REAL ANALYSIS GRADUATE EXAM
Spring 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *upper semicontinuous* (or *u.s.c.*) if for all $x \in \mathbb{R}$ and all $\epsilon > 0$ there exists $\delta > 0$ such that $f(y) < f(x) + \epsilon$ whenever $|y - x| < \delta$.

(i) Show that every u.s.c. function is Borel measurable. HINT: Consider $\{x : f(x) < a\}$.

(ii) Suppose μ is a finite measure on $\mathbb{R}$ and A is a closed subset of $\mathbb{R}$. Using (i) or otherwise, show that the function $x \mapsto \mu(x + A)$ is measurable. Here $x + A = \{x + y : y \in A\}$.

(2) Suppose $\{f_n\}$ and f are measurable functions on $(X, \mathcal{M}, \mu)$ and $f_n \to f$ in measure. Is it necessarily true that $f_n^2 \to f^2$ in measure if:

(a) $\mu(X) < \infty$

(b) $\mu(X) = \infty$.

In each case, prove or give a counterexample.

(3) Suppose $f : [0, 1] \to \mathbb{R}$ is a strictly increasing absolutely continuous function. Let m denote Lebesgue measure. If $m(E) = 0$ show that $m(f(E)) = 0$.

(4) For $n \geq 1$ define h_n on $[0, 1]$ by

$$h_n = \sum_{j=1}^{n} (-1)^j \chi_{(\frac{j-1}{n}, \frac{j}{n}]}.$$

Here χ_E denotes the characteristic function of E. If f is Lebesgue integrable on $[0, 1]$, show that

$$\lim_{n\to\infty} \int_{[0,1]} f h_n \, dm = 0.$$

HINT: First consider f in a suitably smaller function space.

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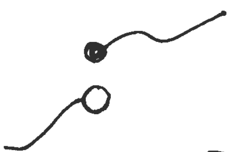
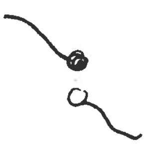
(1) $f : \mathbb{R} \to \mathbb{R}$ upper semicontinuous



if $x_n \to x \Rightarrow \limsup_{x_n \to x} f(x_n) \le f(x)$.

Equivalently, for each x and $\varepsilon > 0$ there exists



$\delta > 0$ s.t. $f(y) - f(x) < \varepsilon$

for all $|x - y| < \delta$.

(i) show f u.s.c. $\Rightarrow$ f Borel.

It suffices to show $f^{-1}(\{x : x < a\})$ is open for all $a \in \mathbb{R}$. Let $f(x) < a$ and set $\varepsilon = a - f(x)$. Choose $\delta > 0$ by the def of u.s.c. so that $|x - y| < \delta \Longrightarrow$ $f(y) - f(x) < \varepsilon = a - f(x)$ i.e. $f(y) < a$ i.e. $y \in f^{-1}(\{x : x < a\})$ i.e. this set is open. $\square$

Suppose μ is finite Borel on $\mathbb{R}$

and A is closed subset of $\mathbb{R}$.

So $X \mapsto \mu(X+A)$ defines a function

$f: \mathbb{R} \to \mathbb{R}$. Show f is Borel.

We show f is u.s.c. by showing

$\limsup_{X_n \to X} f(X_n) \leq f(X)$. We use Fatou.

Observe $-\limsup_n \mu(X_n+A) = -\limsup_n \int \chi_{X_n+A}$

$$= \liminf_n \int -\chi_{X_n+A} \overset{\text{FATOU}}{\geq} \int \liminf_n -\chi_{X_n+A} =$$

$$- \int \limsup_n \chi_{X_n+A} \overset{?}{\geq} - \int \chi_{X+A} =$$

$$-\mu(X+A) \implies \mu(X+A) \geq \limsup_n \mu(X_n+A).$$

It remains to answer "?" for which it suffices to show $\limsup_n \chi_{X_n+A} \leq \chi_{X+A}$.

This is equivalent to saying for all $y \notin x+A$ that $y \in x_n + A$ for only finitely many n. Assume otherwise. Then let $\varepsilon > 0$ and let N be s.t. $|x_n - x| < \varepsilon$ for all $n \geq N$ and take $n \geq N$ s.t. $y \in x_n + A$, say $y = x_n + a$. Then $|y - (x+a)| = |x - x_n| < \varepsilon$. So y is a limit point of $x+A$ which is closed, so $y \in x+A$, violating our assumption. Hence, $\limsup_n \chi_{x_n+A} \leq \chi_{x+A}$ and f is u.s.c. hence Borel by (i). $\square$

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(2) If $f_n \to f$ in measure, show

(i) $m(X) < \infty \implies f_n^2 \to f^2$ in measure,

(ii) $m(X) = \infty \implies f_n^2 \not\to f^2$ in measure, necessarily.

(i) $\lim_{n\to\infty} m(\{x : |f(x) - f_n(x)| \geq \varepsilon\}) = 0 \quad \forall \varepsilon > 0.$

Since $m(X) < \infty$, $\lim_{M\to\infty} m(\{x : |f(x)| > M\}) = 0.$

Fix $\delta, \varepsilon > 0$. NTS $\exists N$ s.t. $n \geq N$

$$\Rightarrow m(\{x : |f^2(x) - f_n^2(x)| > \delta\}) < \varepsilon.$$

$$|f^2(x) - f_n^2(x)| = |ff - ff_n + ff_n - f_nf_n|$$

$$\leq |f||f-f_n| + |f_n||f-f_n|$$

so $\{|f^2 - f_n^2| > \delta\} \subset \{|f||f-f_n| > \frac{\delta}{2}\} \cup$

$$\{|f_n||f-f_n| > \tfrac{\delta}{2}\}.$$

Since $\lim_{M\to\infty} m(\{|f| > M\}) = 0$ choose $M > 0$ s.t.

$$m(\{|f| > M\}) < \varepsilon.$$

So $\{|f||f-f_n| > \frac{\delta}{2}\} = \{|f-f_n| > \frac{\delta}{2M}\} \cup$

$\{|f||f-f_n| > \frac{\delta}{2} \;\&\; |f| > M\}$. Taking

measures, the measure of $\{|f||f-f_n| > \frac{\delta}{2}\}$

is $\leq 2\varepsilon$ taking n sufficiently large.

Note also $\{|f_n| > M\} \subset \{|f-f_n| > \frac{M}{2}\} \cup \{|f| > \frac{M}{2}\}$

so for sufficiently large M & n

$\{|f_n| > M\} < \varepsilon$ and by the same argument

as above $\mu\{|f_n||f-f_n| > \frac{\delta}{2}\} \leq 2\varepsilon$,

possibly increasing n once again. And we are done. $\square$

(ii) Take $f_n(X) = X + \frac{1}{n}$ and $f(X) = X$ on $\mathbb{R}^{\geq 0}$. Then $|f_n(X) - f(X)| = \frac{1}{n} < \varepsilon$ for $n \gg 1$

hence $f_n \to f$ in measure.

Now, $\left| f^2(x) - f_n^2(x) \right| = \left| x^2 - \left(x^2 + \frac{2x}{n} + \frac{1}{n^2}\right) \right|$

$$\overset{\substack{\text{recall} \\ x \geq 0}}{=} \frac{2x}{n} + \frac{1}{n^2} \geq \frac{2x}{n}.$$

But for $x \geq n^2$ then,

$$\left| f^2(x) - f_n^2(x) \right| \geq 2n \geq 2.$$

So for $n \geq 1$, $m(\{x : |f^2(x) - f_n^2(x)| \geq 2\})$

$$\geq m(\{x : x \geq n^2\}) = \infty.$$

So $f_n^2 \not\to f^2$ in measure. $\square$

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(3). $f : [0,1] \to \mathbb{R}$ strictly increasing absolutely continuous. Show if $m(E) = 0$, then $m(f(E)) = 0$.

Since f is strictly increasing and $[0,1]$ is compact and f is continuous, f is a homeomorphism from $[0,1]$ onto its image. So $f(E)$ is Borel.

~~Since f is continuous it is measurable.~~

~~Since f is strictly increasing,~~

~~$f([0,1]) \subset [f(0), f(1)]$ (in fact, an equality).~~

~~So f is bounded, hence is integrable on $[0,1]$.~~

We use the COV theorem for absolutely continuous transformations.

<u>Thm</u> Let $f : [a,b] \longrightarrow \mathbb{R}$ be increasing and absolutely continuous, and let $g : [f(a), f(b)] \to \mathbb{R}$ be integrable. Then

$$\int_{f(a)}^{f(b)} g(y)\,dy = \int_a^b g(f(x))\, f'(x)\, dx .$$

In our situation, let $g(y) = \chi_{f(E)}(y)$ on $[f(0), f(1)]$. Since $f(E)$ is Borel, $g(y)$ is measurable, hence is integrable on $[f(0), f(1)]$. Observe $g(f(x))$

$$= \chi_{f(E)}(f(x)) = \begin{cases} 1 \text{ if } f(x) \in f(E) \\ 0 \text{ if } f(x) \notin f(E) \end{cases}$$

$= \chi_E(x)$. By the theorem, we conclude

$$m(f(E)) = \int_{f(0)}^{f(1)} g(y)\,dy = \int_0^1 g(f(x)) \; f'(x)\,dx$$

$$= \int_0^1 \chi_E(x) f'(x)\,dx = \int_E f'(x)\,dx = 0$$

since $m(E) = 0$. $\square$

(4) For $n \geq 1$ define h_n on $[0,1]$

by $h_n = \sum_{j=1}^{n} (-1)^j \chi_{\left(\frac{j-1}{n}, \frac{j}{n}\right]}$ and $h_n(0) = 0$

(or anything). If f is integrable on $[0,1]$

show $\lim_{n \to \infty} \int f h_n = 0$. $(*)$

It suffices to prove the result for when f is χ_E. Indeed, By linearity, $(*)$ is true for simple functions. ~~If $f \geq 0$ and $\phi_n \leq f$ are simple functions increasing to~~

Then, Fix $\varepsilon > 0$. we show $\left| \int f h_n \right| \to \leq \varepsilon$ as $n \to \infty$.

Since f is integrable, choose a simple function ϕ s.t. $\int |f - \phi| < \varepsilon$. Then $\left| \int f h_n \right| =$

$\left| \int (f-\phi) h_n + \int \phi h_n \right| \leq \int |f - \phi| + \left| \int \phi h_n \right| \leq$

$\varepsilon + \left| \int \phi h_n \right| \longrightarrow \varepsilon + 0$ as $n \to \infty$ since ϕ is simple. We conclude $\lim_{n \to \infty} \int f h_n = 0$.

It remains to prove $\lim_{n\to\infty} \int \chi_E h_n = 0$.

First, observe this is immediate if $E = (a,b)$; for in the integral $\int \chi_{(a,b)} h_n$, the most area that doesn't cancel out is $\frac{1}{n}$, hence $\left| \int \chi_{(a,b)} h_n \right| \le \frac{1}{n} \to 0$ as $n \to \infty$.

It follows that if $U = \bigcup_j (a_j, b_j)$ disjoint, then $\lim_{n\to\infty} \int \chi_U h_n = 0$. Indeed, this is $\lim_{n\to\infty} \int \sum_j \chi_{(a_j,b_j)} h_n = \lim_{n\to\infty} \sum_j \int \chi_{(a_j,b_j)} h_n$

$\left(\text{because } \sum_j \int \left| \chi_{(a_j,b_j)} h_n \right| = \sum_j |b_j - a_j| < \infty \right)$

$= \sum_j \lim_{n\to\infty} \int \chi_{(a_j,b_j)} h_n = 0$

(because if $F_n(j) = \int \chi_{(a_j,b_j)} h_n$, then we have shown $F_n(j) \to 0$ as $n \to \infty$, and also $\left| \int \chi_{(a_j,b_j)} h_n \right| \le \int \chi_{(a_j,b_j)} = |b_j - a_j|$ and $\sum_j |a_j - b_j| < \infty$, hence

$F(j) = |b_j - a_j|$ is a dominating function for the $F_n(j)$; hence, by integrating in counting measure over the j we have

$$\lim_{n\to\infty} \sum_j \int \chi_{(a_j,b_j)} h_n = \sum_j \lim_{n\to\infty} \int \chi_{(a_j,b_j)} h_n$$

$$= \sum_j 0 = 0. \Big)$$

Finally, we use upper regularity to show $\lim\limits_{n\to\infty} \int \chi_E h_n = 0$ for any E.

<u>Lemma</u> If f_n is uniformly bounded and $\lim\limits_{n\to\infty} \int_a^b \chi_u f_n = 0$ for all $U \supset E$, then $\lim\limits_{n\to\infty} \int_a^b \chi_E f_n = 0$.

pf/ Fix $\varepsilon > 0$. By upper regularity, there exists $U = \bigcup_j (a_j, b_j)$ disjoint s.t. $m(U) - m(E) < \varepsilon$. Since $U, E \subset [a,b]$ is finite measure, $m(U - E) < \varepsilon$.

$$\phi_m \to f$$

$$\int f h_n \overset{MCT}{=} \lim_{m\to\infty} \int \phi_m h_n$$

Since f is integrable. Let $\varepsilon > 0$.

$$\int |f - \phi| < \varepsilon$$

$$\left|\int f h_n\right| = \left|\int (f-\phi) h_n + \int \phi h_n\right|$$

$$\le \int |f - \phi| + \left|\int \phi h_n\right| < \varepsilon + \left|\int \phi h_n\right|$$

$\to 0$ as $n \to \infty$.

So $\left| \int \chi_E f_n \right| = \left| \int \chi_U f_n - \int \chi_{U-E} f_n \right|$

$$\leq \left| \int \chi_U f_n \right| + M\varepsilon \quad (|f_n| \leq M)$$

$$\longrightarrow \leq M\varepsilon \text{ as } n \to \infty.$$

Therefore, $\lim_{n\to\infty} \int \chi_E f_n = 0$. ✓

Applying the lemma to $f_n = h_n$, we are done.

$\square$