{j}{k}, \frac{j+1}{k}\right]$. For x in this interval, Then $|f_k(x)| \le k\int_{j/k}^{(j+1)/k} 1 \le 1$. So $\int_{j/k}^{(j+1)/k} |f - f_k| \le \frac{2}{k}$ and similarly for the segment containing b. So $\int |f - f_k| \le \frac{4}{k} \to 0$ as $k \to \infty$. ✓

Next we show the result for $f = \chi_{\bigcup [a_j, b_j]}$

$[a_j,b_j]$ disjoint, except possibly at endpoints.

Since we are only interested in $f \in L^1$, we demand

$\sum_j (b_j - a_j) < \infty$. Fix $\varepsilon > 0$.

Observe $f = \sum_j \underset{\underset{f^j}{|||}}{\chi_{[a_j, b_j]}}$ and $(f)_k = \underbrace{\sum_j (f^j)_k}_{\text{disjoint supports w/ } j}$.

As previously, $|(f^i)_k| \leq 1$, so for large n

$$\sum_{i \geq n} b_j - a_j \leq \varepsilon \quad \text{and} \quad \sum_{i \geq n} \int_{a_i}^{b_i} |f - f_k| \leq 2\varepsilon.$$

On the remaining finitely many intervals we may take n large enough s.t. $\int_{a_i}^{b_i} |f - f_k| \leq \varepsilon$ by the result proved previously. So $\int |f - f_k| \to 0$.

Now we show if f & $\phi \in L^1$, $\int |\phi - \phi_k| \longrightarrow 0$ and $\int |f - \phi| \leq \varepsilon$, then

$\int |f - f_k| \sim \varepsilon$ for $k \gg 1$.

$$\int |f-f_k| \le \underbrace{\int |f-\phi|}_{\le \varepsilon} + \underbrace{\int |\phi-\phi_k|}_{\le \ \varepsilon \text{ for } k \gg 1} + \underbrace{\int |\phi_k - f_k|}_{\substack{\text{remains to}\\ \text{estimate.}\\ \le \varepsilon}}$$

$$\int |\phi_k - f_k| \le \sum_j \int_{j/k}^{(j+1)/k} k \underbrace{\int_{j/k}^{(j+1)/k} |\phi - f|}_{\substack{\text{just a}\\ \text{constant for every } j}}$$

$$= \sum_j \int_{j/k}^{(j+1)/k} |\phi - f| \underbrace{\int_{j/k}^{(j+1)/k} k}_{=1}$$

$$= \sum_j \int_{j/k}^{(j+1)/k} |\phi - f| = \int |\phi - f| \le \varepsilon.$$

$\therefore \int |f-f_k| \sim \varepsilon$ as $k \to \infty$. ✓

Let E be Borel, we show the result is true for $f = \chi_E$. Indeed, let $\underset{E}{\underset{\cup}{\mathcal{U}}} = \bigcup (a_j, b_j)$ disjoint be s.t. $m(\mathcal{U} - E) = \int |\chi_{\mathcal{U}} - \chi_E| \le \varepsilon$. By the previous two results, $\int |f - f_k| \to 0$. ✓

It follows by the triangle inequality that the result holds for simple functions.

Finally, let $f \in L^1$ and let ϕ_n be simple s.t. $\int |f - \phi_n| \leq \frac{1}{n}$.

Then by what we proved earlier, since the result is true for the ϕ_n it is true for f, and we are done. $\square$

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(4) $E \subset \mathbb{R}^n$ Borel

Define $D_E(x) = \lim_{r \to 0} \frac{m(E \cap B(x,r))}{m(B(x,r))}$.

(a) Show $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.

Define the Borel measure λ by $\lambda(F) = m(E \cap F)$

Then clearly $\lambda \ll m$ and λ is σ-finite.

Write $\lambda(F) = \int_F f \, dm$.

Then by the main theorem,

$$\lim_{r\to 0} \frac{\lambda(B(x,r))}{m(B(x,r))} = f(x) \text{ a.e.}$$

Moreover, setting $g = 1$ on E and $g = 0$ on E^c

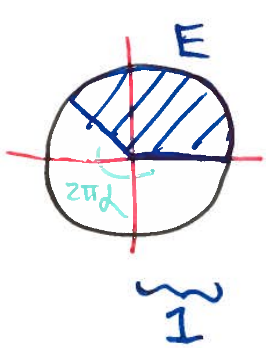
we have $\displaystyle\int_F g\,dm = \int_{F\cap E} dm + \cancelto{0}{\int_{F\cap E^c} g\,dm} = m(F\cap E) = \lambda(F).$

So $f = g$ a.e., hence (a) is proved. $\square$

(b) For $\alpha \in (0,1)$ find E & X s.t.

$D_E(x) = 1$.

$E \subset \mathbb{R}^2$, $E = \left\{ re^{2\pi i\theta} : \begin{matrix} 0 \le r \le 1 \\ 0 \le \theta \le \alpha \end{matrix} \right\}$ and $X = 0$.



Clearly, $\dfrac{m(E\cap B(0,r))}{m(B(0,r))} = \dfrac{\alpha\pi}{\pi} = \alpha$

$\Rightarrow D_E(x) = \alpha.$ $\square$

(c) Show E & x s.t. $D_E(x)$ does not exist

$$E = \bigcup_{\substack{n \geq 1 \\ \text{odd}}}^{\infty} \left(2^{-(n+1)}, 2^{-n}\right).$$

For n odd: $$m\left(E \cap [-2^{-n}, 2^{-n}]\right) = \sum_{\substack{m \geq n \\ m \text{ odd}}}^{\infty} 2^{-m} - 2^{-(m+1)}$$

$$= -\sum_{m \geq n}^{\infty} \underbrace{(-1)^m 2^{-m}}_{=\left(\frac{-1}{2}\right)^m} = -\left(\frac{1}{\underbrace{1-\left(-\frac{1}{2}\right)}_{=3/2}} - \sum_{m=0}^{n-1}\left(-\frac{1}{2}\right)^m\right)$$

$$= \frac{1-\left(-\frac{1}{2}\right)^n}{3/2} - \frac{1}{3/2} = \frac{2}{3}\left(+\frac{(-1)^{n+1}}{2^n}\right) = \frac{2}{3}\frac{1}{2^n}.$$

$$\implies \frac{m\left(E \cap [-2^{-n}, 2^{-n}]\right)}{m\left([-2^{-n}, 2^{-n}]\right)} = \frac{\frac{2}{3}\frac{1}{2^n}}{2^{-n+1}} = \frac{1}{3}.$$

For n even: $\left(2^{-(n+1)}, 2^{-n}\right)$ does not appear in E.

So $$m\left(E \cap [-2^{-n}, 2^{-n}]\right) = \sum_{\substack{m \geq n+1 \\ m \text{ odd}}} 2^{-m} - 2^{-(m+1)} \underset{\text{by above}}{=} \frac{2}{3}\frac{1}{2^{n+1}}.$$

And $$\frac{m\left(E \cap [-2^{-n}, 2^{-n}]\right)}{m\left([-2^{-n}, 2^{-n}]\right)} = \frac{\frac{2}{3}\frac{1}{2^{n+1}}}{2^{-n+1}} = 1/6.$$ So $D_E(x)$ does not exist. $\square$