Intro

Here are my solutions to some of USC's qualifying exams. A lot of the solutions here are ones I came up with myself, but many other ones are adapted from ideas that I found either online or in textbooks, so I definitely don't claim all of the credit for everything here. I've put a question mark (?) next to solutions I didn't feel completely confident in; and although I've done my best to avoid this, some of the other solutions may contain mistakes too, so please keep that in mind. Thanks and good luck! – Alec.

Notation

Below is a guide of notation and terminology you'll find throughout my solutions. If a problem uses the symbols below to mean something else, then I'll do the same for that problem.

- $\mathbb{1}_E$ denotes the indicator function of a measurable set E.
- $\mathcal{B}_X$ denotes the Borel σ-algebra of a topological space X.

Exams

2006, Spring 1

2006, Fall 4

2007, Spring 5

2007, Fall 7

2008, Spring 9

2008, Fall 10

2009, Spring 12

2010, Spring 14

2010, Fall 17

2011, Spring 19

2006, Spring

Problem 1.

- **No**. Consider the function $f : \mathbb{R} \to \mathbb{R}$ consisting of symmetric triangular spikes of height j and base $2j^{-3}$ at each integer $j \geq 2$ along $\mathbb{R}$. Explicitly, f is given by

$$f(x) := \begin{cases} j^4(x-j) & j \geq 2, x \in [j, j+j^{-3}), \\ j^4[(j+2j^{-3})-x] & j \geq 2, x \in [j+j^{-3}, j+2j^{-3}), \\ 0 & \text{else}. \end{cases}$$

The $\mathsf{L}^1(\mathbb{R})$-norm of f is given by the sum of the areas of the triangles,

$$\|f\|_{\mathsf{L}^1(\mathbb{R})} = \sum_{j=2}^{\infty} j \cdot \frac{1}{j^3} = \sum_{j=2}^{\infty} \frac{1}{j^2} < \infty.$$

However, f isn't bounded and $\lim_{x\to\infty} f(x)$ is nonexistent, so neither (i) nor (ii) hold. □

- **Both (i) and (ii) hold** if f' exists everywhere and $|f'| \leq C$ for some $C > 0$.

Assume first that $f(x) \not\to 0$ as $x \to \infty$. Then there's some $\epsilon > 0$ for which we can find a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}$ with $x_j \to \infty$ and $f(x_j) \geq \epsilon$ for each $j \in \mathbb{N}$. We may assume w.l.o.g. that $x_1 \leq x_2 \leq \cdots$ and $|x_{j+1} - x_j| > 2\epsilon/C$ for all $j \in \mathbb{N}$. Fix some $j \in \mathbb{N}$; then $|f(x_j)| \geq \epsilon$, so assume w.l.o.g. that $f(x_j) \geq \epsilon$. For any $y \in (x_j - (\epsilon/C), x_j)$, we have by the mean value theorem that

$$\frac{f(x_j) - f(y)}{x-y} \leq C \implies \epsilon \leq f(x_j) \leq C(x_j - y) + f(y) \implies C(y - x_j) + \epsilon \leq f(y),$$

and similarly $C(x_j - y) + \epsilon \leq f(y)$ for any $y \in (x_j, x_j + (\epsilon/C))$. Then

$$\int_{x_j-(\epsilon/C)}^{x_j+(\epsilon/C)} f(y)\mathsf{d}y \geq \int_{x_j-(\epsilon/C)}^{x_j} [C(y-x_j)+\epsilon]\mathsf{d}y + \int_{x_j}^{x_j+(\epsilon/C)} [C(x_j-y)+\epsilon]\mathsf{d}y = \frac{2\epsilon^2}{C},$$

and so

$$\int_{\mathbb{R}} |f| \geq \sum_{j=1}^{\infty} \int_{x_j-(\epsilon/C)}^{x_j+(\epsilon/C)} |f(y)|\mathsf{d}y \geq \sum_{j=1}^{\infty} \frac{2\epsilon^2}{C} = \infty,$$

contradicting $f \in \mathsf{L}^1(\mathbb{R})$.

Assume next that f is unbounded. If $f(x) \to 0$ as $x \to \infty$, then there some $M > 0$ large enough so that $|f(x)| \leq 1$ for all $x \in \mathbb{R}$ with $|x| > M$. Thus f must be unbounded on the compact set $[-M, M]$, which is impossible since f is continuous. Hence $f(x) \not\to 0$ as $x \to \infty$, which leads to a contradiction as above. □

Problem 2.

(a) For any $x, y > 0$,

$$\frac{1-e^{-yx^2}}{x^2} = \frac{1}{x^2}\left[1 - \sum_{j=0}^{\infty} \frac{(-1)^j y^j x^{2j}}{j!}\right] = -\sum_{j=1}^{\infty} \frac{(-1)^j y^j x^{2(j-1)}}{j!} = -\sum_{j=0}^{\infty} \frac{(-1)^{j+1} y^{j+1} x^{2j}}{(j+1)!}$$
$$\leq y \sum_{j=0}^{\infty} \frac{(-1)^j y^j x^{2j}}{j!} = ye^{-yx^2}$$

and hence by the substitution $s := \sqrt{y}x$,

$$0 \le G(y) \le \int_0^\infty ye^{-yx^2}\,\mathrm{d}x = \sqrt{y}\int_0^\infty e^{-s^2}\,\mathrm{d}s = \frac{\sqrt{\pi y}}{2} < \infty.$$

$\square$

(b) For any $y > 0$,

$$G'(y) = \lim_{z\to y}\frac{G(y)-G(z)}{y-z} = \lim_{z\to y}\int_0^\infty \frac{-e^{-yx^2}+e^{-zx^2}}{(y-z)x^2}\,\mathrm{d}x = -\lim_{z\to y}\int_0^\infty \frac{e^{-yx^2}-e^{-zx^2}}{y-z}\cdot\frac{1}{x^2}\,\mathrm{d}x.$$

Provided that we can justify moving the limit inside the integral, then

$$G'(y) = -\int_0^\infty \lim_{z\to y}\frac{e^{-yx^2}-e^{-zx^2}}{y-z}\frac{\mathrm{d}x}{x^2} = \int_0^\infty \frac{\mathrm{d}e^{-zx^2}}{\mathrm{d}z}\Big|_{z=y}\frac{\mathrm{d}x}{x^2} = \int_0^\infty \frac{-x^2e^{-yx^2}}{x^2}\,\mathrm{d}x = \int_0^\infty e^{-yx^2}\,\mathrm{d}x,$$

and by the substitution $s := \sqrt{y}x$,

$$G'(y) = \frac{1}{\sqrt{y}}\int_0^\infty e^{-s^2}\,\mathrm{d}s = \frac{1}{2}\sqrt{\frac{\pi}{y}},$$

and taking the antiderivative gives $G(y) = \sqrt{\pi y} + c$ for some $c \in \mathbb{R}$. From the definition of G we see that $G(0) = 0$ and now that $G(0) = c$, whereby $c = 0$ and so $G(y) = \sqrt{\pi y}$. To justify exchanging the limit and integration above, it suffices by dominated convergence to bound the integrand by an integrable function. Assume w.l.o.g. that $y < z$. By the mean value theorem, there's some $z_0 \in (y, z)$ with

$$\left|\frac{-e^{-yx^2}+e^{-zx^2}}{(y-z)x^2}\right| = \left|\frac{\partial e^{-zx^2}}{\partial z}\Big|_{z=z_0}\cdot\frac{1}{x^2}\right| \le \sup_{z_1\in(y,z)}\left|\frac{\partial e^{-zx^2}}{\partial z}\Big|_{z=z_1}\cdot\frac{1}{x^2}\right| = \sup_{z_1\in(y,z)}\left|\frac{-x^2e^{-z_1x^2}}{x^2}\right|$$
$$= \sup_{z_1\in(y,z)}\left|1 + z_1x + \frac{(z_1x)^2}{2!} + \frac{(z_1x)^3}{3!} + \frac{(z_1x)^4}{4!} + \cdots\right|^{-1} \le \sup_{z_1\in(y,z)}\frac{2}{z_1^2x^2} \le \frac{2}{y^2x^2},$$

and the right-hand side, when regarded as a function of x on $(0,\infty)$, is integrable. $\square$

Problem 3.

Since $(X, \mathcal{M}, \mu)$ is σ-finite, then $X = \bigsqcup_{j\in J} X_j$ for some countable collection $\{X_j\}_{j\in J} \subset \mathcal{M}$ with $\mu(X_j) < \infty$ for each $j \in J$. Fix some $j \in J$. By Egoroff, for each $k \in \mathbb{N}$, there's a subset $Y_{j,k} \subset X_j$ in $\mathcal{M}$ with $\mu(X_j \setminus Y_{j,k}) < k^{-1}$ and with $f_n \to f$ uniformly on $Y_{j,k}$. We may assume w.l.o.g. that $Y_{j,1} \subset Y_{j,2} \subset \cdots$, so by construction, $Y_{j,k} \nearrow X_j$ (up to a null set) as $k \to \infty$. Setting $F_{j,k} := Y_{j,k} \setminus Y_{j,k-1}$ for each $k \in \mathbb{N}$, we still have $f_n \to f$ uniformly on $F_{j,k}$, and furthermore the collection $\{F_{j,k}\}_{k\in\mathbb{N}}$ is disjoint, so we may write X as the disjoint union

$$X = E_0 \sqcup \bigsqcup_{\substack{j\in J\\ k\in\mathbb{N}}} F_{j,k},$$

where E_0 is the null set $\bigcap_{k=1}^\infty \bigcup_{j\in J}(X_j \setminus Y_{j,k})$. Letting $\{E_\ell\}_{\ell=1}^\infty$ be an enumeration of the countable collection $\{F_{j,k}\}_{j\in J,k\in\mathbb{N}}$, we obtain the desired partition. $\square$

Problem 4.

(a) An equivalent definition for a function $g : \mathbb{R} \to \mathbb{R}$ to be l.s.c. is that $\{x \in \mathbb{R} \mid a < f(x)\}$ is an open set for all $a \in \mathbb{R}$ (see (b)). To see that f has this property, let $a \in \mathbb{R}$ and suppose $a < f(x) = \mathsf{sup}_{j \in \mathbb{N}} f_j(x)$ for some $x \in \mathbb{R}$. Then by definition of sup, there's some $k \in \mathbb{N}$ with $a < f_k(x)$. But f_k is continuous, so there's some $\delta > 0$ such that for all $y \in \mathbb{R}$ with $|x - y| < \delta$, we have $a < f_k(y) \leq \mathsf{sup}_{j \in \mathbb{N}} f_j(y) = f(y)$.

(Note that we in fact only need the f_j's to be l.s.c.) □

(b) This is very similar to problem 1 of 2010, Spring.

2006, Fall

Problem 1.

Let S be the collection of all 1-point subsets of $\mathbb{R}$, and $\sigma(S)$ the σ-algebra generated by S. Now let $\mathcal{F} := \{E \subset \mathbb{R} \mid E \text{ is countable or cocountable}\}$ (it's easy to show that $\mathcal{F}$ is a σ-algebra). We claim that $E \in \sigma(S)$ if and only if $E \in \mathcal{F}$. The inclusion $S \subset \mathcal{F}$ is immediate, so $\sigma(S) \subset \mathcal{F}$. Conversely if $E \in \mathcal{F}$ is countable (resp. cocountable), then it's a countable union (resp. complement of a countable union) of 1-point subsets, and hence $E \in \sigma(S)$; so $\mathcal{F} \subset \sigma(S)$. □

Problem 2.

(a) **True**. By Hölder, $\|f\|_{\mathsf{L}^1(\mu)} \leq \|f\|_{\mathsf{L}^2(\mu)} \|1\|_{\mathsf{L}^2(\mu)} = \|f\|_{\mathsf{L}^2(\mu)} \mu(X)^{1/2} < \infty$. □

(b) **False**. Set $X := (1, \infty)$ with Lebesgue measure μ, and $f(x) := x^{-1}$. Then

$$\|f\|_{\mathsf{L}^1(\mu)} = \int_1^\infty x^{-1} \mathsf{d}x = \infty, \quad \|f\|_{\mathsf{L}^2(\mu)} = \left(\int_1^\infty x^{-2} \mathsf{d}x \right)^{1/2} = 1 < \infty.$$

□

(c) **False**. Set $X := (0, 1)$ with Lebesgue measure μ, and $f(x) := x^{-1/2}$. Then

$$\|f\|_{\mathsf{L}^1(\mu)} = \int_0^1 x^{-1/2} \mathsf{d}x = 2 < \infty, \quad \|f\|_{\mathsf{L}^2(\mu)} = \left(\int_0^1 x^{-1} \mathsf{d}x \right)^{1/2} = \infty.$$

□

(d) **False**. Extend the function f in (c) to all of $X := \mathbb{R}$ by setting $f :\equiv 0$ outside of $(0, 1)$. □

Problem 3 (?)**.**

(a) **No**. We have $|f(x, y)| = |f(y, x)|$ for any $(x, y) \in \mathbb{R}^2$, and so by symmetry

$$\|f\|_{\mathsf{L}^1(\mathbb{R}^2)} = \iint_{\mathbb{R}^2} |f| = 2 \iint_{\{x > y\}} |f(x, y)| \mathsf{d}y \mathsf{d}x = 2 \int_{-\infty}^\infty \underbrace{\int_{-\infty}^x e^{-(x-y)} \mathsf{d}y}_{=1} \, \mathsf{d}x = \infty$$

(the inner integral is equal to 1 by an easy computation). □

(b) **Yes**. Both integrals are equal to 0 by substitution. □

Problem 4.

The function $|f|$ is in $\mathsf{L}^1(\mathbb{R})$, and for each $n \in \mathbb{N}$ we have $|f_n| = |f| \cdot |\sin(x)|^n \leq |f|$, hence

$$\|f_n\|_{\mathsf{L}^1(\mathbb{R})} = \int_{\mathbb{R}} |f_n| \leq \int_{\mathbb{R}} |f| = \|f\|_{\mathsf{L}^1(\mathbb{R})} < \infty.$$

Now $|\sin(x)| < 1$ for a.e. $x \in \mathbb{R}$, so $\lim_{n\to\infty} f_n = 0$ a.e. Then

$$\lim_{n\to\infty} \int_{-\infty}^\infty f_n(x) \mathsf{d}x = \int_{-\infty}^\infty 0 \, \mathsf{d}x = 0$$

by dominated convergence. □

2007, Spring

Problem 1.

Firstly, $\mu() = \lim_{n\to\infty} \mu_n() = \lim_{n\to\infty} 0 = 0$. Now let $\{E_j\}_{j\in J} \subset \mathcal{M}$ be a disjoint collection indexed by a countable set $J \subset \mathbb{N}$, and for each $n \in \mathbb{N}$, let $f_n : \mathbb{N} \to \mathbb{R}$ be given by $f_n(j) := \mu_n(E_j)$. By assumption, $f_1 \leq f_2 \leq \cdots$, and $f_n \nearrow f$ for $f(j) := \mu(E_j)$. If ν is the counting measure on $\mathbb{N}$, then

$$\mu\Big(\bigcup_{j\in J} E_j\Big) = \lim_{n\to\infty} \mu_n\Big(\bigcup_{j\in J} E_j\Big) = \lim_{n\to\infty} \sum_{j\in J} \mu_n(E_j) = \lim_{n\to\infty} \int_{\mathbb{N}} f_n \mathsf{d}\nu = \int_{\mathbb{N}} f \mathsf{d}\nu = \sum_{j\in J} \mu(E_j)$$

by monotone convergence. □

Problem 2.

(a) Let $0 < \alpha < \mu(X)$, and assume the $\inf$ in question is 0. Then we can find a sequence $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$ such that $\mu(E_j) \geq \alpha$ and $\int_X f \mathbb{1}_{E_j} = \int_{E_j} f < j^{-1}$. Then the sequence $\{f\mathbb{1}_{E_j}\}_{j=1}^\infty$ converges to 0 in measure, so there's some subsequence $\{f\mathbb{1}_{E_{j_k}}\}_{k=1}^\infty$ converging to 0 a.e. In this case,

$$0 = \mu\left(\limsup_{k\to\infty} E_{j_k}\right) = \mu\Big(\bigcap_{\ell=1}^\infty \bigcup_{k=\ell}^\infty E_{j_k}\Big)$$

so for any $\epsilon > 0$ there must be some $\ell \in \mathbb{N}$ satisfying the last inequality below,

$$\alpha \leq \mu(E_{j_\ell}) \leq \mu\Big(\bigcup_{k=\ell}^\infty E_{j_k}\Big) < \epsilon.$$

Choosing $\epsilon < \alpha$ gives a contradiction. □

(b) Let $X := (1, \infty)$ with Lebesgue measure μ. The function $f(x) := x^{-2}$ is strictly positive on $(1, \infty)$ and $\int_{(1,\infty)} f = 1$, so $f \in \mathsf{L}^1(\mu)$. However for $\alpha := 1$, the intervals $(j, j+1)$ for $j \in \mathbb{N}$ satisfy $\mu((j, j+1)) = 1$, and for any $\epsilon > 0$, we can choose j large enough so that

$$\int_{(j,j+1)} f = \int_j^{j+1} \frac{\mathsf{d}x}{x^2} = \frac{1}{j^2 + j} < \epsilon.$$

Thus the $\inf$ in question must be 0. □

Problem 3.

Denote by μ the Lebesgue measure on $\mathbb{R}^2$, and let $\epsilon > 0$. Since $[0, 1]$ is compact, f is uniformly continuous, so there's some $0 < \delta < 1$ so that $|f(x) - f(y)| < \epsilon/4$ whenever $|x - y| < \delta$. Let $0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1$ be a partition with $|x_j - x_{j+1}| < \delta$ for each $0 \leq j \leq m-1$ and with $m \in \mathbb{N}$ the smallest integer satisfying $m\delta > 1$. Then $(m-1)\delta \leq 1$ and so $m\delta \leq 1 + \delta < 2$. Our choice of δ yields

$$\mathsf{graph}(f) \subset \bigcup_{j=0}^{m-1} [x_j, x_{j+1}] \times \left[f(x_j) - \frac{\epsilon}{4}, f(x_j) + \frac{\epsilon}{4}\right] \implies \mu(\mathsf{graph}(f)) \leq \sum_{j=0}^{m-1} \delta \cdot \frac{2\epsilon}{4} = m\delta \cdot \frac{\epsilon}{2} < \epsilon.$$

Therefore $\mu(\mathsf{graph}(f)) = 0$. □

Problem 4 (?)**.**

Fix $u \in (0,1)$. Provided we may exchange the order of differentiation and integration, then

$$g'(u) = \int_{-\infty}^{\infty} \frac{\mathsf{d}}{\mathsf{d}u}\left(\frac{x^n e^{ux}}{e^x+1}\right)\mathsf{d}x = \int_{-\infty}^{\infty} \frac{x^{n+1}e^{ux}}{e^x+1}\mathsf{d}x.$$

This exchange is valid if the integrand on the right-hand side is bounded (in magnitude) a.e. by an integrable function. To see this, let $\epsilon > 0$ be such that $u \in (0, 1-\epsilon)$. Then for $x > 0$, we have

$$1 < e^x \implies e^{ux} = (e^x)^u < (e^x)^{1-\epsilon} = e^{(1-\epsilon)x}$$

and for $x < 0$ we have $e^x < 1$. So for any $x \in \mathbb{R}$, we have $e^{ux} < 1 + e^{(1-\epsilon)x}$, whereby

$$\left|\frac{x^{n+1}e^{ux}}{e^x+1}\right| \le \left|\frac{x^{n+1}(1+e^{(1-\epsilon)x})}{e^x+1}\right| \le \left|\frac{x^{n+1}}{e^x+1}\right| + \left|\frac{x^{n+1}e^{(1-\epsilon)x}}{e^{x+1}}\right| \le \left|\frac{x^{n+1}}{e^x+1}\right| + \left|\frac{x^{n+1}}{e^{1+\epsilon x}}\right|.$$

Both summands on the right are integrable, so this completes the proof. □

2007, Fall

Problem 1.

Let $n \in \mathbb{N}$ and $t > 0$. Choose $\epsilon > 0$ small enough so that $t > \epsilon$. By dominated convergence, we may move the operator $\mathsf{d}^n/\mathsf{d}t^n$ inside the given integral since

$$\left|\frac{\mathsf{d}^n}{\mathsf{d}t^n} e^{-tx^2}\right| = \left|(-1)^n x^{2n} e^{-tx^2}\right| \leq \left|x^{2n} e^{-\epsilon x^2}\right|,$$

and the right-hand side, regarded as a function of x on $\mathbb{R}$, is integrable. Hence

$$\int_{-\infty}^{\infty} (-1)^n x^{2n} e^{-tx^2} \mathsf{d}x = \int_{-\infty}^{\infty} \frac{\mathsf{d}^n}{\mathsf{d}t^n} e^{-tx^2} \mathsf{d}x = \frac{\mathsf{d}^n}{\mathsf{d}t^n} \sqrt{\frac{\pi}{t}} = \sqrt{\pi} \cdot \frac{(-1)^n (2n)!}{4^n n!} t^{-(2n+1)/2},$$

whereby setting $t := 1$ gives

$$\int_{-\infty}^{\infty} x^{2n} e^{-x^2} \mathsf{d}x = \frac{(2n)!\sqrt{\pi}}{4^n n!},$$

as desired. □

Problem 2.

(a) Set $f_j := j^2 \mathbb{1}_{(0,j^{-1})}$ for each $j \in \mathbb{N}$. Then

$$\lim_{j\to\infty} \int_{(0,1)} f_j = \lim_{j\to\infty} \int_{(0,j^{-1})} j^2 = \lim_{j\to\infty} j = \infty.$$

However, for any fixed $x \in (0,1)$, for all $j \in \mathbb{N}$ sufficiently large, we have $j^{-1} < x$ and so $f_j(x) = 0$. Thus $\lim_{j\to\infty} f_j(x) = 0$. □

(b) Let $f : [0,1] \to [0,1]$ be the well-known Devil's staircase function. Then f increases continuously from $f(0) = 0$ to $f(1) = 1$. But outside of the measure-0 Cantor set, f' exists and is identically 0, so $f(1) - f(0) = 1 \neq 0 = \int_0^1 f'(x)\mathsf{d}x$. □

Problem 3.

Set $E_j := \{g_j > 2^{-j}\}$ for each $j \in \mathbb{N}$. If $x \in E_j$ for only finitely many $j \in \mathbb{N}$, then there's some $N \in \mathbb{N}$ so that $x \in E_j^{\mathsf{c}}$ for all $j \geq N$, and hence the sum converges for this x,

$$\sum_{j=1}^{\infty} g_j(x) = \sum_{j=1}^{N-1} g_j(x) + \sum_{j=N}^{\infty} g_j(x) < \underbrace{\sum_{j=1}^{N-1} g_j(x)}_{<\infty} + \underbrace{\sum_{j=N}^{\infty} \frac{1}{2^j}}_{<\infty} < \infty.$$

Hence we're done if we can show that the set of those x's belonging to infinitely many E_j's is a null set. This is precisely the set $\limsup_{j\to\infty} E_j$, and we have

$$\mu\left(\limsup_{j\to\infty} E_j\right) = \mu\Big(\bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j\Big) = \lim_{k\to\infty} \mu\Big(\bigcup_{j=k}^{\infty} E_j\Big) \leq \lim_{k\to\infty} \sum_{j=k}^{\infty} \mu(E_j).$$

But each summand on the right is bounded above by 2^{-j}, and the sum $\sum_{j=1}^{\infty} 2^{-j}$ is convergent, whereby the limit on the right is 0. □

Problem 4 (?).

Set $E_t := \mu(\{|g| > t\})$. Integrating by parts,

$$\int_0^\infty \mu(t)\mathsf{d}(t^p) + \int_0^\infty t^p \mathsf{d}\mu(t) = \mu(t)t^p\Big|_0^\infty = \lim_{t\to\infty} \mu(t)t^p.$$

By Fubini, the first integral is equal to

$$\int_0^\infty \left(\int_{\mathbb{R}^d} \mathbb{1}_{E_t}\mathsf{d}x\right) pt^{p-1}\mathsf{d}t = \int_{\mathbb{R}^d}\int_0^\infty \mathbb{1}_{E_t} pt^{p-1}\mathsf{d}t\mathsf{d}x = \int_{\mathbb{R}^d}\int_0^{|g(x)|} pt^{p-1}\mathsf{d}t = \int_{\mathbb{R}^d} |g(x)|^p \mathsf{d}x.$$

Thus the result follows if we can show that $\mathsf{lim}_{t\to\infty}\mu(t)t^p = 0$. Let $\{\varphi_j\}_{j=1}^\infty \subset \mathsf{L}^p(\mathbb{R}^d)$ be a sequence of nonnegative simple functions approaching g with $|\varphi_1| \leq |\varphi_2| \leq \cdots \leq |g|$ a.e. Then for any $t \geq 0$,

$$\{|\varphi_1| > t\} \subset \{|\varphi_2| > t\} \subset \cdots \subset \{|g| > t\} = E_t, \quad E_t = \bigcup_{j=1}^\infty \{|\varphi_j| > t\}.$$

For any $j \in \mathbb{N}$, writing $\varphi_j = \sum_{k=1}^m a_k \mathbb{1}_{A_k}$ for some $a_k \geq 0$ and $A_k \in \mathcal{M}$, the set $\{|\varphi_j| > t\}$ has measure 0 as soon as $t > \mathsf{max}_{1\leq k\leq m} a_k$, whereby

$$\lim_{t\to\infty} \mu(t)t^p = \lim_{t\to\infty}\lim_{j\to\infty} \mu(\{|\varphi_j| > t\})t^p = 0.$$

□

2008, Spring

Incomplete: 4.

Problem 1 (?)**.**

(i) **No**. Suppose the integral exists. Then by Fubini,

$$\int_E \frac{1}{x-y}\mathsf{d}m(x,y) = \int_0^1\int_0^1 \frac{1}{x-y}\mathsf{d}x\mathsf{d}y = \int_0^1 \mathsf{log}\left(1-\frac{1}{y}\right)\mathsf{d}y$$

is well defined. But this is impossible since whenever y belongs to the measure-1 set $[0,1) \subset [0,1]$, we have $1-\frac{1}{y} < 0$ and so $\mathsf{log}\left(1-\frac{1}{y}\right)$ isn't even defined. □

(ii) **Yes**. The integrand is in $\mathsf{L}^+(E,m)$ so by Tonelli,

$$\int_E \frac{1}{x+y}\mathsf{d}m(x,y) = \int_0^1\int_0^1 \frac{1}{x+y}\mathsf{d}x\mathsf{d}y = \int_0^1 \mathsf{log}\left(1+\frac{1}{y}\right)\mathsf{d}y = \mathsf{log}(4)$$

after a routine computation. □

Problem 2.

Let $\mathcal{S} := \{E \subset [0,1] \mid E \text{ compact and } \mu(E) = 1\}$. Firstly if $E_1, E_2 \in \mathcal{S}$, then certainly $E_1 \cup E_2 \subset [0,1]$; so $1 = \mu(E_1) \le \mu(E_1 \cup E_2) \le 1$, whereby

$$\mu(E_1 \cup E_2) = 1 \implies \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cup E_2) = 1 + 1 - 1 = 1.$$

Then inductively, any finite collection $\{E_j\}_{j=1}^m \subset \mathcal{S}$ has measure-1 intersection. We now claim that $\mu(K) = 1$, where K is the (potentially uncountable) intersection $\bigcap_{E\in\mathcal{S}} E$. To see this, let $U \subset [0,1]$ be an open set with $U \supset K$. Then the family of closed sets $\mathcal{T} := \{E \setminus U \mid E \in \mathcal{S}\}$ must satisfy $\bigcap_{E\setminus U\in\mathcal{T}}(E \setminus U) = $. This means that $\mathcal{T}$ doesn't have the finite intersection property, since any family of closed subsets of the compact space $[0,1]$ with this property has nonempty intersection. Thus there's a finite collection $\{E_j \setminus U\}_{j=1}^m \subset \mathcal{T}$ with empty intersection, giving

$$\bigcap_{j=1}^m (E_j \setminus U) = \quad \implies \bigcap_{j=1}^m E_j \subset U \implies 1 = \mu\Big(\bigcap_{j=1}^m E_j\Big) \le \mu(U).$$

Since $U \supset K$ was an arbitrary open set, we have that

$$1 \le \mathsf{inf}\{\mu(U) \mid U \subset [0,1] \text{ open and } U \supset K\} = \mu(K) \le 1$$

by outer regularity of μ. Therefore $\mu(K) = 1$. □

Problem 3.

Neither implication holds.

- Let $f := \mathbb{1}_{(1/2,1]}$, which is continuous a.e. on $[0,1]$, and suppose that there's some continuous $g : [0,1] \to \mathbb{R}$ with $g = f$ a.e. For all $j \ge 3$, the sets $(1/2 - 1/j, 1/2), (1/2, 1/2 + 1/j)$ have positive measure, and thus contain some x_j, y_j, respectively, with $g(x_j) = f(x_j) = 0$ and $g(y_j) = f(y_j) = 1$. Moreover, $x_j \nearrow 1/2$ and $y_j \searrow 1/2$ as $j \to \infty$, so by continuity of g,

$$g\left(\frac{1}{2}-\right) = \lim_{j\to\infty} g(x_j) = \lim_{j\to\infty} 0 = 0, \quad g\left(\frac{1}{2}+\right) = \lim_{j\to\infty} g(y_j) = \lim_{j\to\infty} 1 = 1,$$

  which is impossible since g is continuous at $1/2$. □

- Let $f := \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ and $g :\equiv 0$. Then $f = 0 = g$ outside of the null set $\mathbb{Q} \cap [0,1]$, but f is nowhere continuous on $[0,1]$. □

2008, Fall

Incomplete: 4(b).

Problem 1.

It's enough to show that μ and ν agree on open rectangles, since these generate $\mathcal{B}_{\mathbb{R}^2}$. So, suppose $R = (x_1, x_2) \times (y_1, y_2)$ is such a rectangle, and define the vectors $\boldsymbol{a} := (x_1, y_1), \boldsymbol{b} := (x_2, y_2)$. Let L be the segment $\{t\boldsymbol{a} + (1-t)\boldsymbol{b} \mid t \in (0,1)\}$, let R_1 be the open triangle with endpoints $\boldsymbol{a}, (x_1, y_2), \boldsymbol{b}$, and let R_2 be the open triangle with endpoints $\boldsymbol{a}, (x_2, y_1), \boldsymbol{b}$. Then $R = L \sqcup R_1 \sqcup R_2$, and

$$\mu(R) = \mu(L) + \mu(R_1) + \mu(R_2), \quad \nu(R) = \nu(L) + \nu(R_1) + \nu(R_2).$$

But μ and ν agree on the open triangles R_1, R_2, so we're done if we can show that $\mu(L) = \nu(L)$. Let $\boldsymbol{u}$ be a unit vector orthogonal to $\boldsymbol{b} - \boldsymbol{a}$, and for any $\epsilon > 0$, let L_ϵ be the open triangle with endpoints $\boldsymbol{a} - \epsilon\boldsymbol{u}, \boldsymbol{a} + \epsilon\boldsymbol{u}, \boldsymbol{b}$. Hence we obtain a family of open triangles $\{L_{1/j}\}_{j=1}^\infty$ with $\bigcap_{j=1}^\infty L_{1/j} = L$. Moreover, $\mu(L_1) \le \mu(\mathbb{R}^2) < \infty$ and $\nu(L_1) \le \nu(\mathbb{R}^2) < \infty$, so by continuity from above of the measures μ and ν,

$$\mu(L) = \mu\Big(\bigcap_{j=1}^\infty L_{1/j}\Big) = \lim_{j\to\infty} \mu(L_{1/j}) = \lim_{j\to\infty} \nu(L_{1/j}) = \nu\Big(\bigcap_{j=1}^\infty L_{1/j}\Big) = \nu(L),$$

since μ and ν agree on each of the open triangles $\{L_{1/j}\}_{j=1}^\infty$. $\square$

Problem 2.

For fixed $x > 0$, we have

$$\lim_{n\to\infty} \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n} = \lim_{n\to\infty} \frac{1}{(1+x^2)^n} + \lim_{n\to\infty} \frac{nx^2}{(1+x^2)^n} + \lim_{n\to\infty} \frac{n^2x^4}{(1+x^2)^n}.$$

The first limit is clearly 0. The second and third limits are evaluated via L'Hôspital,

$$\lim_{n\to\infty} \frac{nx^2}{(1+x^2)^n} = \lim_{n\to\infty} \frac{x^2}{\exp(n\log(1+x^2))\log(1+x^2)} = 0,$$
$$\lim_{n\to\infty} \frac{n^2x^4}{(1+x^2)^n} = \lim_{n\to\infty} \frac{2nx^4}{\exp(n\log(1+x^2))\log(1+x^2)} = 0.$$

Then provided that we can justify exchanging the limit and the integral, we have

$$\lim_{n\to\infty} \int_0^\infty \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n} \mathrm{d}x = \int_0^\infty \lim_{n\to\infty} \frac{1 + nx^2 + n^2x^4}{(1+x^2)^n} \mathrm{d}x = 0.$$

To see that this is indeed justified, note that for any $x > 0$, we have

$$\frac{1 + nx^2 + n^2x^4}{(1+x^2)^n} = \frac{1 + nx^2 + n^2x^4}{\sum_{j=0}^n \binom{n}{j} x^{2j}} \le \frac{1 + nx^2 + n^2x^4}{\binom{n}{3} x^{2\cdot 3}} \le \frac{n}{(n-1)(n-2)} \cdot \frac{6(1 + x^2 + x^4)}{x^6}$$

by expanding and rearranging as necessary. Now when $n \ge 3$, we have

$$\frac{\mathrm{d}}{\mathrm{d}n} \frac{n}{(n-1)(n-2)} = \frac{2 - n^2}{(n-1)^2(n-2)^2} \le 0,$$

whereby the function $n/(n-1)(n-2)$ starts to decrease at $n = 3$, yielding

$$\frac{1 + nx^2 + n^2x^4}{(1 + x^2)^n} \leq \frac{3}{(3-1)(3-2)} \cdot \frac{6(1 + x^2 + x^4)}{x^6} = \frac{9(1 + x^2 + x^4)}{x^6}.$$

Regarded as a function of x, the right-hand side is integrable on $(0, \infty)$, and thus we may apply dominated convergence to exchange the limit and integral above as we wished. □

Problem 3.

Let $C > 0$ be such that $|f| \leq C$ a.e. Then using Fubini,

$$\|f\|_{\mathsf{L}^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| \mathsf{d}x = \int_{\mathbb{R}} \int_0^{|f(x)|} \mathsf{d}t\mathsf{d}x = \int_{\mathbb{R}} \int_0^C \mathbb{1}_{\{|f| \geq t\}} \mathsf{d}t\mathsf{d}x = \int_0^C \int_{\mathbb{R}} \mathbb{1}_{\{|f| \geq t\}} \mathsf{d}x\mathsf{d}t$$
$$= \int_0^C m(|f| \geq t) \mathsf{d}t \leq \int_0^C \frac{M}{t^c} \mathsf{d}t = \frac{MC^{1-c}}{1-c} < \infty,$$

as desired. □

Problem 4.

(a) For any $\{x_j\}_{j=0}^m \subset [0, 1]$ with $0 = x_0 < x_1 < \cdots < x_m = 1$,

$$\sum_{j=1}^m |f(x_j) - f(x_{j-1})| = \liminf_{n \to \infty} \sum_{j=1}^m |f_n(x_j) - f_n(x_{j-1})| \leq \liminf_{n \to \infty} T_0^1(f_n).$$

It follows that the desired inequality holds for $T_0^1(f)$, the supremum of the left-hand side over all partitions $\{x_j\}_{j=0}^m \subset [0, 1]$ as above. □

2009, Spring

Incomplete: 3, 4.

Problem 1.

(a) We consider the cases of finite and infinite countable unions separately.

- Suppose $\{E_j\}_{j=1}^m \subset \mathcal{C}$ and let $\epsilon > 0$. For each $1 \leq j \leq m$, there's a set $A_j \in \mathcal{A}$ such that $A_j \subset E_j$ and $\mu(E_j \setminus A_j) < \epsilon/m$. Note that $A := \bigcup_{j=1}^m A_j \in \mathcal{A}$ since $\mathcal{A}$ is an algebra, and we have $A \subset E := \bigcup_{j=1}^m E_j$. Then

$$\mu(E \setminus A) = \mu\Big(\bigcup_{j=1}^m (E_j \setminus A)\Big) \leq \mu\Big(\bigcup_{j=1}^m (E_j \setminus A_j)\Big) \leq \sum_{j=1}^m \mu(E_j \setminus A_j) < \sum_{j=1}^m \frac{\epsilon}{m} = \epsilon,$$

  so $E \in \mathcal{C}$.

- Now suppose $\{E_j\}_{j=1}^\infty \subset \mathcal{C}$ and let $\epsilon > 0$. Letting $F_m := \bigcup_{j=1}^m E_j$ for each $m \in \mathbb{N}$, we have an increasing sequence $F_1 \subset F_2 \subset \cdots$ with $F_m \nearrow E := \bigcup_{j=1}^\infty E_j$ as $m \to \infty$, so by continuity from below, $\mu(F_m) \to \mu(E)$ as $m \to \infty$. Because $\mu(E) \leq \mu(X) < \infty$, we can choose $m \in \mathbb{N}$ large enough so that $\mu(E) - \mu(F_m) < \epsilon/2$, whereby

$$\mu(E) = \mu(E \setminus F_m) + \mu(F_m) \implies \mu(E \setminus F_m) = \mu(E) - \mu(F_m) < \epsilon/2,$$

  the first equality holding since $F_m \subset E$. Moreover, $F_m \in \mathcal{C}$ by the above argument, so we can find some $A \in \mathcal{A}$ with $A \subset F_m \subset E$ and $\mu(F_m \setminus A) < \epsilon/2$. Then

$$\mu(E) = \mu(E \setminus F_m) + \mu(F_m) \implies \mu(E \setminus A) \leq \mu(E \setminus F_m) + \mu(F_m \setminus A) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

  and thus $E \in \mathcal{C}$.

□

(b) Let $X := [0,1]$ with σ-algebra $\mathcal{B}_{[0,1]}$ and Lebesgue measure μ. Let $\mathcal{A} \subset \mathcal{B}_{[0,1]}$ be the algebra generated by all singletons $\{q\}$, $q \in E := \mathbb{Q} \cap [0,1]$, using complements and finite unions. Then $A \in \mathcal{A}$ if and only if A is a finite collection $\{q_j\}_{j=1}^m \subset E$ or A is the complement of such a set. Note that $\{0\} \in \mathcal{A}, \{0\} \subset E$, and $\mu(E \setminus \{0\}) \leq \mu(E) = 0 < \epsilon$ for any $\epsilon > 0$, so E is approximable from inside by $\mathcal{A}$. But observe that any element $A \in \mathcal{A}$ contains at least one rational, while E contains only irrationals, so we can't have $A \subset E^{\mathsf{c}}$, and thus E^{c} isn't approximable from inside by $\mathcal{A}$.

Problem 2.

(a) Both f, g are continuous on the compact set $[a,b]$, so there's some $M > 0$ large enough so that $|f|, |g| \leq M$ on all of $[a,b]$. Now let $\epsilon > 0$ and choose $\delta > 0$ such that for any disjoint collection $\{(a_j, b_j) \subset [a,b]\}_{j=1}^N$, we have

$$\sum_{j=1}^N (b_j - a_j) < \delta \implies \sum_{j=1}^N |f(b_j) - f(a_j)|, \sum_{j=1}^N |g(b_j) - g(a_j)| < \frac{\epsilon}{2M}.$$

Then for any such collection,

$$\sum_{j=1}^{N}|f(b_j)g(b_j)-f(a_j)g(a_j)| \leq \sum_{j=1}^{N}\left[|f(b_j)g(b_j)-f(b_j)g(a_j)|+|f(b_j)g(a_j)-f(a_j)g(a_j)|\right]$$

$$\leq M\Bigg(\underbrace{\sum_{j=1}^{N}|g(b_j)-g(a_j)|}_{<\epsilon/2M}+\underbrace{\sum_{j=1}^{N}|f(b_j)-f(a_j)|}_{<\epsilon/2M}\Bigg) < M\left(\frac{\epsilon}{2M}+\frac{\epsilon}{2M}\right)=\epsilon.$$

□

(b) We've just seen that fg is absolutely continuous, so we have

$$f(b)g(b)-f(a)g(a)=\int_a^b (fg)'=\int_a^b (f'g+fg')=\int_a^b f'g+\int_a^b fg'$$

by the fundamental theorem for Lebesgue integrals. □

(c) Take some $[a,b]\subset\mathbb{R}$ with $b-a\neq 2$, and let $f,g:[a,b]\to\mathbb{R}$ be given by $f(x):=(x-a)/(b-a)$ and $g(x):=\frac{1}{2}\mathbb{1}_{[\frac{b-a}{2},b]}(x)$. Then $f'=1$ and $g'=0$ a.e. on $[a,b]$, but g isn't continuous (in particular, g isn't absolutely continuous). We have

$$\int_a^b \underbrace{f'}_{=1}g+\int_a^b f\underbrace{g'}_{=0}=\int_a^b g=\frac{b-a}{4}\neq\frac{1}{2}=\underbrace{f(b)}_{=1}\underbrace{g(b)}_{=1/2}-\underbrace{f(a)g(a)}_{=0}.$$

□

2010, Spring

Problem 1.

(i) Let f be u.s.c. and $a \in \mathbb{R}$. If $x_0 \in f^{-1}((-\infty, a)) = \{x \in \mathbb{R} \mid f(x) < a\}$, then $f(x_0) + \epsilon < a$ for some $\epsilon > 0$. Then there's some $\delta > 0$ so that $f(x) < f(x_0) + \epsilon < a$ whenever $|x - x_0| < \delta$. Thus $f^{-1}((-\infty, a))$ is open, and in particular Borel. Since sets of the form $(-\infty, a)$ for $a \in \mathbb{R}$ generate $\mathcal{B}_{\mathbb{R}}$, this shows that f is measurable. $\square$

(ii) We first claim that a map $f : \mathbb{R} \to \mathbb{R}$ is u.s.c. if for each $x \in \mathbb{R}$ we have $\limsup_{j\to\infty} f(x_j) \le f(x)$ whenever $\{x_j\}_{j=1}^\infty \subset \mathbb{R}$ satisfies $\lim_{j\to\infty} x_j = x$. (In fact, this is an equivalent definition of upper semicontinuity.)

To establish this, suppose f is u.s.c., but there's some $x \in \mathbb{R}$ and a sequence $\{x_j\}_{j=1}^\infty \subset \mathbb{R}$ converging to x, with $f(x) < a := \limsup_{j\to\infty} f(x_j)$. Let $\epsilon > 0$ be such that $f(x) < a - \epsilon$. By definition of a, there's a subsequence $\{x_{j_k}\}_{k=1}^\infty$ of $\{x_j\}_{j=1}^\infty$ converging to a, so all but finitely many of the x_{j_k}'s belong to $E := \{y \in \mathbb{R} \mid f(y) \ge a - (\epsilon/2)\}$. By inspection, E is closed, so $x = \lim_{k\to\infty} x_{j_k} \in E$, and hence $a - (\epsilon/2) \le f(x) < a - \epsilon$, which is impossible.

Now, define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) := \mu(x + A)$. It's enough to show that f satisfies the above condition. Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}$ converge to some $x \in \mathbb{R}$. Since $|f| \le \mu(\mathbb{R}) < \infty$ on all of $\mathbb{R}$, then

$$\limsup_{j\to\infty} f(x_j) = \limsup_{j\to\infty} \mu(x_j + A) \le \mu\Big(\limsup_{j\to\infty}(x_j + A)\Big)$$

by reverse Fatou's lemma. By definition of $\limsup$, if $y \in \limsup_{j\to\infty}(x_j + A)$, then $y \in x_j + A$ for infinitely many $j \in \mathbb{N}$. Passing to a subsequence of $\{x_j\}_{j=1}^\infty$ if necessary, w.l.o.g. $y = x_j + a_j$, for some $a_j \in A$, for all $j \in \mathbb{N}$, and passing to another subsequence if necessary, w.l.o.g. $\lim_{j\to\infty} a_j$ exists and belongs to A since A is closed. Then $y = x + \lim_{j\to\infty} a_j \in x + A$, whereby we've shown that $\limsup_{j\to\infty}(x_j + A) \subset x + A$. So

$$\limsup_{j\to\infty} f(x_j) \le \mu\Big(\limsup_{j\to\infty}(x_j + A)\Big) \le \mu(x + A) = f(x),$$

and this completes the proof. $\square$

Problem 2.

(a) **True.** Let $\delta, \epsilon > 0$. Since $\mu(X) < \infty$, there's $M > 0$ large enough so that if $E := \{|f| < M\}$, then $\mu(E^{\mathsf{c}}) < \epsilon/3$. Now $\left|f_n^2 - f^2\right| \le \left|f_n^2 - f_n f\right| + \left|f_n f - f^2\right| = |f_n| \cdot |f_n - f_n| + |f| \cdot |f_n - f|$, so

$$\left\{\left|f_n^2 - f^2\right| > \delta\right\} \subset \left\{|f_n| \cdot |f_n - f| > \frac{\delta}{2}\right\} \cup \left\{|f| \cdot |f_n - f| > \frac{\delta}{2}\right\}.$$

Thus $\mu\left(E \cap \left\{\left|f_n^2 - f^2\right| > \delta\right\}\right)$ is bounded above by

$$\mu\left(E \cap \left\{|f_n| \cdot |f_n - f| > \frac{\delta}{2}\right\}\right) + \mu\left(E \cap \left\{|f| \cdot |f_n - f| > \frac{\delta}{2}\right\}\right) + \underbrace{\mu(E^{\mathsf{c}})}_{<\epsilon/3}.$$

For large enough n the second term gives

$$\mu\left(E \cap \left\{|f| \cdot |f_n - f| > \frac{\delta}{2}\right\}\right) < \mu\left(\left\{M|f_n - f| > \frac{\delta}{2}\right\}\right) < \frac{\epsilon}{3}.$$

Moreover $|f_n| \cdot |f_n - f| \le (|f| + |f - f_n|)\,|f - f_n| = |f| \cdot |f_n - f| + |f_n - f|^2$ and so for large enough n the first term gives

$$\mu\left(E \cap \left\{|f_n| \cdot |f_n - f| > \frac{\delta}{2}\right\}\right) \le \mu\left(E \cap \left\{|f| \cdot |f \cdot f_n| > \frac{\delta}{4}\right\}\right) + \mu\left(\left\{|f_n - f|^2 > \frac{\delta}{4}\right\}\right)$$
$$\le \mu\left(\left\{M|f - f_n| > \frac{\delta}{4}\right\}\right) + \mu\left(\left\{|f_n - f| > \frac{\delta^{1/2}}{2}\right\}\right) < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.$$

Hence $\mu\left(E \cap \left\{\left|f_n^2 - f^2\right| > \delta\right\}\right) < \epsilon$. □

(b) **False**. Set $X := (0, \infty)$ with Lebesgue measure μ. If $f_n(x) := x - n^{-1}$ and $f(x) := x$, then for any $\delta > 0$, we have $\mu(\{|f_n(x) - f(x)| > \delta\}) = \mu(\{n^{-1} > \delta\}) \to 0$ and hence $f_n \to f$ in measure. However for any $n \in \mathbb{N}$ and any x in the measure-∞ set $[n, \infty)$,

$$\left|f_n^2(x) - f^2(x)\right| = \left|\left(x^2 - \frac{2x}{n} + \frac{1}{n^2}\right) - x^2\right| = \frac{2x}{n} - \frac{1}{n}^2 \ge 2,$$

whereby $f_n^2 \not\to f^2$ in measure. □

Problem 3.

Let $E \subset [0,1]$ have $m(E) = 0$, and let $\epsilon > 0$. Since f is absolutely continuous, there's some $\delta > 0$ such that for any disjoint collection $\{(a_j, b_j)\}_{j=1}^N$, we have

$$\sum_{j=1}^N (b_j - a_j) < \delta \implies \sum_{j=1}^N [f(b_j) - f(a_j)] < \epsilon.$$

By outer regularity of m, there's an open set $U \subset [0,1]$ with $E \subset U$ and $m(U) < \delta$. We may write U as a disjoint union $U = \bigsqcup_{j \in J} (a_j, b_j)$ for some countable set J. Then for any $N \le |J|$,

$$\sum_{j=1}^N (b_j - a_j) \le \sum_{j \in J} (b_j - a_j) = m(U) < \delta \implies \sum_{j=1}^N [f(b_j) - f(a_j)] < \epsilon,$$

and hence it follows that

$$m(f(E)) = m\Big(\bigcup_{j \in J} (f(a_j), f(b_j))\Big) = \sum_{j \in J} [f(b_j) - f(a_j)] \le \epsilon,$$

where the first inequality used that f was strictly increasing. Hence $m(f(E)) = 0$. □

Problem 4.

- Let $f \in \mathsf{L}^1([0,1])$ and choose any $\epsilon > 0$. We may find a simple function $\varphi = \sum_{k=1}^m a_k \mathbb{1}_{E_k}$ with $\|f - \varphi\|_{\mathsf{L}^1([0,1])} < \epsilon$, where $\{a_k\}_{k=1}^m \subset \mathbb{R}$ and $\{E_k\}_{k=1}^m \subset \mathcal{B}_{[0,1]}$ is a disjoint collection of sets. By discarding countably many singletons if necessary, w.l.o.g. E_k is a disjoint union of intervals for each $1 \le k \le m$. We further assume w.l.o.g. that E_k is a single interval for each $1 \le k \le m$. For each $n \in \mathbb{N}$,

  $$\left|\left|\int h_n f\right| - \left|\int h_n \varphi\right|\right| \le \left|\int h_n (f - \varphi)\right| \le \int \underbrace{|h_n|}_{=1} |f - \varphi| < \epsilon,$$

  so if the result holds for simple functions which are linear combinations of indicators of intervals, then taking the limit as $n \to \infty$ on each side gives $\mathsf{lim}_{n \to \infty} \left|\int h_n f\right| < \epsilon$. Thus we've reduced to the case of simple functions of this form.

- Now suppose $\varphi = \sum_{k=1}^m a_k \mathbb{1}_{E_k}$ is a linear combination of indicators of intervals $E_k \in \mathcal{B}_{[0,1]}$, $1 \le k \le m$. If the result holds for indicators of intervals, then

$$\lim_{n\to\infty} \int h_n \varphi = \sum_{k=1}^m a_k \underbrace{\lim_{n\to\infty} \int h_n \mathbb{1}_{E_k}}_{=0} = 0,$$

so we've further reduced to the case of indicators of intervals.

- Finally, let $E \in \mathcal{B}_{[0,1]}$ be an arbitrary interval, fix $n \in \mathbb{N}$, and let $F_{j_1}, \ldots, F_{j_\ell}$ be those intervals $F_j := (\frac{j-1}{n}, \frac{j}{n}]$ with $F_j \subset E$ (w.l.o.g. $j_1 < \cdots < j_\ell$). Setting $G_0 := F_{j_1-1}$ and $G_1 := F_{j_\ell+1}$, then $E \subset G_0 \cup F_{j_1} \cup \cdots \cup F_{j_\ell} \cup G_1$, so

$$\left|\int_{[0,1]} h_n \mathbb{1}_E\right| = \left|\int_E h_n\right| \le \underbrace{\int_{G_0} 1}_{=1/n} + \left|\sum_{r=1}^{\ell} \int_{F_{j_r}} h_n\right| + \underbrace{\int_{G_1} 1}_{=1/n} = \frac{2}{n} + \left|\sum_{r=1}^{\ell} \frac{(-1)^{j_r}}{n}\right|.$$

The summands on the right alternate signs as r increases, so the entire sum is either 0 or $\pm 1/n$ depending on the parity of ℓ. Whichever is the case,

$$\lim_{n\to\infty} \left|\int_{[0,1]} h_n \mathbb{1}_E\right| \le \lim_{n\to\infty} \left(\frac{2}{n} + \frac{1}{n}\right) = 0.$$

This completes the proof. □

2010, Fall

Problem 1.

Denote by $(X, \mathcal{M}, \mu)$ the measure space, and write X as a countable disjoint union $X = \bigsqcup_{j \in J} X_j$ with $\mu(X_j) < \infty$ for each $j \in J$. Suppose $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ is uncountable. Each A_α has positive measure, so it has $\mu(X_j \cap A_\alpha) > 0$ for some collection of j's in J. Since there are uncountable many $\alpha \in I$ but only countably many $j \in J$, by the pigeonhole principle there must be some $j \in J$ and some uncountable subcollection $I' \subset I$ with $\mu(X_j \cap A_\alpha) > 0$ for all $\alpha \in I'$. But then

$$\infty > \mu(X_j) \geq \mu\Big(\bigsqcup_{\alpha \in I'} (X_j \cap A_\alpha) \Big) = \sum_{\alpha \in I'} \underbrace{\mu(X_j \cap A_\alpha)}_{>0},$$

which is impossible since any uncountable sum of positive numbers in infinite. □

Problem 2.

(a) Let $a > 0$. Consider a simple function $\varphi = \sum_{j=1}^n a_j \mathbb{1}_{E_j}$, with $\{a_j\}_{j=1}^n \subset \mathbb{R}$ and $\{E_j\}_{j=1}^n \subset \mathcal{B}_\mathbb{R}$ a disjoint collection. Observe that $\mathbb{1}_{E_j}(ax) = \mathbb{1}_{a^{-1}E_j}(x)$ for any $1 \leq j \leq n$, whereby

$$\int \varphi(ax)\mathsf{d}x = \sum_{j=1}^n a_j m(a^{-1}E_j) = \frac{1}{a}\sum_{j=1}^n a_j m(E_j) = \frac{1}{a}\int \varphi(x)\mathsf{d}x.$$

Now suppose $f \in \mathsf{L}^1(\mathbb{R})$ is arbitrary. By decomposing $f = f^+ - f^-$, it's enough to consider the case $f \in \mathsf{L}^+(\mathbb{R})$. Let $\{\varphi_j\}_{j=1}^\infty \subset \mathsf{L}^+(\mathbb{R})$ be a sequence of simple functions with $\varphi_1 \leq \varphi_2 \leq \cdots$ and $\lim_{j\to\infty} \varphi_j = f$. Then

$$\int f(ax)\mathsf{d}x = \lim_{j\to\infty} \int \varphi_j(ax)\mathsf{d}x = \lim_{j\to\infty} \frac{1}{a}\int \varphi_j(x)\mathsf{d}x = \frac{1}{a}\int f(x)\mathsf{d}x$$

by applying monotone convergence twice. □

(b) Set $f(x) := nF(x)/x(1+n^2x^2)$. Then by (a),

$$\int f(x)\mathsf{d}x = \frac{1}{n}\int f\left(\frac{x}{n}\right)\mathsf{d}x = \frac{1}{n}\int \frac{nF(x/n)}{(x/n)(1+n^2(x/n)^2)}\mathsf{d}x = \int \frac{1}{1+x^2}\cdot\frac{F(x/n)}{x/n}\mathsf{d}x$$

for any $n \in \mathbb{N}$. Now taking the limit as $n \to \infty$, we may apply dominated convergence since the integrand on the right satisfies

$$\left|\frac{1}{1+x^2}\cdot\frac{F(x/n)}{x/n}\right| \leq \frac{1}{1+x^2}\cdot\frac{nC|x/n|}{|x|} = \frac{C}{1+x^2}$$

and the right-hand side is integrable. Then

$$\lim_{n\to\infty}\int f(x)\mathsf{d}x = \lim_{n\to\infty}\int \frac{1}{1+x^2}\cdot\frac{F(x/n)}{x/n}\mathsf{d}x = \int \frac{1}{1+x^2}\cdot\underbrace{\lim_{n\to\infty}\frac{F(x/n)-F(0)}{(x/n)-0}}_{=F'(0)}\mathsf{d}x = \pi F'(0),$$

where we used that $F(0) = 0$ since $|F(x)| \leq C|x|$ for all $x \in \mathbb{R}$. □

Problem 3.

Assume first that $f \geq 0$. Clearly $1 + f + \cdots + f^n \leq 1 + f + \cdots + f^n + f^{n+1}$ for all $n \in \mathbb{N}$, so by monotone convergence and the geometric series formula,

$$\lim_{n\to\infty} \int_X (1 + f + \cdots + f^n) = \int_X \lim_{n\to\infty} (1 + f + \cdots + f^n) = \int_X \frac{1}{1-f}.$$

The right-hand side always exists since $\mu(X) < \infty$ and $|f| < 1$. Now consider a general measurable function $f = f^+ - f^-$ with $|f| < 1$. We have that $f^j = (f^+ - f^-)^j = (f^+)^j + (-1)^j (f^-)^j$ for any $j \geq 0$ since the product $f^+ f^-$ appearing in the cross terms is always 0. Then

$$\lim_{n\to\infty} \int_X (1 + f + \cdots + f^n) = \lim_{n\to\infty} \int_X [1 + f^+ + \cdots + (f^+)^n] + \lim_{n\to\infty} \int_X [1 - f^- + \cdots + (-1)^n (f^-)^n]$$
$$\leq \lim_{n\to\infty} \int_X [1 + f^+ + \cdots + (f^+)^n] + \lim_{n\to\infty} \int_X [1 + f^- + \cdots + (f^-)^n] = \int_X \frac{1}{1-f^+} + \int_X \frac{1}{1-f^-},$$

and we're done by the nonnegative case since $f^+, f^- \geq 0$. $\square$

Problem 4.

For simplicity, denote $F_0 := F$, and let $j \geq 0$. We may write $\mathsf{d}\mu_{F_j} = \mathsf{d}\nu_j + F_j' \mathsf{d}m$, where m denotes the Lebesgue measure and $\nu_j \perp m$, by Lebesgue-Radon-Nikodym. Thus there is some m-null $N_j \subset [a,b]$ with $\nu_j([a,b] \setminus N_j) = 0$. Then $N := \bigcup_{j=0}^{\infty} N_j$ is also m-null, and for any $E \in \mathcal{B}_{[a,b]}$ disjoint from N, we have by monotone convergence that

$$\int_E \sum_{j=1}^{\infty} F_j' \mathsf{d}m = \sum_{j=1}^{\infty} \int_E F_j' \mathsf{d}m = \sum_{j=1}^{\infty} \int_E \mathsf{d}\mu_{F_j} = \sum_{j=1}^{\infty} \mu_{F_j}(E) = \mu_F(E) = \int_E \mathsf{d}\mu_F = \int_E F' \mathsf{d}m.$$

Since E was arbitrary and N is m-null, we conclude that $\sum_{j=1}^{\infty} F_j' = F'$ m-a.e on $[a,b]$. $\square$

2011, Spring

Incomplete: 2, 3.

Problem 1.

For each $j \in \mathbb{N}$, choose $E_j, F_j \in \mathcal{B}_{\mathbb{R}}$ so $m(A \setminus E_j) \leq m(F_j \setminus E_j) \leq j^{-1}$, and set $E := \bigcup_{j=1}^{\infty} E_j$. Then

$$m(A \setminus E) = m\Big(\bigcup_{j=1}^{\infty}(A \setminus E_j)\Big) = \lim_{j\to\infty} m(A \setminus E_j) \leq \lim_{j\to\infty} \frac{1}{j} = 0.$$

Hence $A = E \sqcup (A \setminus E)$, with $E \in \mathcal{B}_{\mathbb{R}}$ and $A \setminus E$ being m-null. So since m is complete, then $A \in \mathcal{B}_{\mathbb{R}}$ as well. $\square$

Problem 4.

(a) Suppose (w.l.o.g.) that $F_1 \cap F_2 \cap F_3 \cap F_4 = .$ Then $\sum_{j=1}^{7} \mathbb{1}_{F_j} \leq 3$ on all of $[0,1]$, whereby

$$3.5 = \sum_{j=1}^{7} \frac{1}{2} \leq \sum_{j=1}^{7} m(F_j) = \int_{[0,1]} \sum_{j=1}^{7} \mathbb{1}_{F_j} \leq 3m([0,1]) = 3,$$

a contradiction. $\square$

(b) Suppose $\int_{[0,1]} \sup_{n\in\mathbb{N}} f_n < \infty$. Since $f_n \geq 0$ for each $n \in \mathbb{N}$, we have

$$\infty > \int_{[0,1]} \sup_{n\in\mathbb{N}} f_n = \sum_{j=1}^{\infty} \int_{[\frac{1}{j+1}, \frac{1}{j}]} \sup_{n\in\mathbb{N}} f_n.$$

Then because the sum on the right-hand side is convergent, we must have

$$0 = \lim_{N\to\infty} \sum_{j=N}^{\infty} \int_{[\frac{1}{j+1}, \frac{1}{j}]} \sup_{n\in\mathbb{N}} f_n = \lim_{N\to\infty} \int_{[0, \frac{1}{N}]} \sup_{n\in\mathbb{N}} f_n \geq \lim_{N\to\infty} \int_{[0, \frac{1}{N}]} f_N \geq \lim_{N\to\infty} \frac{1}{2} = \frac{1}{2},$$

a contradiction. $\square$