REAL ANALYSIS GRADUATE EXAM
Spring 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $A \subset \mathbb{R}$ and suppose that for each $\epsilon > 0$ there are Lebesgue-measurable sets E, F with $E \subset A \subset F$ and $m(F \backslash E) < \epsilon$. Show that A is Lebesgue measurable.

(2) Let $f > 0$ be a Lebesgue-integrable function on $[0,1]$. Show that

$$\lim_{\epsilon \searrow 0} \frac{1}{\epsilon} \int_{[0,1]} (f^\epsilon - 1)\ dm = \int_{[0,1]} \log f\ dm.$$

Here m denotes Lebesgue measure. HINT: Decompose f (or $\log f$) into two parts.

(3) Suppose $f \in L^1(\mathbb{R})$ is absolutely continuous, and

$$\lim_{h \searrow 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right|\ dx = 0.$$

Show that $f = 0$ a.e.

(4)(a) Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$, and suppose $F_1, \ldots, F_7$ are 7 measurable sets with $\mu(F_j) \geq 1/2$ for all j. Show that there exist indices $i_1 < i_2 < i_3 < i_4$ for which $F_{i_1} \cap F_{i_2} \cap F_{i_3} \cap F_{i_4} \neq \phi$.

(b) Let m denote Lebesgue measure on $[0,1]$, and let $f_n \in L^1(m)$ be nonnegative and measurable with

$$\int_{[0,1/n]} f_n\ dm \geq 1/2$$

for all $n \geq 1$. Show that $\int_{[0,1]} [\sup_n f_n(x)]\ m(dx) = \infty$. HINT: Part (b) does not necessarily use part (a).

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(1) $A \subset \mathbb{R}$. For each $\varepsilon > 0$ there exist E, F w/ $E \subset A \subset F$ and $m(F-E) < \varepsilon$. Show A is measurable.

Choose E_n, F_n corresponding to $\frac{1}{n}$.

Set $E = \cup E_n$ and $F = \cap F_n$. Then $E \subset A \subset F$. Write

$$A = E \cup (A - E).$$

E is measurable. NTS $A - E \subset \overset{\text{measurable}}{N}$, $m(N) = 0$, so that by completeness $A-E$ is measurable. Set $N = F - E \supset A-E$. For each n,

$m(F_n - E_n) < \frac{1}{n}$. Since $F_n - \bigcup_m E_m \subset F_n - E_n$,

$$m\left(F_n - \bigcup_m E_m\right) \leq m(F_n - E_n) < \frac{1}{n}.$$

Since $\bigcap_m F_m - \bigcup_m E_m \subset F_n - \bigcup_m E_m$,

$$m(F - E) = m\left(\bigcap_m F_m - \bigcup_m E_m\right) \leq m\left(F_n - \bigcup_m E_m\right) < \frac{1}{n}.$$

Since n was arbitrary, $m(F - E) = m(N) = 0$, and we are done. $\square$

(2) $f > 0$, $f \in L^1([0,1])$.

Show $\lim_{\varepsilon \downarrow 0} \int_0^1 \frac{f(x)^{\varepsilon} - 1}{\varepsilon} \, dx = \int_0^1 \log f(x) \, dx$

First, $\lim_{\varepsilon \downarrow 0} \frac{f^{\varepsilon}(x) - 1}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{f^{\varepsilon}(x) - f^{(0)}(x)}{\varepsilon - 0}$

$= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} f(x)^{\varepsilon} = \log f(x) f^{(0)}(x) = \log f(x)$.

~~Observe $\pm \frac{d}{d\varepsilon} \frac{f(x)^{\varepsilon} - 1}{\varepsilon} = \pm \frac{\log f(x) f(x)^{\varepsilon} \cdot 1 - f(x)^{\varepsilon} \cdot 1}{\varepsilon^2}$~~

~~$= \pm \frac{f(x)^{\varepsilon} (\log f(x) - 1)}{\varepsilon^2}$~~

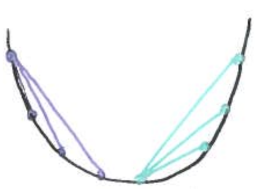
~~$= \pm \frac{\log f(x) f(x)^{\varepsilon} \varepsilon - (f(x)^{\varepsilon} - 1)}{\varepsilon^2}$~~

~~$= \pm \frac{f(x)^{\varepsilon} (\log f(x) \varepsilon - 1) + 1}{\varepsilon^2}$~~

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} f(x)^{\varepsilon} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \log f(x)\, f(x)^{\varepsilon} = \log^2 f(x).$$

~~First, consider $\{x : |f(x)| \leq 1\}$. let x be in this set.~~

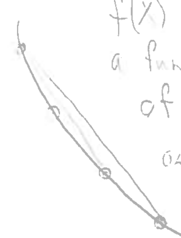
Since the second derivative at 0 $\log^2 f(x)$ is positive (or zero) the slopes of the secant lines $\frac{f^{\varepsilon}(x)-1}{\varepsilon}$ increase w/ ε for $\varepsilon \ll 1$. Moreover, the second derivative $\log^2 f(x) \cdot f(x)^{\varepsilon}$ is always nonnegative, so that $\frac{f^{\varepsilon}(x)-1}{\varepsilon}$ increases for all $\varepsilon > 0$. Hence, the sequence $\left\{\frac{f^{\varepsilon}-1}{\varepsilon}\right\}$ decreases on $[0,1]$ as $\varepsilon \downarrow 0$.



Consider $\{x : f(x) < 1\}$. On this set $\log f(x) < 0$, so the first derivative at zero, $\log f(x)$, is negative; hence $\left\{\frac{f^{\frac{1}{n}}-1}{\frac{1}{n}}\right\}$ is a decreasing set of negative numbers, negative at least for sufficiently large n, x being fixed.

But, in fact, x need not be fixed; the first derivative $\log f(x) \cdot f^{\varepsilon}(x)$ is strictly negative (hence the slopes of the secant lines are always negative) for all x s.t. $0 < f(x) < 1$ and $\varepsilon > 0$ we gather $\left\{ \frac{f^{\frac{1}{n}} - 1}{\frac{1}{n}} \right\}$ is a decreasing sequence of negative functions. Hence we may apply the MCT to $\left\{ \frac{1 - f^{\frac{1}{n}}}{\frac{1}{n}} \right\}$, yielding

$$\lim_{n \to \infty} \int_{\{x : f(x) < 1\}} \frac{1 - f(x)^{\frac{1}{n}}}{\frac{1}{n}} = - \int_{\{x : f(x) < 1\}} \log f(x),$$



as desired, at least on this set.

On $\{x : f(x) \geq 1\}$ the first derivative $\log f(x) \cdot f^{\varepsilon}(x)$ is nonnegative for all $\varepsilon > 0$. So $\left\{ \frac{f^{\frac{1}{n}} - 1}{\frac{1}{n}} \right\}$ is nonnegative and decreasing. Since

$f - 1 = \frac{f^{\frac{1}{1}} - 1}{\frac{1}{1}} \in L^1([0,1])$ (note here we are using $\mu([0,1]) < \infty$!), we may apply the MCT (in its decreasing form), yielding

$$\lim_{n \to \infty} \int_{\{x : f(x) \geq 1\}} \frac{f(x)^{\frac{1}{n}} - 1}{\frac{1}{n}} dx = \int_{\{x : f(x) \geq 1\}} \log f(x) dx.$$

The same arguments apply to any decreasing sequence $x_n \downarrow 0$. We conclude

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int \frac{f^{\varepsilon} - 1}{\varepsilon} = \int \log f \quad . \quad \square$$

~~(Note we did not require the finiteness of $[0,1]$, so our argument is equally valid for all $f > 0$ in $L^1(\mathbb{R})$.)~~ wrong! (see previous page)

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(3) Suppose $f \in L^1(\mathbb{R})$ is absolutely continuous and

$$\lim_{h \downarrow 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right| dx = 0 .$$

(*)

Show $f = 0$ a.e.

It suffices to show $f = \text{constant}$ a.e., for then $f \in L^1(\mathbb{R}) \Rightarrow f = 0$ a.e.

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*) Since $\int_{-n}^{n} \left| \frac{f(x+h) - f(x)}{h} \right| \leq \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right|$,

the hypothesis implies $\lim_{h \downarrow 0} \int_{-n}^{n} \left| \frac{f(x+h) - f(x)}{h} \right| = 0$.

Since $f = \overset{\text{constant}}{}$ a.e. $\iff$ $f = \overset{\text{constant}}{}$ a.e. on $[-n,n]$ $\forall n$

we have reduced to showing f is a constant on $[-n,n]$.

Since f is absolutely continuous on $[-n,n]$, its derivative exists a.e.

So $$\lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| = |f'(x)|$$

converges for a.e. $x \in [-n,n]$.

Note that $f(x+h) \in L^1([-n,n])$ by change of variables ($f \in L^1([-n,n])$ and $x+h$ is increasing absolutely continuous). So

$\left| \frac{f(x+h) - f(x)}{h} \right| \in L^1([-n,n])$ and the integrals make sense.

We forgot to mention $f' \in L^1([-n,n])$, specifically, $f(x) - f(-n) = \int_{-n}^{x} f'(t)\,dt$.

Fix $\varepsilon > 0$. Since m is finite on $[-n,n]$, by Egoroff's theorem $\left| \frac{f(x+h) - f(x)}{h} \right|$ converges uniformly on $[-n,n]$ to $|f'(x)|$ except possibly on a set of measure $< \varepsilon$.

So off of a small set, for h sufficiently small we may write $\left|\frac{f(x+h)-f(x)}{h}\right| \leq |f'(x)| + \varepsilon$.

Since $f' \in L^1$ and we are working on $[-n,n]$, $|f'(x)+\varepsilon|$ serves as a dominating function for $\left|\frac{f(x+h)-f(x)}{h}\right|$ for h sufficiently small and except on a set of measure $< \varepsilon$. Hence, on our uniformly convergent set we have by the DCT

$$\lim_{h\downarrow 0} \int \left|\frac{f(x+h)-f(x)}{h}\right| = \int |f'(x)| = 0$$

from which we conclude $f'(x) = 0$ off of a set of measure $< \varepsilon$. Since ε was arbitrary, $f'(x) = 0$ a.e. on $[-n,n]$. By the FTOC for Lebesgue theory, $f(x) - f(-n) = \int_{-n}^{x} f' = 0$ for a.e. x. Hence, f is actually a constant for a.e. $x \in [-n,n]$, and we conclude $f = 0$ a.e. on $\mathbb{R}$, as discussed at the beginning of the solution. $\square$

(4)(a) $E_i \subset [0,1]$, $i = 1, \ldots, 7$ $\qquad m(E_i) \geq \frac{1}{2}$.

Show $\exists$ distinct $i_1, \ldots, i_4$ s.t.

$$E_{i_1} \cap E_{i_2} \cap E_{i_3} \cap E_{i_4} \neq \emptyset.$$

Assume otherwise. Then

$$\sum_{i=1}^{7} \chi_{E_i} \leq 3 \quad \text{everywhere.}$$

Hence $3 = 3m([0,1]) \geq \int_{[0,1]} \sum_{i=1}^{7} \chi_{E_i} = \sum_{i=1}^{7} m(E_i) \geq 3.5$

a contradiction. $\square$

(b) $f_n \in L^1([0,1])$, $f_n \geq 0$ s.t.

$\int_{[0,\frac{1}{n}]} f_n \geq \frac{1}{2}$. Prove $\int_{[0,1]} \sup_n f_n(x) = \infty$.

$$\int_{[0,1]} \sup_n f_n(x) \underset{\text{positivity}}{=} \sum_{n=1}^{\infty} \int_{[\frac{1}{n+1}, \frac{1}{n}]} \sup_m f_m(x) .$$

Suppose $\int_{[0,1]} \sup_n f_n(x) < \infty$.

Then the tail of the above series goes to zero, i.e.

$$\int_{[0,\frac{1}{n}]} \sup_m f_m(x) = \sum_{k=n}^{\infty} \int_{[\frac{1}{k+1}, \frac{1}{k}]} \sup_m f_m(x)$$

$\longrightarrow 0$ as $n \to \infty$. But then there is an n s.t.

$$\frac{1}{2} > \int_{[0,\frac{1}{n}]} \sup_m f_m(x) \geq \int_{[0,\frac{1}{n}]} f_n(x) \geq \frac{1}{2}$$

a contradiction. $\square$