REAL ANALYSIS GRADUATE EXAM
Fall 2011

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $f \geq 0$ and suppose $f \in L^1([0,\infty))$. Find

$$\lim_n \frac{1}{n} \int_0^n x f(x)\, dx.$$

(2) Suppose $f \geq 0$ is absolutely continuous on $[0,1]$ and $\alpha > 1$. Show that f^α is absolutely continuous.

(3)(a) Let $\{\mu_k\}$ be a sequence of finite signed measures. Find a finite positive measure μ such that $\mu_k \ll \mu$ for all k.

(b) Construct an increasing function whose set of discontinuities is $\mathbb{Q}$. (Prove it is a valid example.)

(4) Let m be Lebesgue measure on $\mathbb{R}$. For $f \in L^1_{loc}$ and $x \in \mathbb{R}^n$, define the function $A_r f$ by

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)\, dy,$$

which is the average value of f on the ball $B(x,r)$ of radius r centered at x, and define the function Hf by $Hf(x) = \sup_{r>0} A_r|f|(x), x \in \mathbb{R}^d$.

(a) Show that for $f \in L^1(\mathbb{R}^n), f \neq 0$, there exist $C, C', R > 0$ such that $Hf(x) \geq C|x|^{-n}$ for all $|x| > R$ and

$$m\Big(\{x : Hf(x) > \alpha\}\Big) \geq \frac{C'}{\alpha} \quad \text{for all sufficiently small } \alpha.$$

(b) Define the function $H^* f$ by

$$H^* f(x) = \sup\left\{\frac{1}{m(B)} \int_B |f(y)|\, dy : B \text{ is a ball containing } x\right\}.$$

Show that $Hf \leq H^* f \leq 2^n Hf$. (Note that unlike Hf, in the definition of $H^* f$ the ball B need not be centered at x.)

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(1) $f \in L^1([0,\infty))$. (Note: we do not need to assume $f \geq 0$).

Find $\lim_{n\to\infty} \frac{1}{n}\int_0^n x f(x)\,dx$.

On $[0,\infty)$ let $f_n = \frac{x}{n}\chi_{[0,n]} f(x)$.

Then $f_n(x) \to 0$ everywhere.

Also, $|f_n| = \frac{x}{n}|f(x)|\chi_{[0,n]}$

$= 0$ on $\{x > n\}$ and $\leq |f(x)|$ on $\{0 \leq x \leq n\}$.

Hence $|f(x)| \in L^1([0,\infty))$ is a dominating function for $\{f_n\}$ on $[0,\infty)$. By the DCT

$$\lim_{n\to\infty} \int_0^n x f(x)\,dx = \lim_{n\to\infty} \int_{[0,\infty)} \frac{x}{n}\chi_{[0,n]} f(x)\,dx = 0.$$

$\square$

(2) $f \geq 0$ is absolutely continuous on $[0,1]$ and $\alpha > 1$. Show f^{α} is absolutely continuous.

We will show y^{α} is Lipschitz on $[0,1]$.

Assume this is true. Fix $\varepsilon > 0$. Choose $\delta > 0$ s.t. if $\sum_{i=1}^{n} |x_i - y_i| < \delta$, then $\sum_{i=1}^{n} |f(x_i) - f(y_i)| < \varepsilon$.

Then $\left| f(x_i)^{\alpha} - f(y_i)^{\alpha} \right| \underset{\text{Lipschitz}}{\leq} C \left| f(x_i) - f(y_i) \right|$ for each i

hence $\sum_{i=1}^{n} \left| f(x_i)^{\alpha} - f(y_i)^{\alpha} \right| \leq C \sum_{i=1}^{n} \left| f(x_i) - f(y_i) \right| < C\varepsilon$.

So f^{α} is absolutely continuous on $[0,1]$.

To see that y^{α} is Lipschitz, we simply note since $\alpha > 1$ that $\alpha y^{\alpha-1} \leq \alpha$ on $[0,1]$, hence by the MVT $\frac{|y^{\alpha} - x^{\alpha}|}{|y - x|} \leq \alpha$ for all $x, y \in [0,1]$

Lipschitz results. $\square$

(3) (a) $\{\mu_k\}$ finite signed measures.

Find a positive finite measure μ s.t. $\mu_k \ll \mu$ for all k.

(b) Find an increasing function that is discontinuous exactly on $\mathbb{Q}$.

(a) Set $\mu(E) = \sum_k \frac{|\mu_k|(E)}{|\mu_k|(X)} 2^{-k}$.

This is w.d. since $|\mu_k|(E) \leq |\mu_k|(X) < \infty \ \forall k$.

This is a measure since if $E = \cup E_j$ disjoint, then

$$\sum_k \frac{|\mu_k|(E)}{|\mu_k|(X)} 2^{-k} = \sum_k \sum_l \frac{|\mu_k|(E_l)}{|\mu_k|(X)} 2^{-k}$$

$$= \sum_l \sum_k \frac{|\mu_k|(E_l)}{|\mu_k|(X)} 2^{-k} = \sum_l \mu(E_l),$$

where the switch of sums is justified by positivity (Tonelli). $\square$

(b) Let $\{x_n\}$ be an enumeration of the rationals, and set

$$f(x) = \sum_{x_n \leq x} 2^{-n} \quad (x \in \mathbb{R})$$

which is certainly increasing.

Let $x \in \mathbb{R} - \mathbb{Q}$. We show f is continuous at x. Since f is increasing, hence can only have jump discontinuities, it suffices choose ~~strictly increasing~~ $x_{n_k} \in \mathbb{Q}$ $\to x$ and show $\lim_{k \to \infty} f(x_{n_k}) = f(x)$.

Indeed, $f(x) = \sum_{x_n \leq x} 2^{-n} = \sup_{\{x_{n_1}, \dots, x_{n_N}\} \subset \{x_n \leq x\}} \sum_i^N 2^{-n_i}$.

For each $\{x_{n_1}, \dots, x_{n_N}\}$ there is M s.t. $x_{n_M} \geq \max_i \{x_{n_i}\}$ (since no x_{n_i} equals x)

hence $\sum_i^N 2^{-n_i} \leq \sum_{x_n \leq x_{n_M}} 2^{-n} = f(x_{n_M})$

hence $f(x) \leq \sup_k f(x_{n_k})$, hence $\sup_k f(x_{n_k}) = f(x)$, as desired.

So f is continuous at x. The same argument shows if $\overset{x_n}{\overset{\shortparallel}{x}} \in \mathbb{Q}$, then

$$f(x) = \sup_k f(x_{n_k}) + 2^{-n},$$

hence is not continuous at x. $\square$

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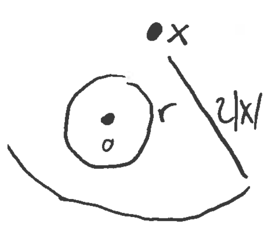
(4) (a)(i) $f \in L^1(\mathbb{R}^n)$, $f \neq 0$.

Show there is $C > 0$, $R > 0$ s.t.

$$Hf(x) \geq C|x|^{-n} \text{ for } |x| > R.$$

Since $f \neq 0$, $\exists c > 0$ s.t. $0 < c < \int |f| < \infty$.

Fix $r > 0$. If $|x| > r$, then $B(x, 2|x|) \supset B(0,r)$.



Hence
$$\frac{1}{2^n C_n |x|^n} \int_{B(x,2|x|)} |f| \geq \frac{1}{2^n C_n |x|^n} \int_{B(0,r)} |f|$$
$$\underset{r \to \infty}{\geq} \frac{c}{2^n C_n |x|^n} \equiv \frac{C}{|x|^n}$$

i.e. $\exists R > 0$ s.t. if $|x| > R$

$$Hf(x) \geq \frac{1}{2^n C_n |x|^n} \int_{B(x,2|x|)} |f| \geq \frac{C}{|x|^n},$$

as desired. $\square$

(a)(ii) Show there is $C' > 0$ s.t.

$$m\left(\{x : Hf(x) > \alpha\}\right) \geq \frac{C'}{\alpha}$$

Want to find an annulus on which $Hf(x) > \alpha$. Consider $R < |x| < R'$, where $R' \equiv \left(\frac{C}{\alpha}\right)^{1/n}$ is $> R$ for $\alpha < \frac{C}{R^n}$.

Then $Hf(x) \geq \frac{C}{|x|^n} \geq \frac{C}{(R')^n} = \alpha$. So

$$m\left(\{x : Hf(x) > \alpha\}\right) \geq m\left(\{x : R < |x| < R'\}\right)$$

$$= C_n\left((R')^n - R^n\right) = C_n\left(\frac{C}{\alpha} - R^n\right)$$

$= \frac{C_n}{\alpha}\left(C - \alpha R^n\right)$. If we assume in addition that $\alpha < \frac{C}{2R^n}$, then this last term is $\geq \frac{C_n}{\alpha}\left(C - \frac{C}{2}\right)$

$= \frac{1}{\alpha}\frac{C_n C}{2} \equiv \frac{C'}{\alpha}$, as desired. $\square$

(b) Let $H^*f(x) = \sup \left\{ \frac{1}{m(B)} \int_B |f(y)|\,dy : \begin{array}{c} B \text{ is a ball} \\ \text{containing } x \end{array} \right\}$.

Show $Hf \le H^*f \le 2^n Hf$.

Since, in particular, $B(x,r)$ is a ball containing x, so $Hf \le H^*f$. The other inequality follows from $x \in B(y,r) \subset B(x,2r)$

as $\begin{cases} |x'-y| < r \\ |x-y| < r \end{cases} \Longrightarrow \begin{aligned} |x-x'| &\le |x-y| + |x'-y| \\ &= 2r \end{aligned}$.

So $$\frac{1}{C_n r^n} \int_{B(y,r)} |f(y)|\,dy \le \frac{2^n}{C_n (2r)^n} \int_{B(x,2r)} |f(y)|\,dy \le 2^n Hf(x).$$

$\square$