REAL ANALYSIS GRADUATE EXAM
Fall 2010

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $\mathcal{A}$ be a collection of pairwise disjoint subsets of a σ-finite measure space, and suppose each set in $\mathcal{A}$ has strictly positive measure. Show that $\mathcal{A}$ is at most countable.

(2)(a) Let m denote Lebesgue measure on $\mathbb{R}$ and let f be an integrable function. Show that for $a > 0$,

$$\int f(ax)\; m(dx) = \frac{1}{a} \int f(x)\; m(dx).$$

HINT: Consider a restricted class of functions f first.

(b) Let F be a measurable function on $\mathbb{R}$ satisfying $|F(x)| \leq C|x|$ for all x, and suppose F is differentiable at 0. Show that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{nF(x)}{x(1+n^2x^2)}\; m(dx) = \pi F'(0).$$

HINT: Use (a).

(3) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$, and let f be a measurable function with $|f| < 1$. Prove that

$$\lim_{n\to\infty} \int_X (1 + f + \cdots + f^n)\; d\mu$$

exists (it may be ∞.) HINT: First consider $f \geq 0$.

(4) Let $\{F_j\}$ be a sequence of nonnegative nondecreasing right-continuous functions on $[a, b]$ and suppose $F(x) = \sum_{j=1}^{\infty} F_j(x)$ is finite for all $x \in [a, b]$. Show that

$$F'(x) = \sum_{j=1}^{\infty} F_j'(x) \quad \text{for } m\text{-a.e. } x \in [a, b].$$

HINT: Consider the corresponding measures μ_F and μ_{F_j}.

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(1) Show if μ is σ-finite and each set in a collection $\mathcal{A}$ consisting of disjoint sets has positive measure, then $\mathcal{A}$ is countable.

Suppose $\mathcal{A}$ is uncountable.

Assume μ is finite. Let

$$S = \left\{ \sum_{A \in \mathcal{A}'} \mu(A) : \begin{array}{l}\mathcal{A}' \subset \mathcal{A} \text{ is countable} \\ \mathcal{A}' \text{ consists of disjoint sets}\end{array} \right\}.$$

Let $m < \infty$ be $\sup S$. Let $\mathcal{A}_n'$ be s.t.

$$m_n = \sum_{A \in \mathcal{A}_n'} \mu(A) \longrightarrow m \quad \text{as } n \to \infty.$$

Set $\mathcal{B}' = \bigcup_{m=1}^{\infty} \mathcal{A}_m'$. Since the $\mathcal{A}_m'$ are countable,

so is $\mathcal{B}'$, hence $\underset{\substack{\vee| \\ m_n}}{\sum_{B \in \mathcal{B}'} m(B)} \leq \sup S = m$.

But since $m_n \to m$, we have

$\sum_{B \in \mathcal{B}'} m(B) = m$. Since $\mathcal{A}$ is uncountable and consists of disjoint sets of positive measure, there exists $B^* \notin \mathcal{B}'$ s.t.

$\sum_{B \in \mathcal{B}' \cup \{B^*\}} \mu(B) > m$, a contradiction. Hence, $\mathcal{A}$ is countable.

We reduce to the finite case. Let $X = \bigcup E_j$ disjoint, $\mu(E_j) < \infty$. (unnecessary: We claim at least one of $\mathcal{A}_j = \{A \cap E_j : A \in \mathcal{A}\}$ is uncountable, if $\mathcal{A}$ is. Otherwise, they are all countable. But $\mathcal{A} = \{\bigcup_j A \cap E_j : A \in \mathcal{A}\}$ is obtained by unioning sets from the $\mathcal{A}_j$, so $\mathcal{A}$ would be countable.) Moreover, we claim at least one of $\mathcal{A}_j' = \{A \cap E_j : A \in \mathcal{A} \text{ and } \mu(A \cap E_j) > 0\}$ is uncountable.

~~Combining w/ the reasoning from before, if all such sets were countable, then $\mathcal{A}$, an uncountable collection of disjoint sets of positive measure, could be reconstructed by taking a countable union of disjoint sets of positive (or zero) measure~~

Each $A \in \mathcal{A}$ is a countable union of sets belonging to the $\mathcal{A}_j$; moreover, this partitions the $\mathcal{A}_j$, so that every $E \in \mathcal{A}_j$ corresponds to exactly one A. If there were only a countable number of sets of positive measure among the $\mathcal{A}_j$ (i.e. $\mathcal{A}_j'$ is countable for all j), then only a countable number of the A can have positive measure, as each $E \in \mathcal{A}_j'$ corresponds to exactly one A and A is a countable union. This is a contradiction. So, there is an E_j s.t. $\mathcal{A}_j$ has an uncountable subset of disjoint positive measure sets; applying the finite case to E_j shows $\mathcal{A}_j'$, and hence $\mathcal{A}_j$, is uncountable.

$\square$

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(2)(a) $f \in L^1(m)$ Lebesgue integrable on $\mathbb{R}$.

Show if $a > 0$, then

$$\int f(ax)\,dm = \frac{1}{a}\int f(x)\,dm.$$

It suffices to prove the result for characteristic functions: by linearity, this extends to nonnegative simple functions; by approximation from below by simple functions + MCT, this extends to nonnegative functions; by linearity, this extends to real functions; by linearity, this extends to complex functions.

So assume $f = \chi_E$, $m(E)<\infty$. Then

$$f(ax) = \begin{cases} 1 & \text{if } ax \in E \\ 0 & \text{if } ax \notin E \end{cases} = \chi_{\frac{E}{a}}(x).$$

Hence, $$\int f(ax) = \int \chi_{\frac{E}{a}} = m\left(\frac{E}{a}\right) = \frac{1}{a} m(E)$$

$$= \frac{1}{a}\int \chi_E = \frac{1}{a}\int f. \checkmark$$

(b) If f is a function on $\mathbb{R}$, $|F(x)| \le C|x|$, and $f'(0)$ exists, then

$$\lim_{n\to\infty} \int \frac{nF(x)}{x(1+n^2x^2)}\, dm = \pi F'(0).$$

First, we show $\int \frac{nF(x)}{x(1+n^2x^2)}$ makes sense.

we have $\int \frac{n|F(x)|}{|x|(1+n^2x^2)} \leq Cn \int \frac{dx}{1+n^2x^2}$

$$= C \tan^{-1}(nx)\Big|_{-\infty}^{\infty}$$

$$= C\pi < \infty.$$

So $\frac{nf(x)}{x(1+n^2x^2)} \in L^1(m)$.

~~Moreover,~~ $\cancel{\frac{Cn}{1+n^2x^2}}$ ~~is a dominating for~~ $\cancel{\{g_n\}}$~~, hence by the DCT~~

$$\cancel{\lim_{n\to\infty}}$$

By part (a), we may effectively use u-substitution, yielding

$$\int \frac{nF(x)}{x(1+n^2x^2)}\,dx = \int \frac{F(\frac{u}{n})}{\frac{u}{n}(1+u^2)}\,du.$$

$u = nx.$

$dx = \frac{du}{n}.$

Since $\int \frac{|F(\frac{u}{n})|}{|\frac{u}{n}|(1+u^2)} \leq C \int \frac{du}{1+u^2} = \pi C,$

we have that $\frac{C}{1+u^2}$ is a dominating function for $\{g_n(u)\}$, $g_n(u) = \frac{F(\frac{u}{n})}{\frac{u}{n}(1+u^2)}$.

Also, since $|F(x)| \le C|x|$, we have $F(0) = 0$, hence

$$F'(0) = \lim_{h \to 0} \frac{F(h)}{h} = \lim_{n \to \infty} \frac{F(\frac{u}{n})}{\frac{u}{n}}.$$

Therefore, by the DCT we conclude

$$\lim_{n \to \infty} \int \frac{F(\frac{u}{n})}{\frac{u}{n}(1+u^2)} = \int \lim_{n \to \infty} \frac{F(\frac{u}{n})}{\frac{u}{n}(1+u^2)}$$

$$= F'(0) \int \frac{du}{1+u^2} = \pi F'(0).$$

$\square$

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(3) $m(X) < \infty$ and $|f| < 1$. Show

$\lim_{n \to \infty} \int 1 + f + \cdots + f^n$ exists. (Possibly ∞.)

~~In this problem we assume f is real-valued.~~

For each n, $\int 1 + f + \cdots + f^n$ makes sense.

Indeed, $\int \left| 1 + f + \cdots + f^n \right| \leq \int 1 + |f| + \cdots + |f|^n$

$\leq (n+1)\mu(X) < \infty$.

~~To say $\lim_{n\to\infty} \int 1 + f + \cdots + f^n$ exists is to say that the series $\sum_{n=0}^{\infty} \int f^n$ converges.~~

Assume the result is proved for $f \geq 0$ and $f \leq 0$. Then the result follows for f real. Indeed, write $f = f_+ - f_-$. Then

$f^n = f_+^n + (-1)^n f_-^n$, so

$$\lim_{n\to\infty} \int 1 + f + \cdots + f^n = \lim_{n\to\infty} \int 1 + f_+ + \cdots + f_+^n \quad +$$

$$\lim_{n\to\infty} \int 1 + (-f_-) + \cdots + (-f_-)^n$$

so long as at least one of these limits is finite; we will show the latter limit is always finite. So let us establish the positive and negative cases.

when $f \geq 0$, $1 + f + \cdots + f^n$ is increasing with n, so the MCT shows $\lim_{n\to\infty} \int 1 + f + \cdots + f^n$

$= \int \lim_{n\to\infty} 1 + f + \cdots + f^n$ exists (possibly ∞).

when $-1 < f \leq 0$, $1 + f + \cdots + f^n = \dfrac{1 - f^{n+1}}{1-f} \leq 1$

since $|f|^{n+1} \leq |f|$. Hence, 1 serves as a dominating function, and we obtain

$$\lim_{n\to\infty} \int 1 + f + \cdots + f^n = \int \lim_{n\to\infty} 1 + f + \cdots + f^n$$

$$= \int \frac{1}{1-f} < \infty$$

(since $\mu(X) < \infty$ and $\frac{1}{1-f} \leq 1$). So we are done. $\square$

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(4) Let $\{F_j\}$ be a sequence of nonnegative nondecreasing right-continuous functions on $[a,b]$ and suppose $F(x) = \sum F_j(x) < \infty$ for all $x \in [a,b]$. Show $F'(x) = \sum F_j'(x)$ for a.e. x.

(and increasing & right continuous)

Since $F \geq 0$ ✓ and $F_j \geq 0$, we may extend F and F_j to functions in NBV by setting $F(x) = \begin{cases} 0, & x < a \\ F(b), & x > b \end{cases}$ and similarly for F_j.

Thus, we may consider the positive Borel measures μ_F and μ_{F_j} satisfying $F(x) = \mu_F((-\infty, x])$ and similarly for F_j. There measures are uniquely determined by this property, hence $\mu_F = \sum \mu_{F_j}$. By Lebesgue-RN let $d\mu_F = d\lambda + f\,dm$ and $d\mu_{F_j} = d\lambda_j + f_j\,dm$.

Now, by the Lebesgue Differentiation Theorem,

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h \to 0} \frac{\mu_F((x, x+h])}{h} = f$$

a.e., i.e. F is differentiable a.e. and $F'(x) = f(x)$. Similarly $F_j'(x) = f_j(x)$ a.e. Observe

$$\lambda(E) + \int_E f\,dm = \mu(E) = \sum_j \mu_j(E) = \sum_j \lambda_j(E) + \sum_j \int_E f_j\,dm$$

$$\overset{\text{positivity}}{=} \sum \lambda_j(E) + \int_E \sum_j f_j\,dm.$$

It is easily shown that since λ_j is mutually singular to m, so is $\sum \lambda_j$. Hence, by the uniqueness in the Lebesgue-RN theorem, $\lambda = \sum_j \lambda_j$ and $f = \sum_j f_j$ a.e. That is,

by passing to a set on which F' and F_j' are mutually defined, $F'(x) = \sum_j F'(x)$ a.e. $\square$