## Notation

- When I say $S_n$, I always mean $\sum_{i=1}^{n} X_n$.
- If $E_n$ are events (or sets), I write $E_n \nearrow E$ to mean $E_n \subset E_{n+1}$ and $\bigcup E_n = E$.
- The notation $a \wedge b$ means $\min(a, b)$, while $a \vee b$ means $\max(a, b)$.
- $X^+ = \max(X, 0)$ and $X^- = -\min(-X, 0)$. Thus, $X = X^+ - X^-$, $|X| = X^+ + X^-$.
- Both $1_A$ and $\mathbf{1}(A)$ refer to the indicator function for the set $A$. Furthermore, $E(X; A)$ means $E(X1_A)$. I will often omit set braces, so for example, all of the below mean the same:

$$E(X1_{\{|X|\leq M\}}) = E(X1_{|X|\leq M}) = E(X\mathbf{1}(|X| \leq M)) = E(X; |X| \leq M)$$

- I use $X_n \implies X$ to mean $X_n$ convreges to $X$ in distribution.
- $o(f(t))$ refers to some function $g(t)$ for which $\lim_{t\to a} \frac{g(t)}{f(t)} \to 0$. The number $a$ depends on context, but is usually either 0 or $\infty$.
- Everyone, including qual writers, makes mistakes. These will be marked in red.
- Problems that I couldn't do will be marked with a ☹, possibly with a partial solution.

# Theorems to Know

In addition to all of the usual theorems (Monotone Convergence Thoerem, Fatou's Lemma, Dominated Convergence Theorem, Fubini's Theorem, Chebyshev's Inequality, Jensen's Inequality, Cauchy-Schwarz Inequality, Borel-Cantelli, Weak Law of Large Numbers, Strong Law of Large Numbers, Kolmogorv's Maximal Inequality, Kolmogorov Three-Series Test, Inversion Formula, Continuity Theorem, Central-Limit Theorem, Linberg Feller Central Limit Theorem), these solutions will assume you know the following theorems:

**Theorem 1** (Relations Between Convergence Concepts)**.** *If $p > q$, then*

$$\begin{array}{ccccc} \xrightarrow{L_p} & \Longrightarrow & \xrightarrow{L_q} & & \\ & & \Downarrow & & \\ \xrightarrow{a.s.} & \Longrightarrow & \xrightarrow{P} & \Longrightarrow & \xrightarrow{\mathcal{D}} \end{array}$$

*Any implication not pictured does not hold in general.*

**Theorem 2.** *If $X_n \to X$ in probability, then there is a subsequence $X_{n_k} \to X$ a.s.*

**Theorem 3.** *$X_n \to X$ a.s. if and only if for all $\varepsilon > 0$, $\sum_1^\infty P(|X_n - X| > \varepsilon) < \infty$.*

**Theorem 4** ("Layer-Cake" Formula)**.**

$$E|X| = \int_0^\infty P(|X| > t)\, dt$$

*and more generally,*

$$E|X|^p = \int_0^\infty p t^{p-1} P(|X| > t)\, dt$$

When $p = 1$, the above is used to prove the following **very** useful fact:

**Theorem 5.** *If $X_1, X_2, \ldots$ i.i.d, then $E|X_1| < \infty$ if and only if $X_n/n \to 0$ a.s.*

The next result is very useful for problems that involve $\max_{1 \le k \le n} X_n$:

**Lemma 1.** *Let $a_n, b_n$ be sequences of numbers where $b_n \to \infty$, and $m_n = \max_{1 \le k \le n} a_n$. If $\frac{a_n}{b_n} \to 0$, then $\frac{m_n}{b_n} \to 0$.*

You may not know the next theorem by this name, but it is taught in 507a:

**Theorem 6** (Skorohod's Reprentation Theorem)**.** *If $X_n \to X$ in distribution, then there exists random variables $X_n', X'$ with the same distributions as $X_n, X$ such that $X_n' \to X'$ a.s.*

**Theorem 7** (Slutsky's Theorem)**.** *If $X_n \implies X$ and $Y_n \implies c$, a constant, then $X_n + Y_n \implies X + c$ and $X_nY_n \implies Xc$.*

For a proof of $X_n + Y_n \implies X + c$ when $c = 0$, see Spring 2008 Problem 2.
For $X_nY_n \implies Xc$ when $c = 1$, see Spring 1997 problem 2.
The next theorem is useful when you what to prove, for example, $\frac{\sum_1^n X_k}{n^p} \to 0$.

**Lemma 2** (Kronecker's Lemma)**.** *If $a_n \to \infty$ and $\sum_1^\infty \frac{x_n}{a_n}$, then*

$$\frac{1}{a_n}\sum_1^n x_k \to 0$$

**Theorem 8.** *If $EX^2 < \infty$, and $\varphi(t) = E^{itX}$, then*

$$\varphi(t) = 1 + i(EX)t - (EX^2)t^2/2 + o(t^2) \qquad \text{as } t \to 0$$

*To make this look cleaner, let $\mu = EX$, $\sigma^2 = Var\ X = EX^2 - \mu^2$. Then*

$$\varphi(t) = 1 + i\mu t - (\sigma^2 + \mu^2)t^2/2 + o(t^2) \qquad \text{as } t \to 0$$

# 1994 Fall

1. (a) Given $\varepsilon > 0$, there exists an $M$ so that $E[|X_n| 1_{|X_n|>M}] < \varepsilon$ for all $n$.
   (b) Let $X_n = n$ with probability $\frac{1}{n}$, $X_n = 0$ with probability $1 - \frac{1}{n}$.
   (c) First, realize that uniform integrability implies that $EX_n$ is bounded as $n \to \infty$, so by Fatou's lemma, $EX \le \liminf EX_n < \infty$. In particular, $E[X 1_{|X|>M}] \to 0$ as $M \to \infty$ (by DCT).
   Thus, given $\varepsilon > 0$, we can choose $M$ so both $E[X_n 1_{X_n > M}] < \varepsilon/2$ for all $n$ and $E[X 1_{X>M}] < \epsilon/2$. Let
   $$Y_n = X_n 1_{X_n \le M} \qquad Z_n = Z_n 1_{X_n > M},$$
   so that $X_n = Y_n + Z_n$, and similarly write $X = Y + Z$.
   Then $|Y_n| \le M$, and $Y_n \to Y$ a.s, so by DCT, $EY_n \to EY$. Thus, as $n \to \infty$,
   $$|EX_n - EX| \le |EY_n - EY| + E|Z_n| + E|Z| \le |EY_n - EY| + \varepsilon/2 + \varepsilon/2 \to \varepsilon$$
   proving $\limsup |EX_n - EX| \le \varepsilon$ for all $\varepsilon > 0$, so $EX_n \to EX$.
   (d) Impossible Problem! What they are asking you to prove is just plain wrong. Let $X_1$ be any variable with $EX_1 = \infty$, and let $X_n = X = 0$, for $n \ge 2$. Then $X_n \to X$ a.s, and $EX_n \to EX$, but $\{X_1, X_2 \dots\}$ is not uniformly integrable since $E[X_1 1_{X_1 \ge M}] = \infty$ for all $M$.
   However, this problem does work with the additional assumptions that $EX_n < \infty$, $EX < \infty$, and $E|X_n - X| \to 0$.
   (e) Typo! They meant to say $Ef(X_n) \le c < \infty$.
   Given $\varepsilon > 0$, choose $M$ so $x > M$ implies $\frac{x}{f(x)} < \varepsilon/c$. Then
   $$E(X_n 1_{X_n > M}) = E\left(f(X_n) \cdot \tfrac{X_n}{f(X_n)} 1_{X_n > M}\right) \le Ef(X_n) \cdot \varepsilon/c \le c \cdot \varepsilon/c = \varepsilon$$
   proving uniform integrability.

2. (a) Typo! The phrase "show that $Y_n \to Y_n'$ converges in distribution" is nonsesnse. They probably meant "show that $Y_n - Y_n'$ converges in distribution."
   To see this, let $\varphi_n(t)$ be the c.f. for $Y_n$. Since $Y_n \to Y$ in distribution, for some $Y$, we have $\varphi_n(t) \to \varphi(t)$, where $\varphi(t) = E^{itY}$. This implies $\varphi_n(t)\varphi_n(-t) \to \varphi(t)\varphi(-t)$. Since $\varphi_n(t)\varphi_n(-t)$ is the c.f. for $Y_n - Y_n'$, and $\varphi(t)\varphi(-t)$ is continuous at zero, by the continuity theorem, we have that $Y_n - Y_n' \to Z$, where $Z$ has c.f. $\varphi(t)\varphi(-t)$.
   (b) The c.f. for $a_n S_n$ is $\exp(-c|a_n t|^\alpha)^n = \exp(-cn|a_n|^\alpha |t|^\alpha)$. If we let $a_n = n^{-1/\alpha}$, then the c.f. for $S_n/n^{1/\alpha}$ becomes $\exp(-c|t|^\alpha)$. Thus, not only will $S_n/n^{1/\alpha}$ converge in distribution, but it will be equal in distribution to $X_1$ for each $n$. So, $Z$ and $X_1$ have the same distribution.

## 1995 Spring

1. Suppose $F_n \implies F$. Then there are r.v.'s $X_n, X$ where $X_n$ (resp. $X$) has distribution $F_n$ (resp. $F$), and that $X_n \to X$ a.s. (Sorokhod's representation theorem). Since $h$ is continuous, this means $h(X_n) \to h(X)$ a.s, and by bounded convergence theorem, $Eh(X_n) \to Eh(X)$, so that $\int h\,dF_n \to \int h\,dF$.

   Suppose $\int h\,dF_n \to \int h\,dF$ for all bounded, continuous $h$. Let $x_0$ be a continuity point of $F$. Given $\varepsilon > 0$, let

$$h(x) = \begin{cases} 1 & x \le x_0 \\ \text{linear} & x_0 \le x \le x_0 + \epsilon \\ 0 & x_0 + \epsilon \le x \end{cases}$$

   Then $1_{x \le x_0} \le h(x) \le 1_{x \le x_0 + \varepsilon}$, so

$$\limsup_{n\to\infty} F_n(x_0) = \limsup_{n\to\infty} \int 1_{x\le x_0}\,dF_n \le \limsup_{n\to\infty} \int h\,dF_n = \int h\,dF \le \int 1_{\{x \le x_0+\epsilon\}} dF = F(x_0+\epsilon)$$

   As $\epsilon \to 0$, this shows $\limsup_{n\to\infty} F_n(x_0) \le F(x_0)$. Doing a very similar argument using

$$h(x) = \begin{cases} 1 & x \le x_0 - \epsilon \\ \text{linear} & x_0 \le x - \varepsilon \le x_0 \\ 0 & x_0 \le x \end{cases}$$

   shows $\liminf_{n\to\infty} F_n(x_0) \ge F(x_0)$. Thus, $F_n(x_0) \to F(x_0)$, so $F_n \implies F$.

2. The condition $E \log X < \infty$ is sufficient and necessary. Suppose $E \log X = \infty$. First, note that $(X_1 \cdots X_n)^{1/n}$ converging a.s. is the same as $S_n/n = \frac{1}{n}(\log X_1 + \cdots + \log X_n)$ converging a.s, since the latter is the log of the former. Now, for $M \ge 0$, let $Y_n^M = (\log X_n) \wedge M$, and $S_n^M = Y_1^M + \cdots + Y_n^M$. Then $S_n \ge S_n^M$, so

$$\liminf S_n/n \ge \liminf S_n^M/n = EY_1^M \qquad (a.s.)$$

   by SLLN. But as $M \to \infty$, $EY_1^M \to E\log X = \infty$ by MCT, so for all $k$, $P(\liminf S_n/n \ge k) = 1$. Thus, $P(\liminf S_n/n = \infty) = P(\bigcap_{k\ge 1}\{\liminf S_n/n \ge k\}) = 1$, so $S_n/n$ cannot converge to a finite limit a.s.

# 1997 Spring

1. (a) First, we show $|X_n|/n^{1/\alpha} \to 0$ a.s. We have

$$\sum_1^\infty P(|X_n|/n^{1/\alpha} > \varepsilon) = \sum_1^\infty P(\tfrac{|X_n|^\alpha}{\varepsilon^\alpha} > n) \le \int_0^\infty P(|X_n|^\alpha/\varepsilon^\alpha > t) = E|X_1|^\alpha/\varepsilon^\alpha < \infty$$

Thus, by Borel Cantelli, $P(|X_n|/n^{1/\alpha} > \varepsilon \text{ i.o.}) = 0$, and intersecting these events for $\varepsilon \searrow 0$ proves $|X_n|/n^{1/\alpha} \to 0$ a.s.

This means that $|X_n|^\alpha/n \to 0$ a.s. as well. Applying the below Lemma, we see that this implies $\max_{1\le k\le n} |X_n|^\alpha/n \to 0$ a.s, so that $\max_{1\le k\le n} |X_n|/n^{1/\alpha} \to 0$

(b) Note that $EX_1$ is finite implies $E|X_1| < \infty$, since $E|X| = EX^+ + EX^-$.

Since $E|X_1| < \infty$, we have that $X_n/n \to 0$ a.s.

Next, we prove that $\max_{1\le i\le n} |X_n|/n \to 0$ a.s. This follows from $|X_n|/n \to 0$ a.s, and the following lemma:

**Lemma**: If a sequence $a_n \ge 0$, and $a_n/n \to 0$, then $\frac{1}{n}\max_{1\le i\le n} a_n \to 0$.

*Proof.* Given $\varepsilon > 0$, choose $k$ so $n > k$ implies $a_n/n < \varepsilon$. Then

$$\limsup_n \frac{\max_{1\le i\le n} a_n}{n} \le \limsup_n \frac{\max(x_1,\ldots,x_k)}{n} + \max_{k\le i\le n} \frac{a_i}{i} \le 0 + \varepsilon$$

This holds for all $\varepsilon > 0$, so $\frac{\max_{1\le i\le n} a_n}{n} \to 0$. □

Finally, let $M_n = \max_{1\le i\le n} |X_n|$. The previous lemma shows that

$$\frac{M_n}{n} \to 0 \qquad \text{a.s.}$$

The SLLN implies $S_n/n \to EX_1 \ne 0$, so

$$\frac{n}{|S_n|} \to \frac{1}{|EX_1|} \qquad \text{a.s.}$$

Thus, the product of these sequences converges to the product of the limits a.s, proving that $M_n/|S_n| \to 0$ a.s.

2. **Lemma 1:** $X_n \implies X$ and $Y_n \implies 0$ implies $X_n + Y_n \implies X$.

*Proof.* Let $x$ be a continuity point of $F_X$, and $\varepsilon > 0$. Since $\{X_n + Y_n \leq x\} \subset \{X_n \leq x + \varepsilon\} \cup \{|Y_n| > \varepsilon\}$ and $\{X_n \leq x - \varepsilon\} \subset \{X_n + Y_n \leq x\} \cup \{|Y_n| > \varepsilon\}$, we have

$$P(X_n \leq x - \varepsilon) - P(|Y_n| > \varepsilon) \leq P(X_n + Y_n \leq x) \leq P(X_n \leq x + \varepsilon) + P(|Y_n| > \varepsilon)$$

Assuming $x \pm \varepsilon$ is also a contiuity point of $F_X$, letting $n \to \infty$ above shows

$$F(x - \varepsilon) \leq P(X_n + Y_n \leq x) \leq F(x + \varepsilon)$$

and letting $\varepsilon \to 0$ completes the proof. □

**Lemma 2:** $X_n \implies X$ and $Y_n \implies 0$ implies $X_n Y_n \implies 0$.

*Proof.* Let $\varepsilon > 0$, $M \in \mathbb{N}$. Then $\{|X_n Y_n| > \varepsilon\} \subset \{|X_n| > \varepsilon M\} \cup \{|Y_n| > \frac{1}{M}\}$, so

$$P(|X_n Y_n| > \varepsilon) \leq P(|X_n| > \varepsilon M) + P(|Y_n| > \tfrac{1}{M})$$

Letting $n \to \infty$, and assuming $\pm\varepsilon M$ is a continuity point of $F_X$, gives

$$\limsup_n P(|X_n Y_n| > \varepsilon) \leq P(|X| > \varepsilon M)$$

and letting $M \to \infty$ gives $\limsup_n P(|X_n Y_n| > \varepsilon) = 0$, so $X_n Y_n \to 0$ in probability, and therefore in distribution. □

Finally, assume $X_n \implies X$ and $Y_n \implies 1$, so that $Y_n - 1 \implies 0$. Lemma 2 implies that

$$X_n(Y_n - 1) \implies 0.$$

This, combined with

$$X_n \implies X$$

and Lemma 1, gives that

$$X_n(Y_n - 1) + X_n \implies X$$

3. (a) The general inversion formula gives, for any $a < b$ (and using the fact that $F_n$ is continuous, so $P(X_n = a) = 0$),

$$\begin{aligned} P(X_n \in (a,b)) &= P(X_n \in (a,b)) + \frac{1}{2}P(X_n \in \{a,b\}) \\ &= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi_n(t)\, dt \\ &= \lim_{T\to\infty} \frac{1}{2\pi} \int 1_{|t|\le T} \frac{e^{-ita} - e^{-itb}}{it} \varphi_n(t)\, dt \qquad (\star) \end{aligned}$$

Since

$$\left| \frac{e^{-ita} - e^{-itb}}{it} \right| = \left| \int_a^b e^{-ity}\, dy \right| \le b - a$$

It follows that the integrand in $(\star)$ is dominated by $(b-a)\varphi_n(t) \in L_1$, so by the DCT,

$$\begin{aligned} P(X \in (a,b)) &= \frac{1}{2\pi} \int \lim_{T\to\infty} 1_{|t|<T} \frac{e^{-ita} - e^{-itb}}{it} \varphi_n(t)\, dt \\ &= \frac{1}{2\pi} \int \frac{e^{-ita} - e^{-itb}}{it} \varphi_n(t)\, dt \\ &= \frac{1}{2\pi} \int \left( \int_a^b e^{-ity}\, dy \right) \varphi_n(t)\, dt \\ &= \int_a^b \frac{1}{2\pi} \int e^{-ity} \varphi_n(t)\, dt\, dy \end{aligned}$$

The last formula implies by definition that $\frac{1}{2\pi}\int e^{-ity}\varphi_n(t)\, dt$ is the density of $X_n$.

(b) We have that

$$|\varphi_n(t+h) - \varphi_n(t)| = |E(e^{i(t+h)X_n} - e^{itX_n})| \le E|e^{i(t+h)X_n} - e^{itX_n}| = E|e^{ihX_n} - 1|$$

since $|e^{itX_n}| = 1$. As $h \to 0$, $e^{ihX_n} - 1 \to 0$, and is dominated by $|e^{ihX_n} - 1| \le 2$, so by the Dominated Convergence Theorem, $E|e^{ihX_n} - 1| \to 0$. Thus, for small $h$, and all $t$, $|\varphi_n(t+h) - \varphi_n(t)| < \varepsilon$, so $\sup_t |\varphi_n(t+h) - \varphi_n(t)| < \varepsilon$.

(c) Typo They meant to say $|\varphi_n(t)| \leq g(t)$ for all $n$ and $t$.

We have that

$$\begin{aligned}\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| &= \sup_{x\in\mathbb{R}} \left| \int e^{-itx}\varphi_n(t)\,dt - \int e^{-itx}\varphi(t)\,dt \right| \\ &\leq \sup_x \int |e^{-itx}(\varphi_n(t) - \varphi(t))|\,dt \\ &= \int |\varphi_n(t) - \varphi(t)|\,dt\end{aligned}$$

Noting that $\varphi_n(t) \to \varphi(t)$ and $|\varphi_n(t)| \leq g(t)$ implies $|\varphi(t)| \leq g(t)$, we get that $|\varphi_n - \varphi| \leq |\varphi_n| + |\varphi| \leq 2g \in L_1$. Since $|\varphi_n(t) - \varphi(t)| \to 0$, by the dominated convergence theorem,

$$\limsup_{n\to\infty} \left( \sup_x |f_n(x) - f(x)| \right) \leq \lim_{n\to\infty} \int |\varphi_n(t) - \varphi(t)|\,dt = 0$$

proving $\sup_x |f_n(x) - f(x)| \to 0$, so $f_n \to f$ uniformly. No need for Arzela-Ascoli.

# 1997 Fall

1. (a) The first is Fatou's Lemma applied to the sequence $1_{A_n}$. The middle is obvious, and the last is Fatou's applied to $1 - 1_{A_n}$: by Fatou's

$$E(\liminf 1 - 1_{A_n}) \leq \liminf E(1 - 1_{A_n}) = \liminf 1 - P(A_n) = 1 - \limsup P(A_n)$$

Then, notice that $E(\liminf 1 - 1_{A_n}) = P((\limsup 1_{A_n})^c) = 1 - P(\limsup 1_{A_n})$.

(b) Let $(\Omega, \mathcal{F}, P)$ be (0,1) with Lebesgue measure, $A_{2k} = (0, 1/3)$, and $A_{2k+1} = (1/3, 1)$, for all $k \in \mathbb{N}$. Then $0 < 1/3 < 2/3 < 1$.

(c) ( $\Longrightarrow$ ) Assume that $P(A_n \text{ i.o.}) = 1$. Let $B$ be an event where $P(B) > 0$. Then

$$\begin{aligned} 1 &= P(A_n \text{ i.o.}) \\ &= P(\{A_n \text{ i.o.}\} \cap B) + P(\{A_n \text{ i.o.}\} \cap B^c) \\ &\leq P(\{A_n \text{ i.o.}\} \cap B) + P(B^c) \end{aligned}$$

so

$$P(\{A_n \text{ i.o.}\} \cap B) \geq 1 - P(B^c) = P(B) > 0.$$

Since the event $\{A_n \text{ i.o.}\} \cap B$ is the same as the event $\{A_n \cap B \text{ i.o.}\}$, the above shows that $P(A_n \cap B \text{ i.o.}) > 0$. By the (contrapositive of the) Borel-Cantelli lemma, this means that $\sum P(A_n \cap B) = \infty$.

($\Longleftarrow$) Assume that, whenever $P(B) > 0$, we have $\sum P(A_n \cap B) = \infty$. Let $B = \{A_n \text{ i.o.}\}^c$, and consider

$$\sum_{n \geq 1} P(A_n \cap B)$$

Notice that only finitely many of the above terms can be nonzero: if $\omega \in B$, then $\omega$ is in only finitely many $A_n$, so only finitely many $A_n \cap B$ are nonempty. Thus, the above sum is finite. Since we assumed the sum would be infinite when $P(B) > 0$, this means $P(B) = 0$, so that $P(B^c) = P(A_n \text{ i.o.}) = 1$.

2. (a) Var $S_n = ES_n^2 = \sum_i EX_i^2 + \sum_{i \neq j} EX_iX_j \leq Kn + 0 = O(n)$.

   (b) By Chebychev's, $S_n^2$, $P(|S_n| > n\varepsilon) = P(S_n^2 > n^2\varepsilon^2) \leq \frac{ES_n^2}{n^2\varepsilon^2} = \frac{O(n)}{\epsilon^2 n^2} = O(\frac{1}{n})$

   (c) Since $\sum P(B_n) = \sum O(\frac{1}{n^2}) < \infty$, by Borel Cantelli, $P(B_n \text{ i.o.}) = 0$.

   (d) We will show that, for all $\varepsilon > 0$, $P(D_n/n^2 > \varepsilon \text{ i.o.}) = 0$, which proves $D_n/n^2 \to 0$ a.s. since $\{D_n/n^2 \to 0\} = \cap_{k \geq 1}\{D/n^2 > \frac{1}{k} \text{ i.o.}\}^c$.

   Note that $\{D_n > n^2\varepsilon\} = \bigcup_{k=n^2+1}^{(n+1)^2-1}\{|S_k - S_{n^2}| > n^2\varepsilon\}$, so

$$P(D_n > n^2\varepsilon) < \sum_{k=n^2+1}^{(n+1)^2-1} P(|S_k - S_{n^2}| > n^2\varepsilon) < \sum_{\ell=1}^{2n} P(|S_{n^2+\ell} - S_{n^2}| > \ell^2\varepsilon)$$

   By the same reasoning as in part (a), we have that Var $(S_{n^2+\ell} - S_{n^2})$ = Var $(X_{n^2+1} + \cdots + X_{n^2+\ell}) = O(\ell)$, so using Chebychev's,

$$P(|S_{n^2+\ell} - S_{n^2}| > \ell^2\varepsilon) \leq \frac{\text{Var } (S_{n^2+\ell} - S_{n^2})}{\ell^4\epsilon^2} = O\left(\frac{1}{\ell^3}\right)$$

   Thus,

$$P(D_n > n^2\varepsilon) < \sum_{\ell=1}^{2n} O\left(\frac{1}{\ell^3}\right) = O\left(\frac{1}{\ell^2}\right)$$

   so by Borel-Cantelli, $P(D_n > n^2\varepsilon \text{ i.o.}) = 0$.

3. (a) Since $\phi'(0) = ia$, we have that

$$\lim_{n\to\infty} \frac{\phi(t/n) - 1}{t/n} = ia$$

   Furthermore, from calculus it is true that $\frac{\log(1+x)}{x} \to 1$ as $x \to 0$, implying $\frac{\log \phi(t/n)}{\phi(t/n)-1} \to 1$ as $n \to \infty$. Multiplying these two limits, we get

$$\lim_{n\to\infty} \frac{\log \phi(t/n)}{t/n} = ia$$

   Taking exp of both sides, we get $\phi(t/n)^n \to e^{iat}$. But $\phi(t/n)^n$ is the c.f. for $S_n/n$, and $e^{iat}$ is the c.f. for $a$, so the continutity theorem implies $S_n/n \to a$ weakly. Finally, one can prove that converging weakly to a constant implies convergence in probability as well, so that $S_n/n \to a$ in probability.

   (b) Since $S_n/n \to a$ in probability, and therefore in distribution, it follows that the c.f.'s also converge, so $\phi(t/n)^n \to e^{iat}$ (uniformly on compact sets). Taking log's,

$$\lim_n \frac{\log \phi(t/n)}{t/n} = \lim_n \frac{\phi(t/n) - 1}{t/n} = ia$$

   also uniformly on compact sets. So, given $\varepsilon > 0$, we can choose $n$ so $|\frac{\phi(t/n)-1}{t/n} - ia| < \varepsilon$ for $|t| \leq 1$, implying $|\frac{\phi(h)-1}{h} - ia| < \varepsilon$ for $|h| < \frac{1}{n}$, so that $\phi'(0) = ia$.

4. (a) For any $\varepsilon > 0$,

$$\sum_n P(|X_n/n| > \varepsilon) = \sum_n P(|X/\varepsilon| > n) \leq \int_0^\infty P(|X/\varepsilon| > x)\,dx = E|X/\varepsilon| < \infty,$$

so by Borel Cantelli, $P(|X_n/n| > \varepsilon \text{ i.o.}) = 0$. Thus,

$$P(|X_n/n| \to 0) = P\left(\bigcap_{k\geq 1}\{|X_n/n| > \tfrac{1}{k} \text{ i.o.}\}^c\right) = 1,$$

so $X_n/n \to 0$ a.s.

(b)

$$\sum_n P(X_n/n > A) = \sum_n P(X/A > n) \geq \int_1^\infty P(X/A > x)\,dx = E(X/A \cdot 1_{X/A>1}) = \infty.$$

Thus, by the second Borel-Cantelli lemma, $P(X_n/n > A \text{ i.o.}) = 1$, so $P(\limsup X_n/n = \infty) = P(\bigcap_{k\geq 1}\{\limsup X_n/n \geq k\}) = 1$.

I'm not sure why what we just proved implies $S_n/n \to \infty$ a.s, but you can prove this as follows. Let $Y_n^M = X_n \wedge M$, and $S_n^M = \sum Y_1^M + \cdots + Y_n^M$. Then

$$\liminf S_n/n \geq \liminf S_n^M/n = EY_1^M \qquad a.s.$$

As $M \to \infty$, by MCT, $EY_1^M \to EX = \infty$, so for all $k$, $P(\liminf S_n/n \geq k) = 1$. Thus, $P(\liminf S_n/n = \infty) = P(\bigcap_{k\geq 1}\{\liminf S_n/n \geq k\}) = 1$, so $S_n/n \to \infty$ a.s.

## 1998 Fall

1. See 1997 Fall 1(c)
2. First note that

$$E(S_n - nf(n))^2 = \text{Var } S_n = \sum \text{Var } X_i \leq n,$$

since $|X_i| \leq 1$. Thus,

$$P(|S_n - nf(n)| > n\varepsilon) \leq \frac{\text{Var } (S_n)}{n^2\varepsilon^2} \leq \frac{n}{\varepsilon^2 n^2} \to 0$$

proving $S_n/n - f(n) \to 0$ in probability.

# 1999 Spring

1. By Borel-Cantelli, $P(X_n \neq c_n \text{ i.o.}) = 0$. With probability 1, only finitely many $X_n$ will not be $c_n$, so the set of values that $S_n$ can take is

$$\bigcup_{n\geq 0} \{b_1 + \cdots + b_n + \sum_{k\geq n+1} c_k : b_j \in B\}$$

This is a countable union of countable sets, so is countable.

2. (a) This is $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} e^{ixt}\, dx = e^{-t^2/2} \int \frac{1}{\sqrt{2\pi}} e^{-(x-it)^2/2}\, dx = \ e^{-t^2/2}$.

(b) We have

$$\begin{aligned}\phi_k(u) &= E(e^{iu(X_k - \frac{1}{k})}) = e^{iu(1-\frac{1}{k})} \cdot \frac{1}{k} + e^{-iu/k} \cdot (1 - \frac{1}{k}) \\ &= \frac{1}{k} \cos \tfrac{u(k-1)}{k} + \frac{k-1}{k} \cos \tfrac{u}{k} + \frac{i}{k} \sin \tfrac{u(k-1)}{k} - \frac{i(k-1)}{k} \sin \tfrac{u}{k}\end{aligned}$$

(c) Since $\sin t = t - o(t^2)$ and $\cos t = 1 - t^2/2 + o(t^2)$, we have

$$\frac{i}{k} \sin \tfrac{t(k-1)}{k} - \frac{i(k-1)}{k} \sin \tfrac{t}{k} = \left(i\frac{(k-1)t}{k^2} + o(t^2)\right) - \left(i\frac{(k-1)t}{k^2} + o(t^2)\right) = o(t^2)$$

$$\begin{aligned}\frac{1}{k} \cos \tfrac{t(k-1)}{k} + \frac{k-1}{k} \cos \tfrac{t}{k} &= \frac{1}{k}\left(1 - \frac{t^2(k-1)^2}{2k^2} + o(t^2)\right) + \frac{k-1}{k}\left(1 - \frac{t^2}{2k^2} + o(t^2)\right) \\ &= 1 - \frac{(k-1)^2 + (k-1)}{k^3} \cdot \frac{t^2}{2} + o(t^2) \\ &= 1 - \frac{k-1}{k^2} \cdot \frac{t^2}{2} + o(t^2)\end{aligned}$$

Thus, adding the above two together, we get

$$\varphi_k(t) = o(t^2) + 1 - \frac{k-1}{k^2} \cdot \frac{t^2}{2} + o(t^2) = 1 - \frac{k-1}{k^2} \cdot \frac{t^2}{2} + o(t^2)$$

(d) Since $S_n - h(n) = \sum X_k - \frac{1}{k}$, and characteristic functions multiply when variables add, the c.f. for $S_n - h(n)$ is $\prod_1^n \phi_k(u)$, implying the c.f. for $(S_n - h(n))/\sqrt{h(n)}$ is

$$\varphi_n^*(u) = \prod_1^n \phi_k(u/\sqrt{h(n)})$$

(e) Writing the previous formula for $\varphi_n^*$ in little oh notation, and using in the third equality that $\log(1+x) = x + o(x)$,

$$\begin{aligned}
\varphi_n^*(u) &= \prod_1^n \left(1 - \frac{k-1}{k^2} \cdot \frac{u^2/h(n)}{2} + o(u^2)/h(n)\right) \\
&= \exp\left(\sum_1^n \log\left(1 - \frac{k-1}{k^2} \cdot \frac{u^2/h(n)}{2} + o(u^2)/h(n)\right)\right) \\
&= \exp\left(\sum_1^n -\frac{k-1}{k^2} \cdot \frac{u^2/h(n)}{2} + o(u^2)/h(n)\right) \\
&= \exp\left(-\frac{u^2}{2} \cdot \left(\frac{1}{h(n)} \sum_1^n \frac{k-1}{k^2}\right) + n \cdot o(u^2)/h(n)\right)
\end{aligned}$$

Since $\sum_1^n \frac{k-1}{k^2} = h(n) - O(1)$, and $n/h(n) \to 0$, it follows that the above approaches $\exp(-u^2/2)$ as $n \to \infty$, as desired.

# 1999 Fall

1. Since $X_n \to X$ a.s, it must be true that $X_n$ is Cauchy almost surely. Since $X_n'$ has the same distrubtion, this means $X_n'$ is Cauchy almost surely, and since Cauchy sequences converege, $X_n'$ converges a.s.

   To elaborate: $(X_1, X_2, \dots)$ and $(X_1', X_2', \dots)$ having the same distribution on $\mathbb{R}^\infty$ means, for any event $E$ in the product sigma algebra on $\mathbb{R}^\infty$, then $P((X_1, X_2, \dots) \in A) = P((X_1', X_2', \dots) \in A)$. Thus,

$$\begin{aligned} 1 = P(X_n \text{ is Cauchy}) &= P\left( \bigcap_{k\geq 0} \bigcup_{M\geq 0} \bigcap_{m,n\geq M} \{|X_n - X_m| \leq \tfrac{1}{k}\} \right) \\ &= P\left( \bigcap_{k\geq 0} \bigcup_{M\geq 0} \bigcap_{m,n\geq M} \{|X_n' - X_m'| \leq \tfrac{1}{k}\} \right) \\ &= P(X_n' \text{ is Cauchy}) \end{aligned}$$

   where the third equality follows since the enclosed event is in the product sigma algebra on $\mathbb{R}^\infty$.

2. Let $f(x)$ be the pdf of $X$, let $\mu_X = f(x)\,dx$ (so $\mu_X(A) = P(X \in A)$, and $\mu_Y$ be the measure that $Y$ induces on $\mathbb{R}$ (namely, $\mu(A) = P(X \in A)$). Then, using Fubini's (allowed since everything is nonnegative):

$$\begin{aligned} P(X+Y \leq z) = \int 1_{x+y\leq z} d\mu_X \times \mu_Y = \int\int 1_{x\leq z-y} d\mu_X d\mu_Y &= \int\int_{-\infty}^{z-y} f(x)\,dx\,d\mu_Y \\ &= \int\int_{-\infty}^{z} f(x-y)\,dx\,d\mu_Y \\ &= \int_{-\infty}^{z}\int f(x-y)\,d\mu_Y\,dx \end{aligned}$$

   Differentiating the last equation with respect to $z$ shows that $X+Y$ has density given by $f_Z(z) = \int f(x-y)\,d\mu_Y$, so $X+Y$ is absolutely continuous.

3. ($\Longrightarrow$) $S_n \to S$ a.s. implies $S_n \to S$ in distribution, so that the c.f. of $S_n$, $\prod_1^n \phi_k(u)$, converges pointwise to the c.f. of $S$, $h(u)$. That $h(u) \neq 0$ in a neighborhood of 0 follows since $h(0) = e^{iS\cdot 0} = 1$, and $h$ is continuous.

   ($\Longleftarrow$) ☹ This problem is very similar to problem 3.3.21 in Durrett (4th edition), and this problem gives a hint that involves looking at other problems.

4. (a) Since $EZ = \frac{1}{2}e^{it} + \frac{1}{2}e^{-it} = \cos t$, the desired c.f. is

$$\prod_1^n \cos(c_k t)$$

(b) It is a standard result that, for $a_n \geq 0$, $\lim_n \prod_1^n (1 - a_n)$ exists and is nonzero if and only if $\sum_1^\infty a_n < \infty$. So, we will show

$$\sum_1^\infty c_k^2 < \infty \iff \sum_1^\infty 1 - \cos c_k t < \infty \text{ for } |t| < t_0$$

This will complete the proof, since the second condition holds iff $\prod_1^n \cos c_k t$ converges for $|t| < t_0$, which as shown in problem 3 holds iff $\sum_1^\infty c_k Z_k$ converges.

Suppose $\sum_1^\infty c_k^2$. Since $1 - \cos c_k \leq \frac{c_k^2 t^2}{2}$, it follows $\sum_1^\infty 1 - \cos c_k t < \infty$ for all $t$.

Suppose $\sum_1^\infty 1 - \cos c_k t < \infty$ for $t < t_0$. Since $\frac{1-\cos x - x^2/2}{x^2} \to 0$ as $x \to 0$, for small enough $t$, we have, for any $0 < \varepsilon < 1$,

$$\frac{1 - \cos c_k t - c_k^2 t^2/2}{c_k^2 t_k^2} > -\varepsilon$$

proving

$$c_k^2 t^2/2 \leq \frac{1 - \cos c_k t}{(1 - \varepsilon)}$$

Since the right hand side has finite sum, so the the left, proving $\sum_1^\infty c_k^2 < \infty$.

# 2000 Spring

1. (a) $\{A_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k$.

   (b) Let $A_1 \supset A_2 \supset \dots$, where $P(A_n) = n^{-1}$. Then $e_n = \sum_1^n k^{-1} \approx \log n$, but

$$f_n = \sum_{i,j} P(A_i \cap A_j) = \sum_{i,j} (\max(i,j))^{-1} = \sum_{k=1}^{n} (2k-1) \cdot k^{-1} \approx 2n - \log n$$

   The third equality follows since there are $2k-1$ pairs $(i,j)$ for which $\max(i,j) = k$. Thus, we see that $f_n/e_n^2 \sim (2n - \log n)/(\log n)^2 \to \infty$.

   (c) Since $EY_n = 1$, we have that

$$1 - E(Y_n Z_n) = E(Y_n - Y_n Z_n) = EY_n(1 - Z_n) = E(Y_n 1_{Y_n \leq \varepsilon}) \leq \varepsilon$$

   so that $E(Y_n Z_n) \geq 1 - \varepsilon$. Using Cauchy-Schwarz,

$$EY_n Z_n \leq EY_n^2 \cdot EZ_n^2 = \frac{EX_n^2}{e_n^2} \cdot EZ_n = \frac{f_n}{e_n^2} EZ_n,$$

   so $EZ_n \geq \frac{e_n^2}{f_n}(1-\varepsilon)$. Letting $n \to \infty$, we get $\limsup_n EZ_n \geq \frac{1-\varepsilon}{\beta}$ Applying Fatou's Lemma to $1 - Z_n$, we get that

$$P(Y_n \geq \varepsilon \text{ i.o.}) = E \limsup Z_n \geq \limsup EZ_n \geq \frac{1-\varepsilon}{\beta}$$

   Finally, realize that $Y_n \geq \varepsilon$ i.o. implies $A_n$ i.o. (if $A_n$ happens finitely often, then $Y_n = X_n/e_n \to 0$, since $e_n \to \infty$). Thus, $P(A_n \text{ i.o.}) \geq P(Y_n \geq \varepsilon \text{ i.o.})$, so the above also implies $P(A_n \text{ i.o.}) \geq \frac{1-\varepsilon}{\beta}$. Letting $\varepsilon \to 0$ proves $P(A_n \text{ i.o.}) \geq \frac{1}{\beta}$.

2. (a) One can prove that, if $E|X|^n < \infty$, then $\varphi(t)$ is $n$ times continuously differentiable, and $\phi^{(n)}(0) = E(iX)^n$. Taylor's theorem then gives that

$$\varphi(t) = 1 + \varphi'(t)t + \frac{\varphi''(t)}{2}t^2 + O(t^3) = 1 + 0 - \frac{\sigma^2 t^2}{2} + O(t^3)$$

(b) The CLT says that, if $X_1, X_2 \ldots$ i.i.d, $EX = \mu$, Var $X = \sigma^2 < \infty$, then

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \implies N(0,1).$$

Here's a sketch of the proof. We can assume $EX = 0$, by applying the theorem to $X_n - \mu$. If $\varphi$ is the c.f. for $X$, then the characteristic function for $S_n/\sqrt{n}$ is

$$\varphi(t/\sqrt{n})^n = (1 - \sigma^2 t^2/2(\sqrt{n})^2 + O(t^3/(\sqrt{n})^3))^n \approx \left(1 - \frac{\sigma^2 t^2}{2n}\right)^n$$

So

$$\lim_{n\to\infty} \varphi(t/\sqrt{n})^n = \lim_{n\to\infty} \left(1 - \frac{\sigma^2 t^2}{2n}\right)^n = e^{-t^2\sigma^2/2}$$

Since $e^{-t^2\sigma^2/2}$ is the c.f. for $N(0,\sigma^2)$, the continuity theorem implies $S_n/\sqrt{n} \implies N(0,\sigma^2)$, which means that $S_n/(\sigma\sqrt{n}) \implies N(0,1)$.

# 2001 Spring

1. (a) $B = \bigcap_{n\geq 1} \bigcup_{k\geq n} \{|X_k| \geq k\}$.

   (b)
$$1 + \sum_1^\infty P(|X_n| \geq n) \geq \int_0^\infty P(|X| > t)\, dt = E|X| = \infty$$
   proving $P(|X_n| \geq n \text{ i.o.}) = 1$ by Borel-Cantelli.

   (c) If $M_n \to m$, then it would be true that $X_{n+1}/(n+1) = M_{n+1} - M_n + M_n/(n+1) \to m - m + 0 = 0$, so that it wouldn't be true $|X_n|/n \geq 1$ i.o..

   (d) $P(A) = P(A \cap B) + P(A \cap B^c) \leq P(\varnothing) + P(B^c) = 0 + 1 - 1 = 0$.

2. (a) To show a set is an interval, you need only show $s, t \in I$ and $s < r < t$ implies $r \in I$. Suppose $s, t \in I$. Let $s < r < t$. If $r > 0$, then $t > 0$ as well, and whenever $X > 0$, we have $e^{rX} < e^{tX}$. When $X < 0$, $e^{rX} < 1$. Using both these bounds,
$$Ee^{rX} = E(e^{rX} 1_{X<0}) + E(e^{rX} 1_{X\geq 0}) \leq 1 + Ee^{tX} 1_{X>0} \leq 1 + Ee^{tX} < \infty$$
   If on the other hand $r < 0$, then
$$Ee^{rX} = E(e^{rX} 1_{X<0}) + E(e^{rX} 1_{X\geq 0}) \leq Ee^{sX} 1_{X<0} + 1 \leq 1 + Ee^{sX} < \infty$$
   Either way, we have $r \in I$, implying $I$ is an interval.

   (b) We use the fact that $f$ is continuous at $x$ if and only if, for every sequence $x_n$ such that $x_n \to x$, it is true that $f(x_n) \to f(x)$.

   Given $t$ in the interior of $I$, let $t_n$ be any sequence in $I$ where $t_n \to t$. Choose some $T^+, T^- \in I$ so that $T^- \leq t_n \leq T^+$ for all $n$. Then $e^{t_n X} \leq e^{T^+ X} 1_{X>0} + e^{T^- X} 1_{X\leq 0}$, and $e^{t_n X} \to e^{tX}$ pointwise, so by the DCT, we have
$$\lim_n Ee^{t_n X} = E \lim_n e^{t_n X} = Ee^{tX}$$
   This proves $M$ is continuous at $t$.

   (c) Let $Y$ be a random variable where $P(Y > y) = \frac{1}{y}$ when $y > 1$, and let $X = \log Y$. For $t > 0$,
$$Ee^{tX} = EY^t = \int_0^\infty t y^{t-1} P(Y > y)\, dy = t \int_0^\infty y^{t-2}\, dy$$
   This integral is only finite for $t < 1$. When $t < 0$, then $Ee^{tX} \leq 1$ since $tX \leq 0$ always. Thus, the interval for which $e^{tX}$ exists is $(-\infty, 1)$.

3. (a) We have that

$$\text{Var } X_k = EX_k^2 = 1^2 \cdot (1 - \tfrac{1}{k^2}) + k^2 \cdot \tfrac{1}{k^2} = 2 - \frac{1}{k^2}$$

Thus,

$$\text{Var } S_n^* = \text{Var } (S_n)/(\sqrt{n})^2 = \frac{1}{n}\sum_1^n \left(2 - \frac{1}{k^2}\right) = 2 - \frac{\sum_1^n k^{-2}}{n} \longrightarrow 2$$

since $\sum_1^n k^{-2} \to \pi^2/6$.

(b) This proof was figured out by Gene Kim.

We first compute the c.f. for $X_n$. This is given by

$$Ee^{iX_nt} = \frac{1}{2}\left(1 - \frac{1}{n^2}\right)(e^{it\cdot 1} + e^{-it\cdot 1}) + \frac{1}{2n^2}(e^{itn} + e^{-itn}) = (1 - \tfrac{1}{n^2})\cos t + \frac{1}{n^2}\cos nt$$

This implies the c.f. for $S_n^*$ is

$$\begin{aligned}\varphi_n^* = Ee^{itS_n/\sqrt{n}} &= \prod_{k=1}^n (1 - \tfrac{1}{k^2})\cos(\tfrac{t}{\sqrt{n}}) + \frac{1}{k^2}\cos(\tfrac{kt}{\sqrt{n}}) \\ &= \cos^n\left(\frac{t}{\sqrt{n}}\right)\prod_{k=1}^n\left(1 + \frac{1}{k^2}\left(\frac{\cos(kt/\sqrt{n})}{\cos(t/\sqrt{n})} - 1\right)\right) \\ &= \cos^n\left(\frac{t}{\sqrt{n}}\right)\exp\left(\sum_{k=1}^\infty 1_{k\le n}\log\left(1 + \frac{1}{k^2}\left(\frac{\cos(kt/\sqrt{n})}{\cos(t/\sqrt{n})} - 1\right)\right)\right)\end{aligned}$$

We will show the enclosed sum approaches zero as $n \to \infty$, for a fixed $t$. Note that $\frac{\cos(kt/\sqrt{n})}{\cos(t/\sqrt{n})} - 1$ is $O(1)$ as $n \to \infty$, and $\log(1 + x)$ is $O(x)$. Thus, we have that $1_{k\le n}\log(\cdots) \le \frac{C_t}{k^2}$, for some constant $C_t$, so by DCT,

$$\begin{aligned}&\lim_{n\to\infty}\sum_{k=1}^\infty 1_{k\le n}\log\left(1 + \frac{1}{k^2}\left(\frac{\cos(kt/\sqrt{n})}{\cos(t/\sqrt{n})} - 1\right)\right) \\ =&\sum_{k=1}^\infty \lim_{n\to\infty} 1_{k\le n}\log\left(1 + \frac{1}{k^2}\left(\frac{\cos(kt/\sqrt{n})}{\cos(t/\sqrt{n})} - 1\right)\right) = \sum_1^\infty 0 = 0.\end{aligned}$$

Next, we consider the $\cos^n(t/\sqrt{n})$. We have

$$\cos^n\left(\frac{t}{\sqrt{n}}\right) = \left(1 - \frac{t^2/2}{n} + o\left(t^2/n\right)\right)^n \to e^{-t^2/2}$$

These last two results imply that $\varphi_n^* \to e^{-t^2/2}$. Since this is the c.f. for $N(0,1)$, we have that $S_n^* \implies N(0,1)$.

# 2001 Fall

1. (a) First, choose constants $M_n$ so $P(|X_n| > M_n) < \frac{1}{n^2}$, then let $c_n = \frac{M_n^2 n^2}{\epsilon^2}$. Letting $Y_n = X_n 1_{|X_n| \le M}$, we have, for any $\varepsilon > 0$,

$$P(|Y_n/c_n| > \epsilon) = P(Y_n^2/\epsilon^2 > c_n^2) \le \frac{\frac{1}{\varepsilon^2} EY_n^2}{c_n^2} \le \frac{M_n^2}{\varepsilon^2 c_n^2} \le \frac{1}{n^2}$$

Thus, by Borel-Cantelli, $P((|Y_n/c_n| > \epsilon \text{ i.o.}) = 0$. This holds for all $\varepsilon > 0$, which allows you to show $Y_n/c_n \to 0$ a.s. Furthermore, since $P(X_n \ne Y_n) < \frac{1}{n^2}$, we have $P(X_n \ne Y_n \text{ i.o.}) = 0$, so that with probability 1 we also have $X_n/c_n \to 0$.

(b) No. Consider the probability space $(0, 1)$, with Lesbesgue measure. Let $\Omega_0$ be set where $P(\Omega_0) = 0$ and whose cardinality is $2^{\aleph_0}$ (for example, the Cantor set). Now, choose $X_n$ so *every* possible sequence of real numbers $c_1, c_2, \ldots$ occurs as $X_1(\omega), X_2(\omega), \ldots$ for some $\omega \in \Omega_0$, and $X_n(\omega) = 0$ for $\omega \notin \Omega_0$. This can be done since the number of such sequences is $(2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} = |\Omega_0|$, and the $X_n$ will indeed be measurable since they are 0 a.e. Then, no matter what constants $c_1, c_2, \ldots$ you choose, there will be some $\omega$ for which $X_n(\omega)/c_n = 1$ for all $n$.

(c) See 1997 Fa, 4(a).

2. (a) The special property is that $\varphi$ will be real. If $X$ and $-X$ have the same distrubtion, then

$$Ee^{itX} = E\cos tX + iE\sin tX$$

But $tX$ is symmetrically positive and negative, and $\sin(tx)$ is an odd function, so $E\sin(tX) = 0$.

Suppose $Ee^{itX}$ is real. Using the inversion formula, we have, for any $a < b$,

$$P(X \in (a, b)) + \frac{1}{2} P(X \in \{a, b\}) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)\, dt$$

Both sides are real, so taking the conjugate of the right preserves equality, resulting in

$$\begin{aligned} P(X \in (a, b)) + \frac{1}{2} P(X \in \{a, b\}) &= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-it(-a)} - e^{-it(-b)}}{-it} \varphi(t)\, dt \\ &= \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-it(-b)} - e^{-it(-a)}}{it} \varphi(t)\, dt \\ &= P(X \in (-b, -a)) + \frac{1}{2} P(X \in \{-b, -a\}) \\ &= P(-X \in (a, b)) + \frac{1}{2} P(-X \in \{a, b\}) \end{aligned}$$

This holds for all $a, b$, proving $X$ and $-X$ have the same distribution.

(b) This is given by $\phi(t/n)^n$.

(c) Since $\phi'(0) = 0$, we have that

$$\lim_{n\to\infty} \frac{\phi(t/n) - 1}{t/n} = 0$$

Furthermore, from calculus it is true that $\frac{\log(1+x)}{x} \to 1$ as $x \to 0$, implying $\frac{\log \phi(t/n)}{\phi(t/n)-1} \to 1$ as $n \to \infty$. Multiplying these two limits, we get

$$\lim_{n\to\infty} \frac{\log \phi(t/n)}{t/n} = 0$$

Taking exp of both sides, we get $\phi(t/n)^n \to 1$. But $\phi(t/n)^n$ is the c.f. for $S_n/n$, and 1 is the c.f. for 0, so the continutity theorem implies $S_n/n \to 0$ weakly. Finally, one can prove that converging weakly to a constant implies convergence in probability as well, so that $S_n/n \to 0$ in probability.

(d) We have

$$E|X| = 2c \int_4^\infty x \cdot \frac{1}{x^2 \log x}\, dx = 2c(\lim_{n\to\infty} \log\log n - \log\log 4) = \infty$$

(e) Since $X$ is symmetric about 0, we have

$$E\frac{e^{itX} - 1}{t} = E\frac{\cos(tX) - 1}{t} = 2c \int_4^\infty \frac{\cos(tx) - 1}{tx^2 \log|x|}\, dx$$

Letting $y = tx$, this becomes

$$E\frac{e^{itX} - 1}{t} = 2c \int_4^\infty \frac{\cos(y) - 1}{t(y/t)^2 \log|y/t|}\, d(y/t) = 2c \int_4^\infty \frac{\cos(y) - 1}{y^2 \log|y/t|}\, dy$$

Since, for $-1 < t < 1$, it's true that $\frac{\cos(y)-1}{y^2 \log|y/t|} \le \frac{\cos(y)-1}{y^2 \log|y|} \in L_1(dy)$, the DCT implies

$$\lim_{t\to 0} 2c \int_4^\infty \frac{\cos(y) - 1}{y^2 \log|y/t|}\, dy = 2c \int_4^\infty \lim_{t\to 0} \frac{\cos(y) - 1}{y^2 \log|y/t|}\, dy = 2c \int_4^\infty 0\, dt = 0$$

Which proves that

$$\lim_{t\to\infty} E\frac{e^{itX} - 1}{t} = \lim_{t\to 0} 2c \int_4^\infty \frac{\cos(y) - 1}{y^2 \log|y/t|}\, dy = 0$$

proving $\phi'(0) = 0$.

# 2002 Spring

1. First, realize that $E|X_1|^2 < \infty$ implies $|X_n|^2/n \to 0$ a.s, which in turn implies $|X_n|/\sqrt{n} \to 0$ a.s. The first fact is proven by using $\sum_{n\geq 1} P(|X_n|^2/n \geq \varepsilon) \leq \int_0^\infty P(|X_1^2/\varepsilon| > t)\, dt = E|X_1/\varepsilon|^2 < \infty$, then using Borel-Cantelli to argue $P(|X_n^2|/n > \varepsilon \text{ i.o.}) = 0$ for all $\varepsilon > 0$, which then gives $X_n^2/n \to 0$ a.s.

   Once you have $|X_n|/\sqrt{n} \to 0$ a.s, we use the below lemma:

   **Lemma** Let $\{a_n\}_{n\geq 0}$ be a nonrandom, nonnegative sequence, where $a_n/\sqrt{n} \to 0$. Let $m_n = \max_{1\leq k\leq n} a_n$. Then $m_n/\sqrt{n} \to 0$.

   *Proof.* Given $\varepsilon > 0$, choose $K$ so $n > K$ implies $a_n/\sqrt{n} < \varepsilon$. Then

   $$\frac{m_n}{\sqrt{n}} \leq \frac{m_K}{\sqrt{n}} + \max_{K\leq j\leq n} \frac{a_j}{\sqrt{n}} \leq \frac{m_K}{\sqrt{n}} + \max_{K\leq j\leq n} \frac{a_j}{\sqrt{j}} \leq \frac{m_K}{\sqrt{n}} + \varepsilon$$

   Letting $n \to \infty$ shows, since $m_K/\sqrt{n} \to 0$, that $\limsup m_n/\sqrt{n} \leq \varepsilon$. This holds for all $\varepsilon > 0$, so $m_n/\sqrt{n} \to 0$. □

   Thus, $|X_n|/\sqrt{n} \to 0$ a.s. implies $\max_{1\leq k\leq n} |X_n|/\sqrt{n} \to 0$ a.s, and therefore in probability.

2. By Borel-Cantelli, $P(|X_n| > \varepsilon_n \text{ i.o.}) = 0$. Thus, with probability 1, there will be some $K$ where $n > K$ implies $|X_n| < \varepsilon_n$, meaning $\sum |X_n| \leq \sum_1^K |X_n| + \sum_{K+1}^\infty \varepsilon_n < \infty$.

# 2002 Fall

1. The desired $\alpha$ is $\alpha = 3$. Let $X_{n,k} = \frac{X_k}{n^3}$. We prove convergence using the Lindberg-Feller CLT. Then, using the fact that Var $(X_k) = \int_{-k}^{k} x^2 \cdot \frac{1}{2k}\, dx = \frac{k^2}{3}$,

$$\sum_{k=1}^{n} EX_{n,k}^2 = \frac{1}{n^3}\sum_{k=1}^{n} \text{Var } X_k = \frac{1}{n^3}\sum_{k=1}^{n}\frac{k^2}{3}$$

Then, since $\sum_{k=1}^{n} \frac{k^2}{3} \approx \int_0^n \frac{x^2}{3}\, dx = \frac{n^3}{9}$, we have that

$$\sum_{k=1}^{n} EX_{n,k}^2 \approx \frac{1}{n^3}\cdot\frac{n^3}{9} \to \frac{1}{9} \qquad\qquad \text{as } n \to \infty$$

The above use of $\approx$ can be made more precise, either by finding an closed form for $\sum_1^n \frac{k^2}{3}$, or by using and upper and lower integral bound.

This gives the first condition of the Lindberg Feller CLT. For the second, we must show

$$\sum_{k=1}^{n} E(X_{n,k}^2 \cdot 1_{|X_{n,k}|>\varepsilon}) = \sum_{k=1}^{n} E(\tfrac{X_k^2}{n^3} \cdot 1_{|X_k|>\varepsilon n^3}) \to 0.$$

Notice that, for large enough $n$, we have that $\varepsilon n^3 > n^2 \geq |X_k|$. Thus, for large $n$, the above sum will be zero, since all the indicator variables $1_{|X_k|>\varepsilon n^3}$ will all be zero.

By the Lindberg Feller CLT, this shows

$$S_n/n^3 = \sum_{k=1}^{n} X_{n,k} \to N(0, \tfrac{1}{9}).$$

2. (a) We first show that $P(Y > n \text{ i.o.}) = 0$. We have

$$\sum_{n\geq 1} P(Y_n > n) \leq \int_0^\infty P(Y > t)\, dt = EY < \infty$$

By Borel Cantelli, $P(Y > n \text{ i.o.}) = 0$.

Thus, with probability one, we have

$$\limsup_n (Y_n)^{1/n} \leq \limsup_n (n)^{1/n} = 1$$

By the root test, the radius convergence of $\sum Y_k \alpha^k$ is at least 1, so that it converges when $|\alpha| < 1$.

(b) Choose $Y$ so that $P(Y > y^y) = \frac{1}{y}$ when $y > 1$. In other words, letting $f(y)$ by the inverse function of $g(y) = y^y$, let $Y$ be the random variable whose distribution is

$$P(Y \le y) = 1 - \frac{1}{f(y)} \qquad (y > 1)$$

Then $\sum P(Y_n > n^n) = \sum \frac{1}{n} = \infty$, so by Borel-Cantelli, $P(Y_n > n^n \text{ i.o.}) = 1$, proving that, with probability one,

$$\limsup_n (Y_n)^{1/n} \ge \limsup (n^n)^{1/n} = \infty.$$

Thus, almost surely the radius of convergence will be 0, proving $S = \infty$.

3. **Proof 1:** Let $\mu$ be the measure on $\mathbb{R}$ induced by $X$, so $\mu(A) = P(X \in A)$, and $\nu$ for $Y$ similarly. Since $E|X+Y|^p < \infty$, using Fubini's theorem we have

$$E|X+Y|^p = \int |x+y|^p d\mu \times \nu = \int \left( \int |x+y|^p d\mu \right) d\nu < \infty$$

This implies $\left(\int |x+y|^p d\mu\right) < \infty$ for $\nu$ a.e. $y$, so there is some $y_0$ for which it holds. Then, using $|x|^p = |x+y_0-y_0|^p \le 2^p(|x+y_0|^p + |-y_0|^p)$,

$$E|X|^p = \int |x|^p d\mu \le \int 2^p|x+y_0|^p + 2^p|y_0|^p \, d\mu = 2^p \int |x+y_0|^p \, d\mu + 2^p|y_0|^p < \infty$$

**Proof 2:** Choose $M$ so $P(|Y| \le M) = \varepsilon > 0$. For all $t$, we have

$$\begin{aligned} P(|X+Y| > t-M) &\ge P(\{|X| > t\} \cap \{|Y| \le M\}) \\ &= P(|X| > t)P(|Y| \le M) \end{aligned}$$

Using this,

$$\begin{aligned} E|X|^p = \int_0^\infty pt^{p-1}P(|X|>t)\,dt &\le \int_0^\infty pt^{p-1}\frac{P(|X+Y|>t-M)}{P(|Y| \le M)}\,dt \\ &= \frac{1}{\varepsilon}\left( \int_0^M pt^{p-1}\,dt + \int_M^\infty pt^{p-1}P(|X+Y|>t-M)\,dt \right) \end{aligned}$$

The first integral, $\int_0^M pt^{p-1}\,dt$, is some $K < \infty$. For the second, we use the chagne of variables $u = t-M$, obtaining

$$E|X|^p \le \frac{1}{\varepsilon}\left( K + \int_0^\infty p(u+M)^{p-1}P(|X+Y|>u)\,du \right)$$

Notice that, when $u > M$, we have $(u+M)^{p-1} \le 2^{p-1}u^{p-1}$, so[1]

$$\begin{aligned} E|X|^p &\le \frac{1}{\varepsilon}\left( K + \int_0^M p(u+M)^{p-1}\,du + 2^{p-1}\int_M^\infty pu^{p-1}P(|X+Y|>u)\,du \right) \\ &\le \frac{1}{\varepsilon}\left( K + \int_0^M p(u+M)^{p-1}\,du + 2^{p-1}E|X+Y|^p \right) < \infty \end{aligned}$$

[1]This only works when $p \ge 1$. When $p < 1$, use the bound $(u+M)^{p-1} \le u^{p-1}$

4. Note that $F_\infty$ being continuous implies that, for some $m$, $P(X_\infty \leq m) = \frac{1}{2}$, implying also that $P(X_\infty \geq m) = P(X_\infty > m) = 1 - \frac{1}{2} = \frac{1}{2}$. This $m$ is a median, so $m = m_\infty$. Furthermore, for any $\varepsilon > 0$, we must have $P(X_\infty \leq m_\infty - \varepsilon) < \frac{1}{2}$: if it equaled $\frac{1}{2}$, that would mean $m_\infty - \varepsilon$ was another median, violating uniqueness. By the same logic, $P(X_\infty \leq m_\infty + \varepsilon) > \frac{1}{2}$.

   For any $\varepsilon > 0$, we have

$$\lim_{n\to\infty} P(X_n \leq m_\infty - \varepsilon) = P(X \leq m_\infty - \varepsilon) < \frac{1}{2}$$

   The above shows that, for large enough $n$, we have $P(X_n \leq m_\infty - \varepsilon) < \frac{1}{2}$, so that for large enough $n$, $m_n \geq m_\infty - \varepsilon$.

   Similarly,

$$\lim_{n\to\infty} P(X_n \leq m_\infty + \varepsilon) = P(X \leq m_\infty + \varepsilon) > \frac{1}{2}$$

   proving $P(X_n \leq m_\infty + \varepsilon) > \frac{1}{2}$ eventually, so that $m_n \leq m_\infty + \varepsilon$ eventually.

   We have shown

$$m_\infty - \varepsilon \leq \liminf_n m_n \leq \limsup_n m_n \leq m_\infty + \varepsilon$$

   for all $\varepsilon > 0$, proving $m_n \to m_\infty$.

## 2003 Spring

1. Since $\phi'(0) = ia$, we have that

$$\lim_{n\to\infty} \frac{\phi(t/n) - 1}{t/n} = ia$$

Furthermore, from calculus it is true that $\frac{\log(1+x)}{x} \to 1$ as $x \to 0$, implying $\frac{\log\phi(t/n)}{\phi(t/n)-1} \to 1$ as $n \to \infty$. Multiplying these two limits, we get

$$\lim_{n\to\infty} \frac{\log\phi(t/n)}{t/n} = ia$$

Taking exp of both sides, we get $\phi(t/n)^n \to e^{iat}$. But $\phi(t/n)^n$ is the c.f. for $S_n/n$, and $e^{iat}$ is the c.f. for $a$, so the continutity theorem implies $S_n/n \to a$ weakly. Finally, one can prove that converging weakly to a constant implies convergence in probability as well, so that $S_n/n \to a$ in probability.

2. Let $a_n = \inf\{x : F_n(x) \geq \frac{1}{2}\}$. This implies $F_n(a_n) \geq \frac{1}{2}$ by right continuity of $F_n$. Since $X_n - X_n' \to 0$ in distribution, we have that $P(|X_n - X_n'| > \varepsilon) \to 0$. Since $X_n > a_n + e$ and $X_n' \leq a_n$ implies $X_n - X_n' > \varepsilon$, we have that

$$\begin{aligned} P(|X_n - X_n'| > \varepsilon) &\geq P(\{X_n > a_n + \varepsilon\} \cap \{X_n' \leq a_n\}) \\ &= P(X_n > a_n + \varepsilon)P(X_n' \leq a_n) \\ &\geq P(X_n > a_n + \varepsilon) \cdot \tfrac{1}{2} \end{aligned}$$

The last inequality follows since $P(X_n' \leq a_n) = P(X_n \leq a_n) = F_n(a_n) \geq \frac{1}{2}$.

Since $P(|X_n - X_n'| > \varepsilon) \to 0$, the displayed string of inequalities implies $P(X_n > a_n + \varepsilon) \to 0$ as well.

By the same logic, we have

$$\begin{aligned} P(|X_n - X_n'| > \varepsilon/2) &\geq P(X_n \leq a_n - \varepsilon)P(X_n' > a_n - \tfrac{\varepsilon}{2}) \\ &= P(X_n \leq a_n - \varepsilon)(1 - P(X_n \leq a_n - \tfrac{\varepsilon}{2})) \\ &\geq P(X_n \leq a_n - \varepsilon) \cdot \tfrac{1}{2} \end{aligned}$$

The last inequlaity follows from the definition of $a_n$: since $a_n - \frac{\varepsilon}{2} < a_n$, and $a_n = \inf\{x : F_n(x) \geq \frac{1}{2}\}$, we must have $P(X_n \leq a_n - \frac{\varepsilon}{2}) < \frac{1}{2}$.

Thus, the above shows that $P(X_n \leq a_n - \varepsilon) \to 0)$. Finally, we have that

$$P(|X_n - a_n| \geq \varepsilon) \leq P(X_n > a_n + \varepsilon) + P(X_n \leq a_n - \varepsilon) \to 0$$

proving $X_n \to a_n$ in probability.

3. Let $a_n = \frac{1}{\alpha}\log n$, and $\beta = 1$. Since $P(X_n > x) = x^{-\alpha}$, we have that

$$P(\tfrac{\log X_n}{(\log n)/\alpha} > 1) = P(X_n > n^{1/\alpha}) = n^{-1}$$

Since $\sum n^{-1} = \infty$, by Borel-Cantelli, $P(\frac{\log X_n}{(\log n)/\alpha} > 1 \text{ i.o.}) = 1$. This proves that $\limsup \frac{\log X_n}{(\log n)/\alpha} \geq 1$ a.s.

Furthermore, for any $e > 0$, we have

$$P(\tfrac{\log X_n}{(\log n)/\alpha} > 1+\varepsilon) = P(X_n > n^{(1+\varepsilon)/\alpha}) = n^{-1-\varepsilon}$$

Since $\sum n^{-1-\varepsilon} < \infty$, by Borel-Cantelli, $P(\frac{\log X_n}{(\log n)/\alpha} > 1+\varepsilon \text{ i.o.}) = 0$. This proves that $\limsup \frac{\log X_n}{(\log n)/\alpha} \leq 1+\varepsilon$ a.s. Since this holds for all $\varepsilon > 0$, this additionally proves that $\limsup \frac{\log X_n}{(\log n)/\alpha} \leq 1$ a.s.

We have proven $\limsup \frac{\log X_n}{(\log n)/\alpha} = 1$ a.s, and would like to prove the same for $M_n$. Since $M_n \geq X_n$, we certainly now know that

$$\limsup \tfrac{\log M_n}{(\log n)/\alpha} \geq 1 \qquad \text{a.s.}$$

For the other inequality, we use the following Lemma:

**Lemma:** Let $\{a_n\}$ be a (nonrandom) sequence, and $\{b_n\}$ be an increasing sequence where $b_n \to \infty$. Let $m_n = \max_{1\leq k\leq n} a_k$. If $\limsup a_n/b_n \leq 1$, then $\limsup m_n/b_n \leq 1$.

*Proof.* Given $\varepsilon > 0$, choose $N$ so $n > N$ implies $a_n/b_n \leq 1+\varepsilon$. Then

$$\frac{m_n}{b_n} \leq \frac{m_N}{b_n} + \max_{N\leq k\leq n} \frac{a_k}{b_n} \leq \frac{m_N}{b_n} + \max_{N\leq k\leq n} \frac{a_k}{b_k} \leq \frac{m_N}{b_n} + 1 + \varepsilon$$

Since $m_N/b_n \to 0$, the above proves $\limsup m_n/b_n \leq 1+\varepsilon$. Letting $\varepsilon \to 0$ completes the proof. □

This lemma shows $\limsup \frac{\log X_n}{(\log n)/\alpha} = 1$ a.s. implies $\limsup \frac{\log M_n}{(\log n)/\alpha} \leq 1$ a.s, so we are done.

4. (i) Let $\|X\|_p$ denote $(EX^p)^{1/p}$. By Minkowski's inequality, $\|X+Y\|_p \leq \|X\|_p + \|Y\|_p$. Therefore,

$$\|X_n - X_m\|_p \leq \|X_n - X\|_p + \|X - X_m\|_p$$

The right side approaches zero since $E|X_n - X|^p \to 0$, proving $\|X_n - X\|_p \to 0$. Raising both sides to $p$ then implies that $E|X_n - X_m|^p \to 0$.

(ii) This proof is due to Gene Kim.

Choose a subsequence $X_{n(k)}$ so that $\|X_{n(k)} - X_{n(k+1)}\|_p < \frac{1}{2^k}$. Let

$$\phi_m = |X_{n(1)}| + \sum_{k=2}^{m} |X_{n(k)} - X_{n(k-1)}| \qquad \phi = \lim_{m\to\infty} \phi_m$$

By the MCT,

$$\|\phi\|_p = \lim_{m\to\infty} \|\phi_m\|_p \leq \|X_{n(1)}\|_p + \sum_{k=2}^{\infty} \|X_{n(k)} - X_{n(k-1)}\|_p \leq \|X_{n(1)}\|_p + \sum_{k=2}^{\infty} \frac{1}{2^k} < \infty$$

Since $\|\phi\|_p < \infty$, it must be true that $\phi < \infty$ almost surely, which proves that the series

$$X = X_{n(1)} + \sum_{k=2}^{\infty} X_{n(k)} - X_{n(k-1)}$$

converges absolultely, and therefore converges. Also,

$$X = \lim_{m\to\infty} X_{n(1)} + \sum_{k=2}^{m} X_{n(k)} - X_{n(k-1)} = \lim_{n\to\infty} X_{n(m)}$$

so $X_{n(m)}$ is a sequence converging almost surely to $X$.

(iii) Letting $X$ be defined as before, for any $m$ we have $X = X_{n(m)} + \sum_{k=m+1}^{\infty} X_{n(k)} - X_{n(k+1)}$, so

$$\|X - X_{n(m)}\|_p \leq \sum_{k=m+1}^{\infty} \|X_{n(k)} - X_{n(k+1)}\|_p \leq \sum_{k=m+1}^{\infty} \frac{1}{2^k} \xrightarrow{m\to\infty} 0$$

proving $X_{n(m)} \to X$ in $L_p$. Since $X_n$ is Cauchy in $L_p$, and has a subsequence converging to $X$, this implies $X_n \to X$ in $L_p$.

## 2003 Fall

1. This proof is due to Gene Kim.

   Let $M_n = \frac{1}{n}\max_{j\le n} X_j$, and let $F_X(x) = P(X \le x)$. Since $M_n \le x$ exactly when each $X_j \le nx$, we have that $P(M_n \le m) = F_X(nx)^n$. Thus,

$$\begin{aligned} EM_n &= \int_0^\infty P(M_n > x)\,dx \\ &= \int_0^\infty 1 - F_X(nx)^n\,dx \\ &= \int_0^\infty \frac{1 - F_X(t)^n}{n}\,dt \\ &= \int_0^\infty (1 - F_X(t))\left(\frac{1 + F_X(t) + F_X(t)^2 + \cdots + F_X(t)^{n-1}}{n}\right)dt \end{aligned}$$

   Since $\left(\frac{1+F_X(t)+F_X(t)^2+\cdots+F_X(t)^{n-1}}{n}\right) \le 1$, the above integrand is bounded by $1 - F_X(t)$, which is integrable since $\int_0^\infty 1 - F_X(t) = EX < \infty$. Thus, by the DCT,

$$\begin{aligned} \lim_{n\to\infty} EM_n &= \int_0^\infty \lim_{n\to\infty} (1 - F_X(t))\left(\frac{1 + F_X(t) + F_X(t)^2 + \cdots + F_X(t)^{n-1}}{n}\right)dt \\ &= \int_0^\infty (1 - F_X(t))1_{\{F_X(t)=1\}}\,dt = \int_0^\infty 0\,dt = 0 \end{aligned}$$

2. Impossible Problem!! Let $U \sim \text{Unif}(0,1)$, and $f(x) = 0$ when $x \le 1$ and $f(x) = x$ when $x > 1$. Then $f(X) = 0$ always, so $X$ and $f(X)$ are independent, but $f$ is not constant.

   The problem is possible when reworded as follows: if $X$ and $f(X)$ are independent, then $f(X)$ is constant a.s.

   Since $X$ is independent of $f(X)$, this implies $f(X)$ is independent of $f(X)$ (this comes from the theorem which says that, if $Y$ independent of $Z$, then $g(Y)$ independent of $h(Z)$). This means that, for any $x \in \mathbb{R}$, the event $\{f(X) \le x\}$ is independent of itself. Thus, $P(f(X) \le x) = 0$ or 1, since $A$ independent of itselft implies $P(A) = P(A \cap A) = P(A)P(A)$. This implies $f(X)$ is constant a.s; if it were nonconstant, there would be some $x$ where $P(f(X) \le x)$ was neither 0 nor 1.

3. Unclear wording: They should have mentioned that $\sigma^2$ was finite.

   (a) Let $S = \sum_1^{N_\lambda} X_i$, and $S_n = \sum_1^n X_i$. We first find the c.f. for $S$. Let $\varphi$ be the c.f. for $X_1$. Then

$$\begin{aligned} Ee^{itS} &= E\sum_{n=0}^{\infty} e^{itS}1_{N_\lambda=n} = \sum_{n=0}^{\infty} E(e^{itS_n}1_{N_\lambda=n}) = \sum_{n=0}^{\infty} E(e^{itS_n})P(N_\lambda=n) \\ &= \sum_{n=0}^{\infty}\varphi(t)^n\frac{e^{-\lambda}\lambda^n}{n!} = e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda\varphi(t))^n}{n!} = e^{-\lambda}e^{\lambda\varphi(t)} = \exp(\lambda(\varphi(t)-1)) \end{aligned}$$

   Since the c.f. for $N_\lambda$ is $\exp(\lambda(e^{it}-1))$, this means the c.f for $\frac{S-N_\lambda\mu}{\sqrt{\lambda}}$ is

$$\begin{aligned} E\left(\exp\left(it\cdot\frac{S-N_\lambda\mu}{\sqrt{\lambda}}\right)\right) &= \exp(\lambda(\varphi(t/\sqrt{\lambda})-1))\cdot\exp(\lambda(e^{-it\mu/\sqrt{\lambda}}-1)) \\ &= \exp\left(\lambda\left(\varphi(\tfrac{t}{\sqrt{\lambda}}) + (e^{-it\mu/\sqrt{\lambda}}-1)-1\right)\right) \end{aligned}$$

   Now, note that that

$$e^{-it\mu/\sqrt{\lambda}}-1 = \frac{-it\mu}{\sqrt{\lambda}} - \frac{t^2\mu^2}{2\lambda} + o(t^2/\lambda)$$

   and

$$\begin{aligned} \varphi(t/\sqrt{\lambda}) &= 1 + it\mu - \frac{t^2}{2}EX^2 + o(t^2/\lambda) \\ &= 1 + \frac{it\mu}{\sqrt{\lambda}} - \frac{t^2}{2\lambda}(\sigma^2+\mu^2) + o(t^2/\lambda) \end{aligned}$$

   Thus,

$$\begin{aligned} E\left(\exp\left(it\cdot\frac{S-N_\lambda\mu}{\sqrt{\lambda}}\right)\right) &= \exp\left(\lambda\left(\frac{it\mu}{\sqrt{\lambda}} - \frac{t^2}{2\lambda}(\sigma^2+\mu^2) + \frac{-it\mu}{\sqrt{\lambda}} - \frac{t^2\mu^2}{2\lambda} + o(t^2/\lambda)\right)\right) \\ &= \exp(-t^2(\sigma^2+2\mu^2)/2 - \lambda o(t^2/\lambda)) \to \exp(-t^2(\sigma^2+2\mu^2)/2) \end{aligned}$$

   The last expression is the c.f. for $N(0,\sigma^2+2\mu^2)$, which is the limit distribution.

   (b) Since the c.f. for $\sqrt{\lambda}\mu$ is $\exp(it\mu\sqrt{\lambda})$, the c.f for $\frac{S-\lambda\mu}{\sqrt{\lambda}}$ is

$$E\left(\exp\left(it\cdot\tfrac{S-\lambda\mu}{\sqrt{\lambda}}\right)\right) = \exp(\lambda(\varphi(t/\sqrt{\lambda})-1))\exp(-it\mu\sqrt{\lambda}) = \exp\left(\lambda\left(\varphi(t/\sqrt{\lambda}) - \frac{it\mu}{\sqrt{\lambda}} - 1\right)\right)$$

   Using the same asymptotics,

$$E\left(\exp\left(it\cdot\tfrac{S-\lambda\mu}{\sqrt{\lambda}}\right)\right) = \exp\left(\lambda\left(\frac{-t^2(\sigma^2+\mu^2)}{2\lambda} + o(t^2/\lambda)\right)\right) \to \exp(-t^2(\sigma^2+\mu^2)/2)$$

   The latter is the c.f. for $N(0,\sigma^2+\mu^2)$, which is therefore the desired limit distrubution.

   (c) The two limit distriubtions are only the same when $\mu = 0$.

4. (a) We have that

$$E[X+Y|X,Y>0] = E[X|X,Y>0] + E[Y|X,Y>0] = E[X|X>0] + E[Y|Y>0]$$
$$= 2E[X|X>0]$$

The second $=$ follows since $X$ is independent of $Y$. We then have

$$E[X|X>0] = \frac{E[X1_{X>0}]}{P(X>0)} = 2\int_0^\infty x\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx = \frac{2}{\sqrt{2\pi}}\left(-e^{-x^2/2}\right)\Big|_0^\infty = \sqrt{2/\pi}$$

Thus, $E[Z|X,Y>0] = 2\sqrt{2/\pi}$.

(b) This problem is a little misleading: you can't really get a closed form for the dsitribution of $Z$. However, you can get an expression in terms of the distribution of $X$.

$$P(Z \le z|X,Y>0) = \frac{P(Z\le z, X>0, Y>0)}{P(X>0,Y>0)},$$

Let $T$ be the event that $Z \le z$, $X>0$, $Y>0$. Let $S$ be the event that $(X,Y)$ is in the square with vertices $(\pm z, 0)$ and $(0, \pm z)$. By symmetry, $P(T) = \frac{1}{4}P(S)$. Now, let $S'$ be the event that $(X,Y)$ is in this same square, but rotated 45 degrees about the orgin; this is the square with vertices $(\pm\frac{z}{\sqrt{2}}, \pm\frac{z}{\sqrt{2}})$. Since the pdf of $(X,Y)$ is

$$\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\cdot\frac{1}{\sqrt{2\pi}}e^{-x^2/2} = \frac{1}{2\pi}e^{-r^2/2},$$

where $r^2 = x^2+y^2$, it follows that the pdf has rotational symmetry, so that $P(S) = P(S')$. Finally,

$$\begin{aligned}P(S') &= P(|X| \le \tfrac{z}{\sqrt{2}})P(|Y|\le \tfrac{z}{\sqrt{2}})\\ &= \left(\frac{1}{\sqrt{2\pi}}\int_{-z/\sqrt{2}}^{z/\sqrt{2}} e^{-x^2/2}\,dx\right)^2\\ &= 4\left(\frac{1}{\sqrt{2\pi}}\int_0^{z/\sqrt{2}} e^{-x^2/2}\,dx\right)^2 = 4(F_X(z/\sqrt{2})-\tfrac{1}{2})^2\end{aligned}$$

so

$$P(Z|\le zX,Y>0) = \frac{\frac{1}{4}P(S')}{P(X>0)P(Y>0)} = P(S') = 4(F_X(z/\sqrt{2})-\tfrac{1}{2})^2$$

Differentiating with respect to $z$ gives the density $f_Z(z)$ of $Z$:

$$f_Z(z) = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2\pi}}e^{-(z/\sqrt{2})^2/2}\cdot 8(F_X(z/\sqrt{2})-\tfrac{1}{2}) = \frac{4}{\sqrt{\pi}}e^{-z^2/4}\cdot(F_X(z/\sqrt{2})-\tfrac{1}{2})$$

# 2004 Spring

**The $\pi - \lambda$ theorem:** A $\pi$-system is a collection of subsets of $\Omega$ which is closed under intersection. A $\lambda$-system, $\mathcal{L}$, is a collection of subsets of $\Omega$ where

(i) $\Omega \in \mathcal{L}$

(ii) if $A, B \in \mathcal{L}$, $A \subset B$, then $B \setminus A \in \mathcal{L}$

(iii) if $A_n \nearrow A$, and each $A_n \in \mathcal{L}$, then $A \in \mathcal{L}$.

The $\pi - \lambda$ theorem says that, if $\mathcal{P}$ is a $\pi$-system, $\mathcal{L}$ is a $\lambda$-system, and $\mathcal{P} \subset \mathcal{L}$, then $\sigma(P) \subset \mathcal{L}$, where $\sigma(P)$ is the sigma algebra generated by $P$.

1. (a) Let $\mathcal{A}$ be the sets of the form $\{X \leq x\}$, for $x \in [-\infty, +\infty]$, and $B$ be sets of the form $\{Y \leq y\}$. Note that $\mathcal{A}$ is a $\pi$-system, since $\{X \leq a\} \cap \{X \leq b\} = \{X \leq a \wedge b\}$. Let

$$\mathcal{L} = \{E \in \sigma(X) : P(E \cap B) = P(E)P(B) \text{ for all } B \in \mathcal{B}\}$$

Note that by assumtion, $\mathcal{A} \subset \mathcal{L}$.

We will show $\mathcal{L}$ is a Lambda system, by checking each of the above three conditions

(i) $P(\Omega \cap B) = P(B) = P(\Omega)P(B)$, so $\Omega \in \mathcal{L}$.

(ii) If $E, F \in \mathcal{L}$, and $E \subset F$, then

$$\begin{aligned} P((E \setminus F) \cap B) &= P(E \cap B) - P(F \cap B) = P(E)P(B) - P(F)P(B) \\ &= (P(E) - P(F))P(B) = P(E \setminus F)P(B) \end{aligned}$$

so $E \setminus F \in \mathcal{L}$.

(iii) If $E_n \nearrow E$, then $E_n \cap B \nearrow E \cap B$, proving that $P(E_n \cap B) = P(E_n)P(B) \nearrow P(E \cap B)$. Since we also have $P(E_n)P(B) \nearrow P(E)(B)$, this implies $P(E \cap B) = P(E)P(B)$.

Applying the $\pi - \lambda$ theorem gives that $\sigma(\mathcal{A}) = \sigma(X) \subset \mathcal{L}$. We the apply the $\pi - \lambda$ theorem *again* to

$$\mathcal{L}' = \{E \in \sigma(Y) : P(E \cap A) = P(E)P(A) \text{ for all } A \in \sigma(X)\}$$

Since $\mathcal{B} \subset \mathcal{L}'$, we have that $\sigma(B) = \sigma(Y) \subset \mathcal{L}'$. Now, notice that $\sigma(Y) \subset \mathcal{L}$ means that, for all $A \in \sigma(X)$, and all $B \in \sigma(Y)$, $P(A \cap B) = P(A)P(B)$, proving that $X, Y$ are independent.

(b) It is sufficient to show that, for all $k$,

$$P(B_1 = b_1, \ldots, B_k = b_k) = P(B_1 = b_1) \cdots P(B_k = b_k)$$

since the sets $\{B_i = b_i\}$, for $b_1 = 0, 1$, generate $\sigma(B_i)$. Note that the right hand side is $(1/2)^k$, since $\lfloor 2^k U \rfloor$ will be odd half the time. The left hand side is also $(1/2)^k$, since the event $\{B_1 = b_1, \ldots, B_k = b_k\}$ is exactly the event that the first $k$ binary digits of $U$ are $b_1, \ldots, b_k$, and the set of possible values of $U$ for which that occurs form an interval of length $(1/2)^k$.

2. Note that $s_n^2 = \sum EX_i^2 = 1 + 1 + 2 + \cdots + 2^{n-2} = 2^{n-1}$. This means that

$$X_n/s_n \sim N(0, \tfrac{2^{n-2}}{s_n^2}) = N(0, \tfrac{1}{2}),$$

so that $P(|X_n|/s_n > \varepsilon)$ is constant in $n$, so $P(|X_n|/s_n > \varepsilon) \not\to 0$. Thus,

$$\sum_{k=1}^{n} \int_{|X_n| > \varepsilon s_n} X_n^2 \, dP \geq \int_{|X_n|/s_n > \varepsilon} X_n^2 \, dP \geq \varepsilon^2 P(|X_n|/s_n \geq \varepsilon) \not\to 0$$

so the Lindberg condtion doesn't hold.

Note that, if $Z_1 \sim N(0, \sigma_1^2)$ and $Z_2 \sim N(0, \sigma_2^2)$, then $Z_1 + Z_2 \sim N(0, \sigma_1^2 + \sigma_2^2)$. This is because the c.f. for $N(0, \sigma^2)$ is $\exp(-t^2\sigma^2/2)$, so the c.f. for $Z_1 + Z_2$ is

$$\exp(-t^2\sigma_1^2/2) \cdot \exp(-t^2\sigma_2^2/2) = \exp(-t^2(\sigma_1^2 + \sigma_2^2)/2)$$

This means that

$$S_n \sim N(0, 1 + 1 + 2 + \cdots + 2^{n-2}) = N(0, 2^{n-1})$$

so $S_n/s_n \sim N(0, 1)$. So, not only does $S_n/s_n \to N(0, 1)$ in distribution, but in fact each $S_n/s_n$ is *equal* to $N(0, 1)$ in distribution!

3. Recall Kronecker's Lemma: if $a_n \nearrow \infty$, and $\sum_1^\infty \frac{x_n}{a_n}$ converges, then $\frac{1}{a_n}\sum_1^n x_k \to 0$. Thus, it suffices to show that $\sum_1^\infty \frac{X_n^2}{n^2}$ converges. To do this, we use the Kolmogorov 3-series test. Let $Y_n = \frac{X_n^2}{n^2}\mathbf{1}\left(\frac{X_n^2}{n^2} \le 1\right) = \frac{X_n^2}{n^2}\mathbf{1}(X_n \le n)$. We must check that

(i) $\sum_1^\infty P(\frac{X_n^2}{n^2} > 1) < \infty$ (ii) $\sum_1^\infty EY_n$ converges (iii) $\sum_1^\infty \text{Var } Y_n < \infty$

(i) This is true since $EX_1 < \infty$, which holds if and only if $\sum_1^\infty P(X_k > k) < \infty$, which is the same as $\sum_1^\infty P(X_k^2/k^2 > 1) < \infty$.

(ii) The below computation uses many clever tricks. For the first equality, we are using $X_1 1_{X_1 \le n} = \sum_1^n X_1 1_{\{k-1<X_1\le k\}}$. For the second, we use Fubini's, vaild since all summands are positive. For the third, we bound $\sum_{n=k}^\infty n^{-2} \le \int_k^\infty x^{-2}\,dx = \frac{1}{k}$. For the fourth, note that $X_1^2 1_{(k-1,k]} \le kX_1 1_{(k-1,k]}$.

$$\begin{aligned}\sum_{n=1}^\infty E(\tfrac{X_n^2}{n^2}; |X| \le n) = \sum_{n=1}^\infty \sum_{k=1}^n \frac{1}{n^2} E(X_1^2 1_{\{k-1<X\le k\}}) &= \sum_{k=1}^\infty E(X_1^2; 1_{(k-1,k]}) \sum_{n=k}^\infty \frac{1}{n^2} \\ &\le \sum_{k=1}^\infty E(X_1^2; 1_{(k-1,k]}) \frac{1}{k} \\ &\le \sum_{k=1}^\infty E(X_1; 1_{(k-1,k]}) \\ &= EX_1 < \infty\end{aligned}$$

(iii) To show $\sum \text{Var } Y_n < \infty$, we show $\sum EY_n^2 < \infty$, using the same tricks.

$$\begin{aligned}\sum_{n=1}^\infty E(\tfrac{X_n^4}{n^4}; |X| \le n) = \sum_{n=1}^\infty \sum_{k=1}^n n^{-4} E(X_1^4 1_{(k-1,k]}) &= \sum_{k=1}^\infty E(X_1^4 1_{(k-1,k]}) \sum_{n=k}^\infty n^{-4} \\ &\le \sum_{k=1}^\infty E(X_1^4 1_{(k-1,k]}) \frac{3}{k^3} \\ &\le 3\sum_{k=1}^\infty E(X_1 1_{(k-1,k]}) = 3EX_1 < \infty\end{aligned}$$

This completes the proof!

# 2004 Fall

**Lemma** If $y_n$ is a sequence of real numbers, and every subsequence has a further subsequence converging to $y$, then $y_n \to y$.

*Proof.* Suppose $y_n \not\to y$. Then there is an $\varepsilon > 0$, and a subsequence $y_{n(k)}$ where $|y - y_{n(k)}| > \varepsilon$. This means no subsequence of $y_{n(k)}$ can approach $y$, contradicting the assumtion. □

1. (a) $\implies$ (b) We are given that $X_n \to 0$ in probability, which implies every subsequence $X_{n(k)}$ has a further subsequence $X_{n(k_m)}$ converging almost surely to 0. Since $f$ is continuous, this means $f(X_{n(k_m)}) \to f(0)$ a.s, and since $f$ is bounded, by DCT, $Ef(X_{n(k_m)}) \to f(0)$. We have shown every subsequence of $Ef(X_n)$ has a further subsequence converging to $f(0)$: by the above lemma, this implies $Ef(X_n) \to f(0)$.

   (b) $\implies$ (a) Given $\varepsilon > 0$, let $h(x) = (|x|/\varepsilon) \wedge 1 = \min(|x|/\varepsilon, 1)$. The idea is that $h$ is bounded, continuous, and $1_{|x| \geq \varepsilon} \leq h(x)$. Thus,

   $$P(|X_n| > \varepsilon) = E1_{|X_n| > \varepsilon} \leq Eh(X_n)$$

   So letting $n \to \infty$, we get

   $$\limsup_n P(|X_n| > \varepsilon) \leq \lim_n Eh(X_n) = h(0) = 0.$$

2. (a) The c.f. of $S_n/n$ is always $\varphi(t/n)^n$, so in this case, $(e^{-|t/n|})^n = e^{-|t|}$.

   (b) The law of large numbers does not hold since $E|X_1| = \infty$.

   Also, the law of large numbers would imply $S_n/n \to \mu$, but the previous result, and the continuity theorem, show that $S_n/n \to X_1$ in distribution.

3. (a) We have that $P(X_n \geq \log n) = e^{-\log n} = n^{-1}$, and $\sum n^{-1} = \infty$, so by Borel-Cantelli, $P(X_n/\log n \geq 1 \text{ i.o.}) = 1$, which proves $P(\limsup_n X_n/\log n \geq 1) = 1$. For any $\varepsilon > 0$, we have $P(X_n/\log n > 1+\varepsilon) = n^{-(1+\varepsilon)}$, which is now summable, so again by Borel Cantelli, $P(X_n/\log n > 1+\varepsilon \text{ i.o.}) = 0$. This shows

$$\limsup_n X_n/\log n \leq 1+\varepsilon \qquad \text{a.s.}$$

Letting $L = \limsup X_n/\log n$, since $\{L \leq 1\} = \bigcap_{k\geq 1}\{L \leq 1+\frac{1}{k}\}$, the above implies $L \leq 1$ a.s, so we have shown $L = 1$ a.s.

(b) We first show:

**Lemma** Given a (non random) sequence $a_1, a_2, \ldots$, where $a_n \geq 0$, and $\limsup_n \frac{a_n}{\log n} = 1$, let $m_n = \max_{1\leq k\leq n} a_k$. Then $\limsup_n \frac{m_n}{\log n} \leq 1$.

*Proof.* Given $\varepsilon > 0$, choose $K$ so $n > K$ implies $\frac{a_n}{\log n} < 1+\varepsilon$. Then

$$\frac{m_n}{\log n} \leq \frac{m_K}{\log n} + \max_{K+1\leq j\leq n} \frac{a_j}{\log j} \leq \frac{m_K}{\log n} + 1 + \varepsilon$$

Letting $n \to \infty$, we have $m_K/\log n \to 0$, so the above shows $\limsup \frac{m_n}{\log n} \leq 1+\varepsilon$. □

Thus, using $\limsup \frac{X_n}{\log n} = 1$ a.s. and the Lemma proves $\limsup \frac{M_n}{\log n} \leq 1$ a.s. Secondly, we show $\liminf \frac{M_n}{\log n} \geq 1$ a.s. For any $\varepsilon > 0$, we have

$$P(M_n/\log n < 1-\varepsilon) = P(X_i \leq (1-\varepsilon)\log n)^n = (1-e^{-(1-\varepsilon)\log n})^n = \left(1 - \frac{n^\varepsilon}{n}\right)^n \leq e^{-n^\varepsilon}$$

Since $\sum(\frac{1}{e^\varepsilon})^n < \infty$, this implies that $P(M_n/\log n < 1-\varepsilon \text{ i.o.}) = 0$. Thus, almost surely we will have $M_n/\log n$ is eventually greater than $1-\varepsilon$, so $\liminf M_n/\log n \geq 1-\varepsilon$ a.s, so $\liminf M_n/\log n \geq 1$ a.s.

# 2006 Spring

1. (a) The condition is $p_n \to 0$, since $P(|X_n| > \varepsilon) = P(X_n = 1) = p_n$, so $X_n \to 0$ in probability iff $p_n \to 0$.

   (b) The condition is $\sum p_n < \infty$, since

$$X_n \to 0 \text{ a.s.} \iff P(X_n = 1 \text{ i.o.}) = 0 \iff \sum P(X_n = 1) < \infty$$

   with the last $\iff$ following from Borel-Cantelli.

2. (a) Note that $EI_1 = P(Y_1 \le f(X_1)) = J$ (since $(X_1, Y_1)$ is uniform over the unit square, and the area for which $y \le f(x)$ is $J$), and $Ef(X_1) = \int_0^1 f(x)\,dx = J$. Thus, by SLLN, $\frac{1}{n}\sum I_i$ and $\frac{1}{n}f(X_i)$ both converge to $J$ a.s.

   (b) Since $J_n - J = \frac{1}{n}\sum_1^n (I_i - J)$, and each $I_i - J$ has mean 0, we have

$$E(J_n - J)^2 = \text{Var }(J_n - J) = \frac{1}{n^2}\sum_1^n \text{Var }(I_i - J) = \frac{n}{n^2}\text{Var }(I_1) = \frac{1}{n}(EI_1^2 - (EI_1)^2) = \frac{1}{n}(J - J^2)$$

   The last step follows since $I_i^2 = I_i$ (it is always 0 or 1).
   In the same vein,

$$E(J_n^* - J) = \frac{n}{n^2}\sum \text{Var } f(X_i) = \frac{1}{n}(Ef(X_i)^2 - (Ef(X_i))^2) = \frac{1}{n}\left(\int_0^1 f(x)^2\,dx - J^2\right)$$

   Thus, in order to prove $E(J_n^* - J) \le E(J_n - J)^2$, it suffices to prove $\int_0^1 f(x)^2\,dx \le J = \int_0^1 f(x)\,dx$, which is true since $f(x) \in [0, 1]$, so that $f(x)^2 \le f(x)$. In the previous inequality, equality only holds when $f(x)$ is 0 or 1, and the only two continuous functions which are always 0 or 1 are $f(x) = 0$ and $f(x) = 1$.

   (c) Note this distribution of $\frac{\sqrt{n}(J_n - J)}{\sigma}$ is approximately the standard normal, for large $n$, where $\sigma = \text{Var } I_i = J - J^2$. Thus,

$$P\left(\frac{\sqrt{n}|J_n - J|}{\sigma} < 3\right) \approx 0.95$$

$$P(|J_n - J| < 3(J - J^2)/\sqrt{n}) \approx 0.95$$

   Solving $3(J - J^2)/\sqrt{n} = 0.01$ for $n$, we get $n = 90,000 \cdot (J - J^2) \le 90,000$, so choosing $n = 90,000$ should sort of work.

3. (a) $X_n \to X$ in probability if, for all $\varepsilon > 0$, $P(|X_n - X| > \varepsilon) \to 0$ as $n \to \infty$.

$X_n \to X$ in distribution if, for any $x$ for which the function $F_X(x) = P(X \leq x)$ is continuous at $x$, we have $P(X_n \leq x) \to P(X \leq x)$ as $n \to \infty$.

(b) It does not converge in probability, since $P(|X_n - Y| > \varepsilon) = P(|X - (1 - X)| > \varepsilon) = P(|2X - 1| > \varepsilon) = 1 \not\to 0$.

It does converge in distribution, since $P(X_n \leq x) = P(Y \leq x)$ for all $n$.

(c) It is a well known fact that convergence in probability implies that in distribution. To see this, suppose $Z_n \to Z$ in probability, and let $z$ be a continuity point of $F_Z(z) = P(Z \leq z)$. Using the fact that

$$\{Z_n \leq z\} \subset \{Z \leq z + \varepsilon\} \cup \{|Z - Z_n| > \varepsilon\}$$

we have

$$P(Z_n \leq z) \leq P(Z \leq Z + \varepsilon) + P(|Z - Z_n| > \varepsilon)$$

Using $\{Z \leq z - \varepsilon\} \subset \{Z_n \leq z\} \cup \{|Z_n - Z| > \varepsilon\}$, we also have

$$P(Z_n \leq z) \geq P(Z \leq z - \varepsilon) - P(|Z - Z_n| > \varepsilon)$$

letting $n \to \infty$, the above two inequalities imply

$$P(Z \leq z - \varepsilon) = F_Z(z - \varepsilon) \leq \lim_{n\to\infty} P(Z_n \leq z) \leq F_Z(z + \varepsilon) = P(Z \leq z + \varepsilon)$$

then letting $\varepsilon \to 0$ gives $\lim_{n\to\infty} P(Z_n \leq z) = F_Z(z)$.

Since $Y_n \to Y$ in probability, we have $Y_n \to Y$ in distribution. But $X$ has the same distribution as $Y$, and convergence in distribution only depends on distrubtion, proving that $Y_n \to X$ in distribution as well.

# 2007 Spring

1. (i) For any $\varepsilon$, $\{X_n/n > \varepsilon\} = \{X_n > n\varepsilon\} \searrow \{X_n = \infty\}$, so $P(X_n/n > \varepsilon) \searrow P(X_n = \infty) = 0$.

   (ii) Using the inequalities

$$\sum_{n\geq 1} P(|X_n|/\varepsilon > n) \leq E|X_1/\varepsilon| \leq \sum_{n\geq 0} P(|X_n|/\varepsilon > n)$$

   We have

$$\begin{aligned} E|X_1| < \infty &\iff \sum P(|X_n/n| > \varepsilon) < \infty \\ &\iff P(|X_n/n| > \varepsilon \text{ i.o.}) = 0 \\ &\iff X_n/n \to 0 \text{ a.s.} \end{aligned}$$

   The second $\iff$ is Borel-Cantelli, and the third follows by intersecting $\{|X_n/n| > \varepsilon_k \text{ i.o.}\}$ for $\varepsilon_k \searrow 0$.

   (iii) Using $X_n/\sqrt{n} \to 0 \iff X_n^2/n \to 0$ and the previous problem, the desired condtion is $EX_1^2 < \infty$.

2. (i) We have, using Fubini's theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\phi(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ikt}\sum_{x\in\mathbb{Z}} e^{itx}P(X=x)\,dt = \sum_{x\in\mathbb{Z}} P(X=x)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it(x-k)}\,dt$$

   Consider $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it(x-k)}\,dt$. When $x = k$, this is clearly 1. When $x \neq k$, breaking the complex exponential into its sinusoidal real and imaginary parts shows that the integral is zero. Thus, the only positive contribution to the sum is when $X = k$, so the sum is $P(X = k)$.

   (ii) The c.f. for $S_n$ is $\phi_X(t)^n$, so

$$P(S_n = k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itk}\phi_X(t)^n\,dt$$

3. (i) ($\Longleftarrow$) We have, for $M > \sup|\mu_n|$,

$$P(|X_n| > M) \leq P(||X_n - \mu_n| > M - |\mu_n|) \leq \frac{\sigma_n^2}{(M - |\mu_n|)^2} \leq \frac{\sup \sigma_n^2}{(M - \sup|\mu_n|)^2}$$

so

$$\sup_n P(|X_n| > M) \leq \frac{\sup \sigma_n^2}{(M - \sup|\mu_n|)^2} \to 0 \qquad \text{as } M \to \infty$$

($\Longrightarrow$) Suppose $\sup|\mu_n| = \infty$. Then for any $M$, there will be some $X_N$ for which $|\mu_N| > M$, implying by symmetry of the normal distribution that $P(|X_N| > M) > \frac{1}{2}$, meaning $\limsup_n P(|X_n| > M) \geq \frac{1}{2} \not\to 0$.

Suppose $\sup|\mu_n| = C < \infty$, but $\sup \sigma_n = \infty$. Recall that for a normal distrubtion, $P(|X_n - \mu_n| > \sigma_n) \approx .32$. For any $M$, there will be some $X_N$ for which $\sigma_N > M + C$, so

$$\begin{aligned}\limsup_n P(|X_n| > M) &\geq P(|X_N| > M) \\ &\geq P(|X_N - \mu_N| > M + |\mu_N|) \\ &\geq P(|X_N - \mu_N| > \sigma_N) > 0.3 \not\to 0\end{aligned}$$

(ii) ($\Longleftarrow$) If $\mu_n \to \mu$ and $\sigma_n \to \sigma$, then $e^{i\mu_n t} \to e^{i\mu t}$ and $e^{-t^2\sigma_n^2/e} \to e^{-t^2\sigma^2/2}$ pointwise, so $e^{i\mu_n t}e^{-t^2\sigma_n^2/2} \to e^{i\mu t}e^{-t^2\sigma^2/2}$. Note that $e^{i\mu_n t}e^{-t^2\sigma_n^2/2}$ is the c.f. of $X_n$. Since the limit function is continuous at zero, this implies $X_n \to$ some $X$ in distribution, by the continuity theorem.

($\implies$) Suppose $X_n \to X$ weakly. This implies the c.f.'s of $X_n$ converge pointwise, so $e^{i\mu_n t} e^{-t^2\sigma_n^2/2} \to \varphi(t)$. Taking magnitudes,

$$|e^{i\mu_n t} e^{-t^2\sigma_n^2/2}| = e^{-t^2\sigma_n^2/2} \to |\varphi(t)|,$$

Since $X_n \to X$ weakly implies the $X_n$ are tight, by part (i), $\sup \sigma_n < \infty$, meaning we must have $|\phi(t)| > 0$. Setting $t = 1$, we get $\sigma_n \to \sqrt{-2\log|\varphi(1)|} = \sigma$.

We now have

$$e^{i\mu_n t} = \varphi(t) e^{t^2\sigma_n^2/2} \to \varphi(t) e^{t^2\sigma^2/2} = \rho(t), \tag{1}$$

where $|\rho(t)| = |e^{i\mu_n t}| = 1$. From part $(i)$, we know $\sup|\mu_n| < \infty$, so $\{\mu_n\}_{n\geq 0}$ has at least one accumulation point. When $t = 1$ in (1), $e^{i\mu_n} \to \rho(1)$ implies that all accumulation points of $\{\mu_n\}_{n\geq 0}$ are of the form $\arg\rho(1) + 2\pi k$.

Suppose, by way of contradiction there were at least two accumulation points. This would imply there were subsequences $\mu_{h(n)}$ and $\mu_{\ell(n)}$ so that

$$\mu_{h(n)} \to \arg\rho(1) + 2\pi k_1 \qquad \text{and} \qquad u_{\ell(n)} \to \arg\rho(1) + 2\pi k_2$$

where $k_1 \neq k_2$ are integers. Now, setting $t = 2\pi$ in (1), so that $e^{i2\pi\mu_n} \to \rho(2\pi)$, we can find further subsquences $h'(n)$ of $h(n)$ and $\ell'(n)$ of $\ell(n)$ so that

$$\mu_{h'(n)} \to \frac{1}{2\pi}\arg\rho(2\pi) + k_1' \qquad \text{and} \qquad u_{\ell(n)} \to \frac{1}{2\pi}\arg\rho(2\pi) + k_2'$$

for some $k_1', k_2' \in \mathbb{Z}$. Setting corresponding limits of subsequences equal to each other, we get

$$\arg\rho(1) + 2\pi k_1 = \frac{1}{2\pi}\arg\rho(1) + k_1'$$
$$\arg\rho(1) + 2\pi k_2 = \frac{1}{2\pi}\arg\rho(1) + k_2'$$

so that

$$2\pi = \frac{k_1' - k_2'}{k_1 - k_2}$$

contradicting the irrationality of $\pi$.

Thus, there is only one acculumation point, $\mu$, of $\{\mu_n\}_{n\geq 0}$. Since $\{\mu_n\}$ is bounded, every subsequence of $\mu_n$ has a further convergent subsequence. Since these subsussequences always converge to $\mu$, it follows $\mu_n \to \mu$.

# 2007 Fall

1. Let $A_n$ be the event $\{L_n > \log n + \theta \log\log n\}$. Then

$$P(A_n) = \frac{1}{2}^{\log n + \theta \log\log n} = \frac{1}{n(\log n)^\theta}$$

Since $\sum P(A_n) < \infty$ (use the integral test), by Borel-Cantelli, $P(A_n \text{ i.o.}) = 0$.

2. The continuous form of the inversion formula implies, since $\int |\phi_n| < \infty$, that $X_n$ have densities for $n < \infty$, given by $f_n(x) = \frac{1}{2\pi} \int e^{-itx} \varphi_n(t)\,dt$ (for a proof of this fact, see Spring 1997, problem 3). Furthermore, $|\varphi_n(x)| \le g(x)$ and $\varphi_n(x) \to \varphi_\infty(x)$ implies $|\varphi_\infty(x)| \le g(x)$, so we also have that $\varphi_\infty$ is integrable, implying the density $f_\infty$ exists, and is given by a similar formula.

We have that

$$\begin{aligned} \sup_{x\in\mathbb{R}} |f_n(x) - f(x)| &= \sup_{x\in\mathbb{R}} \left| \int e^{-itx}\varphi_n(t)\,dt - \int e^{-itx}\varphi_\infty(t)\,dt \right| \\ &\le \sup_x \int |e^{-itx}(\varphi_n(t) - \varphi_\infty(t))|\,dt \\ &= \int |\varphi_n(t) - \varphi_\infty(t)|\,dt \end{aligned}$$

Since $|\varphi_n - \varphi| \le 2g \in L_1$, and $|\varphi_n(t) - \varphi(t)| \to 0$, by the dominated convergence theorem,

$$\limsup_{n\to\infty} \left( \sup_x |f_n(x) - f(x)| \right) \le \lim_{n\to\infty} \int |\varphi_n(t) - \varphi_\infty(t)|\,dt = 0$$

proving $\sup_x |f_n(x) - f(x)| \to 0$, so $f_n \to f$ uniformly.

3. Choose $A_0$ so that $\sup_n \frac{E(X_n^2; |X_n|>A)}{EX_n^2} < \frac{1}{2}$ when $A > A_0$. Then for these $A$,

$$EX_n^2 = E(X_n^2; |X_n| \le A) + E(X_n^2; |X_n| > A) \le A^2 + \frac{1}{2}EX_n^2$$

so rearranging, we get

$$\frac{EX_n^2}{A^2} \le 2$$

Thus, using Chebychev's inequality, for $A > A_0$,

$$\begin{aligned}\sup_n P(|X_n| > A) &\le \sup_n \frac{E(X_n^2; |X_n| > A)}{A^2} \\ &= \sup_n \frac{E(X_n^2; |X_n| > A)}{EX_n^2} \cdot \frac{EX_n^2}{A^2} \\ &\le \sup_n \frac{E(X_n^2; |X_n| > A)}{EX_n^2} \cdot 2\end{aligned}$$

Letting $A \to \infty$, the right hand side approaches 0 (by assumption), proving

$$\lim_{A\to\infty} \sup_n P(|X_n| > A) = 0,$$

which means the $X_n$, and therefore their distributions $F_n$, are tight.

4. (a) Take expectations of both sides of the inequality $\varphi(t)1_{Y>t} \le \varphi(Y)$.

   (b) Using (a), with $\varphi(t) = e^{\lambda t}$,

$$P(S_n > nx) \le \frac{Ee^{\lambda S_n}}{e^{\lambda nx}}$$

   Since $e^{\lambda S_n} = e^{\lambda X_1} \times \cdots \times e^{\lambda X_n}$, and each factor is independent, with the same expectation, we have

$$P(S_n > nx) \le \frac{(Ee^{\lambda X_1})^n}{e^{\lambda nx}} = \left(\frac{M(\lambda)}{e^{\lambda x}}\right)^n$$

   Taking logs,

$$\log P(S_n > nx) \le n(\log M(\lambda) - \lambda x)$$

   so rearranging and taking the inf over $\lambda > 0$,

$$\frac{1}{n}\log P(S_n > nx) \le \inf_{\lambda>0} -(\lambda x - M(\lambda)) = -\sup_{\lambda>0}(\lambda x - M(\lambda)) = -I(x)$$

# 2008 Spring

1. (a) Let $S_n = X_1 + \cdots + X_n$. We have

$$\varphi_\varepsilon = Ee^{itS_\varepsilon} = \sum_{n\geq 0} E[e^{itS_\varepsilon}|N_\varepsilon = n]P(N_\varepsilon = n) = \sum_{n\geq 0} E[e^{itS_n}] \cdot \frac{e^{-\lambda/\varepsilon^2}(\lambda/\varepsilon^2)^n}{n!}$$

Note that $E[e^{itS_n}] = (\cos\varepsilon t)^n$, since $\cos\varepsilon t$ is the c.f. for $X_n$, and adding random variable makes their c.f's multiply. Thus,

$$\varphi_\varepsilon = e^{-\lambda/\varepsilon^2}\sum_{n\geq 0}\frac{(\lambda/\varepsilon^2\cdot\cos\varepsilon t)^n}{n!} = e^{-\lambda/\varepsilon^2}e^{\lambda/\varepsilon^2\cdot\cos\varepsilon t} = e^{\lambda(\cos\varepsilon t-1)/\varepsilon^2}$$

(b) As $\varepsilon \to 0$, using, L'Hoptial's rule twice, $\frac{\cos\varepsilon t-1}{\varepsilon^2} \to \frac{-t\sin\varepsilon t}{2\varepsilon} \to \frac{-t^2}{2}$, so $\varphi_\varepsilon \to e^{-\lambda t^2/2}$. This is the c.f. of $N(0,\lambda)$, proving $\varphi_\varepsilon$ converges in distribution to $N(0,\lambda)$.

2. Let $x$ be a continuity point of $F_X$, and $\varepsilon > 0$. Since $\{X_n + Y_n \leq x\} \subset \{X_n \leq x+\varepsilon\} \cup \{|Y_n| > \varepsilon\}$ and $\{X_n \leq x-\varepsilon\} \subset \{X_n + Y_n \leq x\} \cup \{|Y_n| > \varepsilon\}$, we have

$$P(X_n \leq x-\varepsilon) - P(|Y_n| > \varepsilon) \leq P(X_n + Y_n \leq x) \leq P(X_n \leq x+\varepsilon) + P(|Y_n| > \varepsilon)$$

Assuming $x \pm \varepsilon$ is also a contiuity point of $F_X$, letting $n \to \infty$ above shows

$$F(x-\varepsilon) \leq \liminf_n P(X_n + Y_n \leq x) \leq \limsup_n P(X_n + Y_n \leq x) \leq F(x+\varepsilon)$$

and letting $\varepsilon \to 0$ shows $P(X_n + Y_n \leq x) \to F(x)$, completing the proof.

3. (a) Note that $V_n$ can be written as a function of the $U_i$ for which $a_{n-i} \neq 0$, and $V_{n+1}$ as a function of the $U_i$ for which $a_{n+1-i} \neq 0$. This means that $V_n$ and $V_{n+1}$ are functions of disjoint sets of independent variables, since for all $i$, $a_{n-i}a_{n-i+1} = 0$, so at least one of $a_{n-i}$ and $a_{n-i+1}$ is zero, meaning there is no $U_i$ which both $V_n$ and $V_{n+1}$ both depend on. Since $V_n, V_{n+1}$ are functions of independent vectors, they are independent.

   (b) Note that $V_n \sim N(0, a_0^2 + \cdots + a_{n-1}^2)$. This is because, when $X \sim N(0, \sigma^2)$ and $Y \sim N(0, \rho^2)$, then $X + Y \sim N(0, \sigma^2 + \rho^2)$, which can be proven by looking at characteristic functions.

   Let $A_n = \sum_0^{n-1} a_i^2$, and $A = \sum_0^\infty a_i^2$. Then $V_n \sim N(0, a_1^2 + \cdots + a_n^2)$, so $V_n/\sqrt{A_n}$ is standard normal, so (for large enough $x$),

$$P(V_n \geq x\sqrt{A}) \leq P(V_n/\sqrt{A_n} \geq x) \leq \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{x^2}{2}\right) \leq \exp\left(-\frac{x^2}{2}\right)$$

   Letting $x = \sqrt{2(1+\varepsilon)\log n}$,

$$P\left(\frac{V_n}{\sqrt{\log n}} \geq \sqrt{2(1+\varepsilon)A}\right) \leq \exp\left(-\frac{\left(\sqrt{2(1+\varepsilon)\log n}\right)^2}{2}\right) = n^{-1-\varepsilon}$$

   Since $\sum n^{-1-\varepsilon} < \infty$, Borel-Cantelli implies $P\left(\frac{V_n}{\sqrt{\log n}} \geq \sqrt{2(1+\varepsilon)A} \text{ i.o.}\right) = 0$. This means that $\limsup \frac{V_n}{\sqrt{\log n}} \leq \sqrt{2(1+\varepsilon)A}$ a.s. Letting $\varepsilon \to 0$ proves that $\limsup \frac{V_n}{\sqrt{\log n}} \leq \sqrt{2A}$ a.s.

4. The appropriate choice of $t$ is $t = \frac{1}{c}$. We have

$$P(X \geq c) \leq P((X+\tfrac{1}{c})^2 \geq (c+\tfrac{1}{c})^2) \leq \frac{E(X+\frac{1}{c})^2}{(c+\frac{1}{c})^2} = \frac{EX^2 + \frac{2}{c}EX + \frac{1}{c^2}}{(c+\frac{1}{c})^2} = \frac{1+\frac{1}{c^2}}{(c+\frac{1}{c})^2} = \frac{1}{c^2+1}$$

This solution of course doesn't help show you how to approach the problem correctly. Assuming you didn't know what $t$ was, you would have

$$P(X \geq c) \leq P((X+t)^2 \geq (c+t)^2) \leq \frac{E(X+t)^2}{(c+t)^2} = \frac{1+t^2}{(c+t)^2}$$

You want to find a $t$ so that $\frac{1+t^2}{(c+t)^2} \leq \frac{1}{c^2+1}$. Cross multiplying and simplyifying that inequality is how you find $t = \frac{1}{c}$.

## 2008 Fall

1. It does follows that $E \log X_n \to E \log X$. Since $X_n \to X$, in distribution, there exist variables $Y_n, Y$ with the same distribution as $X_n, X$, and where $Y_n \to Y$ almost surely.

   By Fatou's Lemma, we have that $\liminf E \log Y_n \geq E \log Y$.

   Since $EY_n \to c$, we must have that $EY_n \leq K$ for some constant $K$ and large enough $n$. Given $\varepsilon > 0$, choose $M$ so $x > M$ implies $\frac{\log y}{y} \leq \frac{\varepsilon}{K}$ and so $P(Y = M) = 0$. Then

   $$E(\log Y_n 1_{Y_n > M}) = E\left(\frac{\log Y_n}{Y_n} \cdot Y_n 1_{Y_n > M}\right) \leq E\left(\frac{\varepsilon}{K} \cdot Y_n 1_{Y_n > M}\right) \leq \frac{\varepsilon}{K} EY_n \leq \varepsilon$$

   so

   $$E \log Y_n \leq E(\log Y_n 1_{Y_n \leq M}) + E(\log Y_n 1_{Y_n > M}) \leq E(\log Y_n 1_{Y_n \leq M}) + \varepsilon$$

   Taking limits above, we get

   $$\limsup_n E \log Y_n \leq \varepsilon + \limsup_n E(\log Y_n 1_{Y_n \leq M}) \overset{DCT}{=} \varepsilon + E(\log Y 1_{Y \leq M}) \leq \varepsilon + E \log Y$$

   To justify the middle equality, realize that $Y_n \to Y$ a.s. and $P(Y = M) = 0$ implies $\log Y_n 1_{Y_n \leq M} \to \log Y 1_{Y \leq M}$ a.s, and the $\log Y_n 1_{Y_n \leq M}$ are dominated by $\log M$.

   Letting $\varepsilon \to 0$ above, we have shown that

   $$E \log Y \leq \liminf E \log Y_n \leq \limsup_n E \log Y_n \leq E \log Y$$

   which implies $E \log X_n = E \log Y_n \to E \log Y = E \log X$.

2. ☹ First, we get an upper lower bound on $P(X_n \geq \alpha)$:

   $$P(X_n \geq \alpha) = \sum_{k=\alpha}^{\infty} \frac{\lambda^k}{k!}$$

   Let $a_n$ be the integer closest to $\frac{\log n}{\log \log n}$, so $a_n = \frac{\log n}{\log \log n}(1 + o(1))$. Using Sterling's approximation, which says that $\log(k!) = k \log k + O(k)$, and the fact that $O(a_n)$ implies $o(\log n)$,

   $$\begin{aligned} P(X_n = a_n) &= \frac{e^{-\lambda} e^{a_n \log \lambda}}{a_n!} \\ &= \exp(-a_n \log a_n + a_n(1 + \log \lambda) + o(a_n)) \\ &= \exp\left(-\frac{\log n}{\log \log n} \cdot (\log \log n - \log \log \log n) + o(\log n)\right) \\ &= \exp(-\log n + o(\log n)) = n^{-1+o(1)} \end{aligned}$$

   The above computation is useless, since $\sum n^{-1+o(1)}$ can be either finite or infinite.

3. (a) The special property is that $\varphi$ will be real. If $X$ and $-X$ have the same distrubtion, then

$$Ee^{itX} = E\cos tX + iE\sin tX$$

But $tX$ is symmetrically positive and negative, and $\sin(tx)$ is an odd function, so $E\sin(tX) = 0$.

Suppose $Ee^{itX}$ is real. Using the inversion formula, we have, for any $a < b$,

$$P(X \in (a,b)) + \frac{1}{2}P(X \in \{a,b\}) = \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it}\varphi(t)\,dt$$

Both sides are real, so taking the conjugate of the right preserves equality, resulting in

$$\begin{aligned}
P(X \in (a,b)) + \frac{1}{2}P(X \in \{a,b\}) &= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-it(-a)} - e^{-it(-b)}}{-it}\varphi(t)\,dt \\
&= \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-it(-b)} - e^{-it(-a)}}{it}\varphi(t)\,dt \\
&= P(X \in (-b,-a)) + \frac{1}{2}P(X \in \{-b,-a\}) \\
&= P(-X \in (a,b)) + \frac{1}{2}P(-X \in \{a,b\})
\end{aligned}$$

This holds for all $a, b$, proving $X$ and $-X$ have the same distribution.

(b) This is given by $\phi(t/n)^n$.

(c) Since $\phi'(0) = 0$, we have that

$$\lim_{n\to\infty} \frac{\phi(t/n) - 1}{t/n} = 0$$

Furthermore, from calculus it is true that $\frac{\log(1+x)}{x} \to 1$ as $x \to 0$, implying $\frac{\log\phi(t/n)}{\phi(t/n)-1} \to 1$ as $n \to \infty$. Multiplying these two limits, we get

$$\lim_{n\to\infty} \frac{\log\phi(t/n)}{t/n} = 0$$

Taking exp of both sides, we get $\phi(t/n)^n \to 1$. But $\phi(t/n)^n$ is the c.f. for $S_n/n$, and 1 is the c.f. for 0, so the continuity theorem implies $S_n/n \to 0$ weakly. Finally, one can prove that converging weakly to a constant implies convergence in probability as well, so that $S_n/n \to 0$ in probability.

(d) We have

$$E|X| = 2c\int_4^\infty x \cdot \frac{1}{x^2\log x}\,dx = 2c(\lim_{n\to\infty} \log\log n - \log\log 4) = \infty$$

(e) Since $X$ is symmetric about 0, we have

$$E\frac{e^{itX}-1}{t} = E\frac{\cos(tX)-1}{t} = 2c\int_4^\infty \frac{\cos(tx)-1}{tx^2\log|x|}\,dx$$

Letting $y = tx$, this becomes

$$E\frac{e^{itX}-1}{t} = 2c\int_4^\infty \frac{\cos(y)-1}{t(y/t)^2\log|y/t|}\,d(y/t) = 2c\int_4^\infty \frac{\cos(y)-1}{y^2\log|y/t|}\,dy$$

Since, for $-1 < t < 1$, it's true that $\frac{\cos(y)-1}{y^2\log|y/t|} \le \frac{\cos(y)-1}{y^2\log|y|} \in L_1(dy)$, the DCT implies

$$\lim_{t\to 0} E\frac{e^{itX}-1}{t} = \lim_{t\to 0} 2c\int_4^\infty \frac{\cos(y)-1}{y^2\log|y/t|}\,dy = 2c\int_4^\infty \lim_{t\to 0}\frac{\cos(y)-1}{y^2\log|y/t|}\,dy = 2c\int_4^\infty 0\,dt = 0$$

proving $\phi'(0) = 0$.

## 2009 Spring

1. The only thing let to prove is when $\mu = \pm\infty$. Assume WLOG $\mu = \infty$. Given $M \in \mathbb{N}$, let $X_n^M = X_n \wedge M$. Note that $E|X_n^M| < \infty$, since $(X_n^M)^+ < M$, and $E(X_n^M)^- = EX_n^- < \infty$ since $EX_n = EX_n^+ - EX_n^- = \infty$. Thus, letting $S_n^M = \sum_1^n X_i^M$, and using the regular SLLN,

$$\liminf_n S_n/n \geq \lim_n S_n^M/n = EX_1^M \qquad \text{a.s.}$$

As $M \to \infty$, by MCT, $EX_1^M \to EX_1 = \infty$. Using this, and the fact that the intersection of countably many almost sure events is almost sure, we have

$$P(S_n/n \to \infty) = P\left(\bigcap_{M=1}^{\infty} \liminf_n S_n/n > EX_1^M\right) = 1$$

so $S_n/n \to \infty = \mu$ a.s.

2. You actually only need to assume $X_n \to 0$ in probability to to this problem.

Since $X_n \to 0$ a.s. implies, for any $k$, that $P(X_n > k^{-2}) \to 0$, we have that for each $k$, there exists an $n_k$ such that $P(X_{n_k} > k^{-2}) < k^{-2}$. By Borel-Cantelli, $P(X_{n_k} > k^{-2} \text{ i.o.}) = 0$, implying that, almost surely, only finitely many $X_{n_k}$ will exceed $k^{-2}$, meaning $\sum_1^\infty X_{n_k}$ will be finite. Thus, almost surely, $\lim_m Y_m = \sum_1^\infty X_{n_k}$ will be finite.

3. ☹

   (a)

   (b)

   (c)

4. The first step is to prove that $|X_n|/n \to 0$ a.s. The fact that $E|X_n| < \infty$ and $X_n$ i.i.d implies $|X_n|/n \to 0$ a.s. has been proven many times in these answers, see for example 1997 Fall, 4(a), or 2007 Spring 1(ii).

   Next, we prove that $\max_{1 \le i \le n} |X_n|/n \to 0$ a.s. This follows from $|X_n|/n \to 0$ a.s, and the following lemma:

   **Lemma**: if $a_n \ge 0$ is a sequence of numbers, and $a_n/n \to 0$, then $\frac{1}{n} \max_{1 \le i \le n} a_n \to 0$.

   *Proof.* Given $\varepsilon > 0$, choose $k$ so $n > k$ implies $a_n/n < \varepsilon$. Then

$$\limsup_n \frac{\max_{1 \le i \le n} a_n}{n} \le \limsup_n \frac{\max(x_1, \ldots, x_k)}{n} + \max_{k \le i \le n} \frac{a_i}{i} \le 0 + \varepsilon$$

   This holds for all $\varepsilon > 0$, so $\frac{\max_{1 \le i \le n} a_n}{n} \to 0$. □

   Finally, let $M_n = \max_{1 \le i \le n} |X_n|$. We have, using what we just showed and the SLLN, that

$$\frac{M_n}{n} \to 0 \qquad \text{a.s.} \qquad \textit{and} \qquad \frac{n}{|S_n|} \to \frac{1}{|EX_1|} \qquad \text{a.s.}$$

   Thus, the product of these sequences converges to the product of the limits a.s, proving that $M_n/|S_n| \to 0$ a.s.

## 2009 Fall

1. See 2011 Fall, problem 2.

2. Note Var $X_n = n^{-2\alpha}$, so $\sum \text{Var } X_n < \infty \iff \alpha > \frac{1}{2}$. It follows, by the "Kolmogorov 1-series theorem", that $\alpha > \frac{1}{2}$ implies $\sum X_n$ converges a.s. When $\alpha \le \frac{1}{2}$, the more subtle 3-series theorem is needed. To check the conditions of this theorem are satisfied, it suffices to realize that, for any $A > 0$, if $Y_n = X_n 1_{\{|X_n| \le A\}}$, then $\sum \text{Var } Y_n = \infty$, which follows since $Y_n = X_n$ for large enough $n$.

   Note $|X_n| = n^{-\alpha}$ with probability 1, so $\sum X_n$ converges exactly when $\alpha > 1$.

3. (i) You can prove, by induction, that $V_{n-1}$ is independent of $U_{n+k}$ for all $k \ge 0$. It holds when $n = 2$, since $V_1 = U_1$ is independent of all other $U_i$. Assuming $V_{n-1}$ is independent of all $U_{n+k}$, the inductive step follows since $V_n$ is a function of $V_{n-1}$ and $U_n$, both of which are independent of $U_{n+1+k}$ for $k \ge 0$.

   (ii) This problem is unfair, since it requires knowledge of conditional expectation, which is not covered until 507b. However, you should be able to prove equation $(*)$, shown in the next part, and this is all you need in order to do part (iii).

   Let $A = \{V_{n-1} \in [0, \frac{1}{2}]\}$ and $B = \{V_{n-1} \in [\frac{1}{2}, 1]\}$. Then

$$\begin{aligned} V_n &= 2V_{n-1}U_n 1_A + (2V_{n-1} - 1)U_n 1_B \\ &= U_n(2V_{n-1}(1_A + 1_B) - 1_B) \\ &= U_n(2V_{n-1} - 1_B) \end{aligned}$$

   Thus, using the independence of $U_n$ and $V_{n-1}$,

$$E[V_n|V_{n-1}] = E[U_n|V_{n-1}] \cdot E[2V_{n-1} - 1_B|V_{n-1}] = E[U_n] \cdot (2V_{n-1} - 1_B) = \frac{1}{2}(2V_{n-1} - 1_B)$$

   (iii) Taking the expectation of the equation $E[V_n|V_{n-1}] = V_{n-1} - 1_B$, we get

$$EV_n = EV_{n-1} - P(V_{n-1} \in [\tfrac{1}{2}, 1]) \qquad (*)$$

   which gives

$$EV_n = EV_1 + \sum_{k=2}^{n} EV_k - EV_{k-1} = \frac{1}{2} - \sum_{k=2}^{n} P(V_{k-1} \in [\tfrac{1}{2}, 1])$$

   Thus, for all $n$, $\sum_{k=2}^{n} P(V_{k-1} \in [\frac{1}{2}, 1]) = \frac{1}{2} - EV_n \le \frac{1}{2}$ (since $V_n \ge 0$), proving in particular that $P(V_{k-1} \in [\frac{1}{2}, 1]) \to 0$ as $k \to \infty$, so $P(V_{k-1} < \frac{1}{2}) \to 1$.

## 2010 Spring

1. (a)

$$P(|\eta_n| > \varepsilon) = P\left(\bigcap_1^n X_i > 0\right) = (1 - e^{-\lambda})^n \to 0 \qquad \text{as } n \to \infty$$

   (b) They are asking if there is a sebsequence converging in $L_1$ to some $\eta$, implying convergence in probability as well. Since every subsequence converges in probability to 0, we would need $\eta = 0$, so $E\eta_{n_k} \to 0$. Since $E\eta_{n_k} = \lambda^{n_k}$, this is only possible when $\lambda < 1$.

2. Suppose $\sup X_n < \infty$ a.s. Then $\{\limsup_n X_n < A\} \nearrow \{\sup_n X_n < \infty\}$ as $A \to \infty$, since if $\sup_n X_n < \infty$, then $\limsup_n X_n$ is certainy less than some $A$. It follows that, for some $A$, $P(\limsup_n X_n < A) > 0$. Since $\limsup X_n < A$ implies $X_n$ will be more than $A$ only finitely many times, this implies $P(X_n > A \text{ i.o.}) < 1$. Finally, by Borel Cantelli, $\sum P(X_n > A) = \infty$ would imply $P(X_n > A \text{ i.o.}) = 1$, we have that $\sum P(X_n > A) < \infty$.

   Suppose that $\sum P(X_n > A) < \infty$. By Borel-Cantelli, $P(X_n > A \text{ i.o.}) = 0$. Thus, with probability 1, the sequence $X_n$ will be greater that $A$ only finitely times, meaning $\sup X_n < \infty$ (since $\sup X_n$ will be $\max(X_{n_1}, \ldots, X_{n_k}, A)$, where $n_1, \ldots, n_k$ are the indices for which $X_n > A$). Thus, $\sup X_n < \infty$ a.s.

3. We first show that $S_{N_n}/\sigma\sqrt{a_n} - S_{a_n}/\sigma\sqrt{a_n} \to 0$ in probability. For any $\varepsilon, \delta > 0$,

$$
\begin{aligned}
P(|S_{N_n} - S_{a_n}|/\sigma\sqrt{a_n} > \varepsilon) &= P(|S_{N_n} - S_{a_n}| > \varepsilon\sqrt{a_n}\sigma) \\
&\le P(\{|S_{N_n} - S_{a_n}| > \varepsilon\sqrt{a_n}\sigma\} \cap \{|N_n - a_n| \le \delta a_n\}) + P(|\tfrac{N_n}{a_n} - 1| > \delta) \to 0 \\
&\le P(\max_{-a_n\delta \le k \le a_n\delta} |S_k - S_{a_n}| > \varepsilon\sqrt{a_n}\sigma)
\end{aligned}
$$

The above could use some explaining. The first $\le$ follows from $P(A) = P(A \cap B) + P(A \cap B^c) \le P(A \cap B) + P(B^c)$, and in this case, $P(B^c) \to 0$ means that $P(B^c) \to 0$ as $n \to \infty$, which follows since $N_n/a_n \to 1$ in probability. Finally, given that the random $N_n$ is at most $a_n\delta$ away from $a_n$, the event $|S_{N_n} - S_{a_n}| > \varepsilon a_n \sigma$ that this holds when $N_n =$ some $k$.

We know use Kolmogorov's maximal inequality, which says that, given $X_1, X_2 \ldots$ independent, $EX_i = 0$, then $P(\max_{1\le k\le n} |S_n| > x) \le x^{-2}\text{Var } S_n$. Thus, applying this to $X_{a_n\delta}, X_{a_n\delta+1}, \ldots$ and $X_{a_n\delta}, X_{a_n\delta-1}, \ldots$, we have

$$
\begin{aligned}
P(|S_{N_n} - S_{a_n}|/\sigma\sqrt{a_n} > \varepsilon) &\le P(\max_{1\le k\le a_n\delta} |S_k - S_{a_n}| > \varepsilon\sqrt{a_n}\sigma) + P(\max_{1\le -k\le a_n\delta} |S_k - S_{a_n}| > \varepsilon\sqrt{a_n}\sigma) \\
&\le \frac{2}{\varepsilon^2\sigma^2 a_n}\text{Var } (S_{a_n+\delta a_n} - S_{a_n}) = \frac{2}{\varepsilon^2\sigma^2 a_n} \cdot \delta a_n \cdot \text{Var } X_i \le \frac{2\delta}{\varepsilon^2}
\end{aligned}
$$

Letting $\delta \to 0$ proves that $P(|S_{N_n} - S_{a_n}|/\sigma\sqrt{a_n} > \varepsilon) \to 0$ as $n \to \infty$, proving

$$\frac{S_{N_n}}{\sigma\sqrt{a_n}} - \frac{S_{a_n}}{\sigma\sqrt{a_n}} \to 0$$

in probability.

Furthermore,

$$S_{a_n}/\sigma\sqrt{a_n} \to N(0,1)$$

in distribution by the CLT. Thus, using Slutsky's to add the last two sequences gives

$$S_{N_n}/\sigma\sqrt{a_n} \to N(0,1)$$

in distribution.

# 2010 Fall

1. It will converge to zero a.s. We have
$$P(|X_n/n| > \varepsilon) \le \frac{EX_n^2}{n^2\varepsilon^2} \le \frac{1}{n^2\varepsilon^2}$$
Thus, by Borel Cantelli, $P(|X_n/n| > \varepsilon \text{ i.o.}) = 0$, so intersecting the events $\{|X_n/n| > \varepsilon_k \text{ i.o.}\}^c$ for some $\varepsilon_k \searrow 0$ givens $X_n/n \to 0$ a.s.

2. Let $Y_{n,i} = \frac{X_i}{\sqrt{n\log n}} \cdot 1_{\{|X_i| < \sqrt{n\log n}\}}$. The Lindberg-Feller CLT has two conditions. For the first, we find
$$\begin{aligned} EY_{n,i}^2 &= \frac{1}{n\log n} \cdot 2\int_1^{\sqrt{n\log n}} y^2 \cdot \frac{1}{y^3}\,dy \\ &= \frac{2}{n\log n} \cdot \log(\sqrt{n\log n}) \\ &= \frac{1}{n} \cdot \left(1 + \frac{\log\log n}{\log n}\right) \end{aligned}$$
Thus, we get that $\sum_{i=1}^n EY_{n,i}^2 = nEY_{n,1}^2 = 1 + \frac{\log\log n}{\log n} \to 1$. Since this limit is nonzero, we can apply Lindeberg, and since it is 1, we have that $\sigma^2 = 1$.

   Secondly, we compute
$$\begin{aligned} E\big(Y_{n,i}^2 \cdot 1_{|Y_{n,i}|>\varepsilon}\big) &= \frac{1}{n\log n} \cdot 2\int_{\varepsilon\sqrt{n\log n}}^{\sqrt{n\log n}} y^2 \cdot \frac{1}{y^3}\,dy \\ &= \frac{1}{n\log n} \cdot 2(\log(\sqrt{n\log n}) - \log(\varepsilon\sqrt{n\log n})) \\ &= \frac{2}{n\log n} \cdot \log(1/\varepsilon) \end{aligned}$$
So, we get $\sum_{i=1}^n EY_{n,i}^2 1_{|Y_{n,i}|>\varepsilon} = n \cdot EY_{n,1}^2 1_{|Y_{n,i}|>\varepsilon} = n \cdot \frac{2}{n\log n} \cdot \log(\frac{1}{\varepsilon}) \to 0$, as required. Thus, we can apply Lindeberg-Feller CLT to obtain
$$\sum_{i=1}^n Y_{n,i} \implies N(0,\sigma^2) = N(0,1)$$
Next, we show that $\sum_1^n \frac{X_i}{\sqrt{n\log n}} - \sum_{i=1}^n Y_{n,i} \to 0$ in probability. Note that this difference is given by $\sum_1^n X_i 1_{|X_i|>\sqrt{n\log n}}$, so we compute
$$P\left(\left|\sum_1^n X_i 1_{|X_i|>\sqrt{n\log n}}\right| > \varepsilon\right) \le P\left(\bigcup_1^n \{|X_i| > \sqrt{n\log n}\}\right) \le n \cdot P(|X_1| > \sqrt{n\log n})$$
But $P(|X_1| > \sqrt{n\log n}) = 2\int_{n\log n}^\infty \frac{1}{x^3}\,dx = \frac{1}{n\log n}$, so the above is at most $\frac{1}{\log n} \to 0$, proving convergence in probability.

   It can be proven that if $A_n \implies A$ and $B_n \to b$ (a constant) in probability, than $A_n + B_n \implies A + B$. Using this, combined with $\sum_{i=1}^n Y_{n,i} \implies N(0,1)$ and $\sum_1^n \frac{X_i}{\sqrt{n\log n}} - \sum_{i=1}^n Y_{n,i} \to 0$ in probability gives the desired result.

3. Let $X^+ = \max(X, 0)$. I claim $EX^+ < \infty$. If not, then for all $M \in \mathbb{N}$, we would have $EX^+/M = \infty$, so that

$$\sum_{n=0}^{\infty} P(X_n^+/n > M) = \sum_{n=0}^{\infty} P(X_n^+/M > n) > \int_0^\infty P(X^+/M > t)\,dt = EX^+/M = \infty$$

implying $P(X_n^+/n > M \text{ i.o.}) = P(\limsup X_n^+/n > M) = 1$. Since this holds for all $M$, it follows that $\limsup X_n^+/n = \infty$ almost surely, contradicting the problem statement.

Finally, using SLLN,

$$\limsup_n \frac{\sum X_k}{n} \le \limsup_n \frac{\sum X_k^+}{n} \overset{a.s.}{=} EX_k^+ < \infty$$

4. It does follow that $E|X| < \infty$.

**Proof 1:** Choose $M$ so $P(|Y| \le M) = \varepsilon > 0$. For all $t$, we have

$$P(|X+Y| > t - M) \ge P(\{|X| > t\} \cap \{|Y| \le M\}) = P(|X| > t)P(|Y| \le M)$$

Using this,

$$\begin{aligned} E|X| = \int_0^\infty P(|X| > t)\,dt &\le \int_0^\infty \frac{P(|X+Y| > t - M)}{P(|Y| \le M)}\,dt \\ &= \frac{1}{\varepsilon}\left(M + \int_0^\infty P(|X+Y| > t)\,dt\right) \\ &= \frac{1}{\varepsilon}(M + E|X+Y|) < \infty \end{aligned}$$

**Proof 2:** Let $\mu$ be the measure on $\mathbb{R}$ induced by $X$, so $\mu(A) = P(X \in A)$, and $\nu$ for $Y$ similarly. Since $E|X+Y| < \infty$, using Fubini's theorem we have

$$E|X+Y| = \int |x+y| d\mu \times \nu = \int \left(\int |x+y| d\mu\right) d\nu < \infty$$

This implies $\left(\int |x+y| d\mu\right) < \infty$ for $\nu$ a.e. $y$, so there is some $y_0$ for which it holds. Then

$$E|X| = \int |x| d\mu \le \int |x+y_0| + |y_0|\,d\mu = \int |x+y_0|\,d\mu + |y_0| < \infty$$

# 2011 Spring

1. Impossible Problem! You need the additional assumption $a_n \geq 0$ for this problem to work; if infinitely many $a_n$ are negative, then $\sum P(|X_n| > a_n)$ would be $\infty$!

   Asssuming additionally each $a_n \geq 0$, then

   $$|S_n/a_n| = |X_n/a_n + \tfrac{a_{n-1}}{a_n}\tfrac{S_{n-1}}{a_{n-1}}| \geq |X_n/a_n| - |\tfrac{a_{n-1}}{a_n}| \cdot |\tfrac{S_{n-1}}{a_{n-1}}| \geq |X_n/a_n| - C|\tfrac{S_{n-1}}{n-1}|$$

   so

   $$\limsup_n |X_n/a_n| \leq \limsup_n |S_n/a_n| + C \cdot |S_{n-1}/a_{n-1}| = 0 \qquad a.s.$$

   In particular, this shows that $P(|X_n/a_n| > 1 \text{ i.o.}) = 0$, because $|X_n/a_n|$ i.o. would imply $\limsup_n |X_n/a_n| \geq 1$. By Borel-Cantelli, we must have $\sum P(|X_n| > a_n) < \infty$.

2. Typo! They meant to say $P(X_n = 1) = p$, $P(X_n = -1) = 1 - p$.

   (a) By SSLN, $S_n/n \to EX_1 = 2p - 1 \neq 0$ a.s, so with probability 1, for some $N$, $S_{N+k}$ will be bounded away from 0 for all $k \geq 0$.

   (b) Note that, using $\sqrt{n}(n/e)^n < n! < e\sqrt{n}(n/e)^n$,

   $$P(S_{2n} = 0) = \frac{1}{2^{2n}}\binom{2n}{n} > \frac{1}{4^n}\left(\frac{(2n/e)^{2n}\sqrt{n}}{((n/e)^n\sqrt{n}e)^2}\right) = \frac{1}{e^2\sqrt{n}}$$

   Thus, $\sum_{n\geq 1} P(S_{2n} = 0) = \infty$, so $P(S_{2n} = 0 \text{ i.o.}) = 1$. This shows $P(\tau < \infty) = 1$, since $\tau = \infty$ implies $S_{2n} = 0$ not infinitely often. We now compute $E\tau$. In order for $\tau$ to be $2k + 2$, the path has to start by moving to 1 (or $-1$), stay at or above 1 (below $-1$), then return to 0. The number of ways the middle step can happen is counted by the Catalan numbers, $\frac{1}{k+1}\binom{2k}{k}$. Thus,

   $$E\tau = \sum_{k\geq 0}(2k + 2)P(\tau = 2k + 2) = \sum_{k\geq 0}(2k + 2)\frac{1}{2^{2k+2}} \cdot \frac{2}{k + 1}\binom{2k}{k}$$

   Using the same approximation as before, this sum is infinite.

3. (a) Without loss of generality, we can assum $EX_n = 0$ by replacing $X_n$ with $X_n' = X_n - EX_n$.

   Using Chebychev's,

   $$P(|S_n/n| > \epsilon) < \frac{E(S_n^4)}{n^4\varepsilon^4}$$

   When $S_n^4$ is expanded out, it contains summands like $X_i^4$, $X_i^2X_j^2$, $X_i^3X_j$, $X_i^2X_jX_k$, and $X_iX_jX_kX_\ell$. Only the first two have nonzero expectation (since distinct $X_i$ are independent, and $EX_i = 0$). Thus, letting $\sup EX_n^4 = M$,

   $$P(|S_n/n| > \epsilon) < \frac{\sum EX_i^4 + \sum_{i\neq j} EX_i^2EX_j^2}{n^4\varepsilon^4} \leq \frac{n \cdot M + n(n-1)M}{n^4\varepsilon^4} \in O(1/n^2)$$

   Using Borel Cantelli, we then have $P(|S_n/n| > \varepsilon \text{ i.o.}) = 0$. This holds for all $\varepsilon$, so intersecting these events for some sequence $\varepsilon_k \searrow 0$ gives $S_n/n \to 0$ a.s.

   (b) If $E|X_1| < \infty$, then $S_n/n \to EX_1$ a.s.

# 2011 Fall

1. (a) $X_n \to X$ a.s. if $P(\{\omega : X_n(\omega) \to X(\omega)\}) = 1$. $X_n \to X$ in $L_1$ if $E|X_n - X| \to 0$.

   (b) i. Let $X_1, X_2 \dots$ be independent, where $P(X_n = n^2) = \frac{1}{n^2} = 1 - P(X_n = 0)$. Then $X_n \to 0$ a.s. (since $P(X_n > 0 \text{ i.o.}) = 0$ by Borel-Cantelli) but $EX_n = 1 \not\to 0$.

   ii. On the probability space $[0,1]$, with Lesbegue measure, let $X_{n,k} = 1_{[\frac{k-1}{n}, \frac{k}{n}]}$, for $n \geq 0$, and $1 \leq k \leq n$. Then let $X'_m$ be the sequence

   $$X_{1,1}, X_{2,1}, X_{2,2}, X_{3,1}, X_{3,2}, X_{3,3}, \dots$$

   i.e. the result of ordeing $X_{n,k}$ lexicographically by $(n,k)$. Since $E|X_{n,k}| = \frac{1}{n} \to 0$ as $n \to \infty$, it follows $X'_m \to 0$ in $L_1$. However, $X'_m(\omega) \not\to 0$ for any $\omega \in [0,1]$, since any $\omega$ will be contained in at least one of the intervals $[\frac{k-1}{n}, \frac{k}{n}]$ for each each $n$.

   (c) For any $\varepsilon > 0$, we have $P(|X_n - X| > \varepsilon) \leq \frac{E|X_n - X|}{\varepsilon}$. Thus, $\sum P(|X_n - X| > \varepsilon) \leq \frac{1}{\varepsilon} \sum E|X_n - X| < \infty$, so $P(|X_n - X| > \varepsilon \text{ i.o.}) = 0$ by Borel Cantelli. This shows that $X_n \to X$ a.s.

2. First, note that

   $$P(-\log X_n / \log n \geq 1) = P(X_n \leq n^{-1}) = 1/n$$

   Thus, $\sum P(-\log X_n / \log n \geq 1) = \infty$, so $P(-\log X_n / \log n \geq 1 \text{ i.o.}) = 1$, so $\limsup_n -\log X_n / \log n \geq 1$ a.s.

   Now, for any $\varepsilon > 0$, we similarly have that

   $$\sum P(-\log X_n / \log n \geq 1 + \varepsilon) = \sum \frac{1}{n^{1+\varepsilon}} < \infty$$

   So $P(\frac{-\log X_n}{\log n \geq 1} + \varepsilon \text{ i.o.}) = 0$, so $\limsup_n \frac{-\log X_n}{\log n} \leq 1 + \varepsilon$ a.s. Intersecting the events $\{\limsup_n \frac{-\log X_n}{\log n} \leq 1 + \frac{1}{k}\}$ for $k \in \mathbb{N}$ shows that $\limsup_n \frac{-\log X_n}{\log n} \leq 1$ a.s.

3. (a) Note the constant that $X+Y$ equals must be 1, since $EX+Y = EX+EY = \frac{1}{2}+\frac{1}{2}$. Thus, the $i^{\text{th}}$ bit of $X$ is the opposite of that of $Y$.

   (b) Suppose that, for each $i$, vector $(X_i, Y_i, Z_i)$, where $X_i$ is the $i^{\text{th}}$ **trinary** digit of $X$, is uniformly distrubted over the 6 permutations of $(0,1,2)$. Then $X, Y, Z$ are each uniformly distrubted over $[0,1]$ since each of their trinary digits are 0,1 or 2 with equal probability, and $X + Y + Z$ is always equal to $1 + \frac{1}{3} + \frac{1}{3^2} + \cdots = \frac{3}{2}$.

## 2012 Spring

1. (a) Let $X = \sum X_i$. By MCT, $EX = \sum \lambda_i < \infty$, so we must have $P(X = \infty) = 0$. Alternatively, $P(X_n > 0) = 1 - e^{-\lambda_n} \leq \lambda_n$, so $\sum P(X_n > 0) < \infty$, so $P(X_n > 0 \text{ i.o.}) = 0$, implying only finitely many $X_n$ are nonzero a.s.

   (b) $P(X_n > 0) = 1 - e^{-\lambda_n} \geq (\lambda_n/2) \wedge \frac{1}{2}$, where $a \wedge b = \min(a, b)$. Therefore, $\sum P(X_n > 0) \geq \sum (\lambda_n/2) \wedge \frac{1}{2} = \infty$, so $P(X_n \geq 1 \text{ i.o.}) = 1$, so $\sum X_n = \infty$ a.s.

2. Note that $\text{Var } X = EX^2 = \frac{1}{3}$. By CLT,

$$\frac{\sum_1^n X_i}{\sqrt{n}} \implies N(0, 1/3) \tag{2}$$

By SLLN,

$$\frac{\sum_1^n X_i^2}{n} \stackrel{\text{a.s.}}{\to} EX^2 = 1/3$$

so

$$\frac{\sqrt{n}}{\sqrt{\sum_1^n X_i^2}} \stackrel{\text{a.s.}}{\to} \sqrt{3} \tag{3}$$

Using Slutsky's theorem ($X_n \implies X$ and $Y_n \to c$ in probability implies $X_n Y_n \to cX$), along with (2) and (3) gives

$$\frac{\sum_1^n X_i}{\sqrt{\sum_1^n X_i^2}} \implies N(0, 1)$$

3. **Remark:** As far as I can tell, this problem is ridiculously hard, using tricks that aren't that common or intuitive. The $\implies$ direction is reasonable, but I'm almost certain no one got the $\impliedby$ when this test was given.

   $(a) \implies (b)$ Letting $T_n = n^{-1/p} \sum_1^n \xi_n$, we have

$$\frac{\xi_n}{n^{1/p}} = T_n - T_{n-1} \cdot \frac{(n-1)^{1/p}}{n^{1/p}}$$

   Letting $n \to \infty$ above, since $T_n \to T$ a.s, and $\frac{(n-1)^{1/p}}{n^{1/p}} \to 1$, we get

$$\frac{\xi_n}{n^{1/p}} = T_n - T_{n-1} \cdot \frac{(n-1)^{1/p}}{n^{1/p}} \to T - T \cdot 1 = 0$$

   so that $\xi_n/n^{1/p} \to 0$ a.s. This means $P(|\xi_n|/n^{1/p} > 1 \text{ i.o.}) = P(|\xi_n|^p > n \text{ i.o.}) = 0$, so (using Borel Cantelli on the last inequality),

$$E|\xi|^p = \int_0^\infty P(|\xi|^p > t)\,dt \leq \sum_{n \geq 0} P(|\xi_n|^p > n) < \infty$$

   proving $E|\xi|^p < \infty$. Now, suppose by way of contradiction that $p > 1$ and $E\xi \neq 0$. Using Jensen's, $(E|\xi|)^p \leq E|\xi|^p < \infty$, so $E|\xi| < \infty$. By SLLN,

$$\frac{\sum_{k=1}^n \xi_n}{n} \to E\xi \neq 0$$

   almost surely as $n \to \infty$. We also have, since $p > 1$, that

$$\frac{1}{n^{1/p-1}} \to \infty$$

   Multiplying the two above limits implies that

$$\frac{\sum_{k=1}^n \xi_n}{n^{1/p}} \to \infty \qquad \text{a.s.}$$

   contradicting that the limit was finite. Thus, we must have $p \leq 1$ or $E\xi = 0$.

$(b) \implies (a)$ First, suppose that $p \le 1$. We can actually assume $p < 1$, since $p = 1$ follows from SLLN. We will show that $\sum_1^\infty \frac{|\xi_n|}{n^{1/p}}$ converges a.s. This implies $\sum_1^\infty \frac{\xi_n}{n^{1/p}}$ converges a.s., which by Kronecker's Lemma implies $n^{-1/p} \sum_1^n \xi_k \to 0$ a.s., the desired result.

To show $\sum_1^\infty \frac{|\xi_n|}{n^{1/p}}$, we use the Kolmogorov 3-series test. Let $Y_n = \frac{\xi_n}{n^{1/p}} \mathbf{1}(|\xi_n|^p \le n)$. We must check that

(i) $\sum_1^\infty P(|\xi_n|^p > n) < \infty$ (ii) $\sum_1^\infty EY_n$ converges (iii) $\sum_1^\infty \text{Var } Y_n < \infty$

(i) This is true since $E|\xi|_1^p < \infty$, which holds if and only if $\sum_1^\infty P(|\xi|_1^p > k) < \infty$.

(ii) The below computations uses many clever tricks. For the first equality, we are using $|\xi_1| 1_{|\xi_1|^p \le n} = \sum_1^n |\xi_1| 1_{\{k-1 < |\xi_1|^p \le k\}}$. For the second, we use Fubini's, being careful with the indices. For the third, we bound $\sum_{n=k}^\infty n^{-1/p} \le \int_k^\infty x^{-1/p}\, dx$. For the fourth, realize that when $|xi|^p \le k$, then $|\xi_1|^{1-p} = (|\xi_1|^p)^{(1/p)-1} \le k^{(1/p)-1}$.

$$\begin{aligned}
\sum_{n=1}^\infty E(\tfrac{|\xi_n|}{n^{1/p}}; |\xi|^p \le n) &= \sum_{n=1}^\infty \sum_{k=1}^n \frac{1}{n^{1/p}} E(|\xi_1| 1_{\{k-1 < |\xi|^p \le k\}}) \\
&= \sum_{k=1}^\infty E(|\xi_1| 1_{\{k-1<|\xi|^p \le k\}}) \sum_{n=k}^\infty \frac{1}{n^{1/p}} \\
&\le \sum_{k=1}^\infty E(|\xi_1|^p \cdot |\xi_1|^{1-p} 1_{\{k-1<|\xi|^p\le k\}}) \frac{k^{1-1/p}}{1/p - 1} \\
&\le \frac{1}{1/p-1} \sum_{k=1}^\infty E\Big(|\xi|^p \left(k^{1/p-1}\right) 1_{\{k-1<|\xi|^p\le k\}}\Big) \cdot k^{1-1/p} \\
&= \frac{1}{1/p-1} E|\xi_1|^p < \infty
\end{aligned}$$

(iii) To show $\sum \text{Var } Y_n < \infty$, we show $\sum EY_n^2 < \infty$, using the same tricks.

$$\begin{aligned}
\sum_{n=1}^\infty E(\tfrac{|\xi_1|^2}{n^{2/p}}; |\xi| \le n) &= \sum_{n=1}^\infty \sum_{k=1}^n n^{-2/p} E(|\xi|^2 1_{\{k-1<|\xi|^p\le k\}}) \\
&= \sum_{k=1}^\infty E(|\xi|^2 1_{\{k-1<|\xi|^p\le k\}}) \sum_{n=k}^\infty n^{-2/p} \\
&\le \sum_{k=1}^\infty E(|\xi|^p \cdot |\xi_1|^{2-p} 1_{\{k-1<|\xi|^p\le k\}}) \frac{k^{1-2/p}}{2/p-1} \\
&\le \frac{1}{2/p-1} \sum_{k=1}^\infty E(|\xi_1|^p 1_{\{k-1<|\xi|^p\le k\}}) = \frac{EX_1}{2/p-1} < \infty
\end{aligned}$$

This completes the proof in the case $p \le 1$.

Now, suppose $E\xi_i = 0$ and $p \in (1,2)$. Let $Y_k = \xi_k 1_{\{|\xi|_k \le k^{1/p}\}}$, and let $T_n = Y_1 + \cdots + Y_n$. Since

$$\sum P(|\xi_k| > k^{1/p}) \le \int_0^\infty P(|\xi_1|^p > t)\, dt = E|\xi|^p < \infty,$$

it follows that $P(\xi_k \ne Y_k \text{ i.o.}) = 0$, so it suffices to prove $T_n/n^{1/p} \to 0$. We compute

$$\begin{aligned}
\sum_{k=1}^\infty \text{Var }(Y_k/k^{1/p}) &\le \sum_{k=1}^\infty E(Y_k^2)/k^{2/p} \\
&= \sum_{k=1}^\infty \int_0^{k^{1/p}} \frac{2y}{k^{2/p}} P(Y_k > y)\, dy \\
&\le \sum_{k=1}^\infty \sum_{n=1}^k \int_{(n-1)^{1/p}}^{n^{1/p}} \frac{2y}{k^{2/p}} P(|\xi| > y)\, dy \\
&\overset{\text{Fubini}}{=} \sum_{n=1}^\infty \int_{(n-1)^{1/p}}^{n^{1/p}} 2yP(|\xi| > y) \left( \sum_{k=n}^\infty \frac{1}{k^{2/p}} \right) dy
\end{aligned}$$

We can bound $\sum_{m=n}^\infty \frac{1}{k^{2/p}}$ by an integral:

$$\sum_{k=n}^\infty \frac{1}{k^{2/p}} \le \int_{n-1}^\infty x^{-2/p}\, dx = \frac{(n-1)^{(p-2)/p}}{(2-p)/p} \le Cy^{p-2},$$

for any $y \in [(n-1)^{1/p}, n^{1/p}]$, and some constant $C$. Therefore,

$$\sum_{k=1}^\infty \text{Var }(Y_k/k^{1/p}) \le \int_0^\infty 2Cy^{p-1} P(|\xi| > y)\, dy < \infty,$$

with the last inequality following since $E|\xi|^p = \int_0^\infty py^{p-1}P(|\xi| > y)\, dy < \infty$. By Kolmogorov's theorem for the convergence of random series, letting $\mu_k = EY_k$, we have $\sum_1^\infty (Y_k - \mu_k)/k^{1/p} < \infty$ a.s, which by Kronecker's Lemma implies

$$n^{-1/p} \sum_1^n Y_k - \mu_k \to 0 \qquad a.s.$$

To show that $n^{-1/p}\sum_1^n Y_k \to 0$ a.s, completing the proof, we need only show $n^{-1/p}\sum_1^n \mu_k \to 0$. Since $\mu_k + E(\xi_k; |\xi| > k^{1/p}) = E\xi_k = 0$, we have that

$$\begin{aligned}
|\mu_k| \le E(|\xi|; |\xi| > k^{1/p}) &= k^{1/p} E(|\xi|/k^{1/p}; |\xi| > k^{1/p}) \\
&\le k^{1/p} E(|\xi|^p/k; |\xi| > k^{1/p}) \\
&= k^{-1+1/p} E(|\xi|^p; |\xi| > k^{1/p})
\end{aligned}$$

Since $\sum_1^n k^{-1+1/p} \le Kn^{1/p}$ and $E(|\xi|^p; |\xi| > k^{1/p}) \to 0$ as $k \to \infty$ (by DCT), it follows that $n^{1/p} \sum \mu_k \to 0$, completing the proof.

# 2012 Fall

1. (a) For any $0 < x < 1$, we have

$$P(X_n \le x) = \int_0^x 1 + \sin 2\pi nt \, dt = x + \tfrac{1-\cos 2\pi nx}{2\pi n} \to x + 0$$

as $n \to \infty$. Thus, $X_n \implies X$, where $P(X \le x) = x$, i.e, $X$ is uniform on $[0, 1]$.

(b) Let $a_n = -\log n$. Then

$$P(\tfrac{1}{a_n} \log X_n > 2) = P(X_n < n^{-2}) = n^{-2} + \tfrac{1-\cos(2\pi n \cdot n^{-2})}{2\pi n} = n^{-2} + O(n^{-3})$$

Notice $\sum P(\frac{1}{a_n} \log X_n > 2) < \infty$. By Borel-Cantelli, $P(\frac{1}{a_n} \log X_n > 2 \text{ i.o.}) = 0$, proving $\limsup_n \frac{1}{a_n} \log X_n \le 2$ a.s. Furthermore,

$$P(\tfrac{1}{a_n} \log X_n > 1) = P(X_n < n^{-1}) = n^{-2} + \tfrac{1-\cos(2\pi)}{2\pi n} = n^{-1}$$

So by Borel-Cantelli again, $P(\frac{1}{a_n} \log X_n > 1 \text{ i.o.}) = 1$, so the limsup will be at least 1 almost surely.

2. (a) possibly wrong solution: The following proof did not at any point use $\sup \text{Var } X_n < \infty$, so I suspect I made a mistake. Please check to make sure my logic is correct. Given $n$, for each $m$ we can variables i.i.d. $X_m^1, \dots, X_m^n$ so

$$X_m^1 + \cdots + X_m^n \overset{d}{=} X_m^1$$

We first show that the sequence $X_1^1, X_2^1, X_3^1 \dots$ is tight. Since $X_m^i > A$ for each $i$ implies that $\sum_i X_m^i \ge nA$, and $X_m^1 \overset{d}{=} \sum_1 X_m^i$, we have

$$P(X_m^1 > A)^n = P\left(\bigcap_1^n X_m^i > A\right) \le P(X_m > nA) \le P(|X_m| > nA).$$

Similarly, $P(X_m^1 < -A)^n \le P(|X_m| > nA)$, so

$$\sup_m P(|X_m^1| > A) = \sup_m P(X_m^1 > A) + P(X_m^1 < -A) \le \sup_m 2P(|X_m| > nA)^{1/n}$$

By tightness of $X_m$, the right hand side of above approaches 0 as $A \to \infty$, proving the left does as well, so the sequence $\{X_m^1\}_{m \to \infty}$ is tight.
By Helly's selection theorem, there exists a subsequence $X_{m_k}^1$ and a random variable $X^1$ so that $X_{m_k}^1 \implies X^1$. Since $X_m^i \overset{d}{=} X_m^1$, this means $X_{m_k}^i \implies X^i$, where $X^i \overset{d}{=} X^1$. Since $Z_n \implies Z$, $Y_n \implies Y$ and $Z_n, Y_n$ being independent implies $Z_n + Y_n \implies Z + Y$ (to prove this, look at characteristic functions), it follows that

$$X_{m_k} \overset{d}{=} \sum_1^n X_{m_k}^i \implies \sum_1^n X^i.$$

But we also have $X_{m_k} \implies X$ so we must have $X \overset{d}{=} \sum_1^n X^i$. This shows $X$ has been written as a sum of $n$ iid random variables, so $X$ is infinitely divisible.

(b) In general, if $X$ is any varible where $|X| \leq 1$ a.s, then $X$ is not infintiely divisible. If $X_1 + \ldots X_n \stackrel{d}{=} X$, then it must mean that each $X_i \leq \frac{1}{n}$ a.s. If not, for some $\varepsilon > 0$ then there would be a possibility that each $X_i > \frac{1}{n} + \varepsilon$, implying $\sum X_i > 1$, which is a contradiction, since $X$ has the same distribution as $\sum X_i$, and $X \leq 1$ always. Similarly, $X_i \geq -\frac{1}{n}$ a.s, so $|X_i| \leq \frac{1}{n}$ a.s, implying

$$\text{Var } X_i \leq EX_i^2 \leq \frac{1}{n^2} \tag{1}$$

However, we also have

$$\text{Var }(X) = \sum \text{Var }(X_i) = n\text{Var }(X_1)$$

so that

$$\text{Var }(X_i) = \frac{\text{Var } X}{n} \tag{2}$$

But (1) and (2) are in contradiction for large enough $n$, so $X$ is not infinitely divisible.

(c) We could just run through the same argument above to show that $U$ is not infinitely divisible.

I think they were going for this argument: if $U'$ has the same distribution as $U$, and is independent of $U$, then $U + U' \stackrel{d}{=} X$ (you can check this). Thus, if you could divide $U$ into any number of parts, $n$, then you could do the same for $U'$, and then use this to divide $X \stackrel{d}{=} U + U'$ into $2n$ parts. This, doesn't *quite* contradict the fact that $X$ is non infinitely divisible, but it's close.

3. ☹

# 2013 Spring

1. (a) We have that

$$E(X_{i,n}^2\mathbf{1}(|X_{i,n}|>\varepsilon)) = E((\tfrac{X_i}{\sqrt{n}})^2;\mathbf{1}(|\tfrac{X_i}{\sqrt{n}}|>\varepsilon)) = \frac{1}{n}E(X_1^2\mathbf{1}(|X_1|>\varepsilon\sqrt{n}))$$

so

$$L_{n,\varepsilon} = \sum_1^n E(X_{i,n}^2\mathbf{1}(|X_{i,n}|>\varepsilon)) = E(X_1^2\mathbf{1}(|X_1|>\varepsilon\sqrt{n}))$$

Since $X_1^2\mathbf{1}(|X_1|>\varepsilon\sqrt{n}) \to 0$ almost surely as $n\to\infty$, and $EX_1^2<\infty$, by the DCT, the last quantiy approaches 0 as $n\to\infty$.

(b) Using Jensen's inequality, $E|X_{i,n}|^p = E((X_{i,n}^2)^{p/2}) \geq (EX_{i,n}^2)^{p/2} \geq EX_{i,n}^2$, so

$$L_{n,\varepsilon} = \sum_1^n E(X_{i,n}^2\mathbf{1}(|X_{i,n}|>\varepsilon)) \leq \sum_1^n E|X_{i,n}|^p \to 0$$

(c) Let $X_{i,n}$ have normal dsitribution $N(0,\frac{2^{k-2}}{2^{n-1}})$ when $i\geq 2$, and $X_{1,n}$ have distribution $N(0,\frac{1}{2^{n-1}})$. Then because $Z_1\sim N(0,\sigma_1^2)$ and $Z_2\sim N(0,\sigma_2^2)$ implies $Z_1+Z_2\sim N(0,\sigma_1^2+\sigma_2^2)$, we have that

$$W_n \sim N\left(0,\frac{1+1+2+\cdots+2^{n-2}}{2^{n-1}}\right) = N(0,1)$$

so that not only does $W_n\to N(0,1)$ in distribution, but each $W_n$ is equal to $N(0,1)$ in distibution.

However, the Lindeberg condition does not hold, since $X_{n,n}\sim N(0,\frac{2^{n-2}}{2^{n-1}}) = N(0,\frac{1}{2})$, so

$$\sum_1^n E(X_{i,n}^2;\mathbf{1}(|X_{i,n}|>\varepsilon)) \geq E(X_{n,n}^2;\mathbf{1}(|X_{n,n}|>\varepsilon)) \geq \varepsilon P(X_{n,n}>\varepsilon) \not\to 0$$

where the last quantity does not approach zero since each $X_{n,n}$ have the same $N(0,\frac{1}{2})$ distribution, so $P(X_{n,n}>\varepsilon)$ is constant in $n$.

2. (a) By definition, a matrix $M$ is nonneggative semidefinite if $x^T Mx \geq 0$ when $x$ is any column vector. Given a column vector $x = [a_0\ a_1\ \ldots\ a_{n-1}]$, expand out the right side of the inequality

$$0 \leq E\Big((a_0 + a_1X + a_2X^2 + \ldots a_{n-1}X^{n-1})^2\Big)$$

then distribute the $E$ over all of the terms, so each $X^k$ becomes $m_k$. You will see that the result is exactly $x^T H_n x$, proving $x^T H_n x \geq 0$, so $H_n$ is nonnegative semidefinite.

(b) First of all, what does $\Delta^k m_n$ mean? First of all, they don't just mean $\Delta m_n = m_{n+1} - m_n$, they mean that for any sequence $a_n$, $\Delta a_n = a_{n+1} - a_n$. So, $\Delta a_n$ is itselt a sequence, and you can apply $\Delta$ to that, getting $\Delta^2 a_n$. For example,

$$\Delta^2 m_n = \Delta(m_{n+1} - m_n) = (m_{n+2} - m_{n+1}) - (m_{n+1} - m_n) = m_{n+2} - 2m_{n+1} + m_n$$

$$\Delta^3 m_n = m_{n+3} - 2m_{n+2} + m_{n+1} - (m_{n+2} - 2m_{n+1} + m_n) = m_{n+3} - 3m_{n+2} + 3m_{n+1} - m_n$$

$$\Delta^4 m_n = m_{n+4} - 4m_{n+3} + 6m_{n+2} - 4m_{n+1} + m_n$$

Fans of combinatorics will notice Pascal's triangle appearing on the RHS of each equation. In fact, you can prove by induction that

$$\Delta^k m_n = \sum_{j=0}^{k} \binom{k}{j} (-1)^{j+k} m_{n+k}$$

Using this, and the binomial theorem, we have that

$$0 \leq EX^n(1-X)^k = E\sum_{j=0}^{k} \binom{k}{j} (-1)^j X^{n+j} = (-1)^k \sum_{j=0}^{k} \binom{k}{j} (-1)^{j+k} m_{n+k} = (-1)^k \Delta^k m_n$$

3. (a) First, we find the c.f. for $Y_k$, which has pdf $e^{-x}$:

$$\phi(t) = Ee^{itY_k} = \int_0^\infty e^{ity} e^{-y}\, dy = \frac{1}{it-1} e^{y(it-1)}\Big|_0^\infty = \frac{1}{1-it}$$

This means that the c.f. for $\frac{Y_k - 1}{k} = \frac{1}{1-it/k} e^{-it/k}$.

Let $W_n = \gamma + \sum_{k=1}^n \frac{Y_k-1}{k}$. Since $W_n \to W$ a.s, so that $e^{itW_n} \to e^{itW}$, and each $|e^{itW_n}| \leq 1$, it follows by DCT that

$$\varphi(t) = Ee^{itW} = \lim_n Ee^{itW_n} = \lim_n e^{i\gamma t} \prod_1^n \frac{e^{-it/k}}{1-it/k} = e^{i\gamma t} \prod_1^\infty \frac{e^{-it/k}}{1-it/k}$$

As far as I can tell, this is the only way to express the characteristic function.

(b)

$$|\varphi(t)| = \left| e^{i\gamma t} \prod_{1}^{\infty} \frac{e^{-it/k}}{1 - it/k} \right| = \left| e^{i\gamma t} \right| \prod_{1}^{\infty} \frac{\left| e^{-it/k} \right|}{|1 - it/k|} = \prod_{1}^{\infty} \frac{1}{\sqrt{1^2 + t^2/k^2}}$$
$$= \exp\left( \sum_{k=1}^{\infty} -\frac{1}{2} \log(1 + t^2/k^2) \right) \leq \exp\left( -\frac{1}{2} \log(1 + t^2) - \frac{1}{2} \log(1 + t^2/4) \right)$$

Using the concavity of log, so that $\log x$ lies above the secant line joining $(1, 0)$ and $(1 + t^2, \log(1 + t^2))$, for any $1 \leq x \leq t^2$ is true that

$$\log x \geq \frac{\log(1 + t^2) - \log(1)}{1 + t^2 - 1} \cdot (x - 1) = \frac{\log(1 + t^2)}{t^2}(x - 1),$$

and setting $x = 1 + t^2/4$ implies $\log(1 + t^2/4) \geq \frac{\log t^2}{4}$,so

$$|\varphi(t)| \leq \exp\left( -\frac{1}{2}\Big( \log(1 + t^2) + \frac{\log(1 + t^2)}{4} \Big) \right) = \exp(\log(1+t^2)^{-5/8}) = (\sqrt{1 + t^2})^{-5/4}$$

Since $\sqrt{1 + t^2} \geq \max(1, t)$ it follows that

$$\int |\varphi(t)|], dt < \int_{-\infty}^{\infty} (\sqrt{1 + t^2})^{-5/4} \leq \int_{-\infty}^{\infty} \min\left( 1, \frac{1}{|t|^{5/4}} \right) < \infty.$$

(c) It does follow that $W$ has an absolutely continuous distribution.

(d) ☹ The inversion formula gives

$$f_W(w) = \int_{-\infty}^{\infty} e^{-itw} \varphi(t)\, dt = \int_{-\infty}^{\infty} e^{-itw} \varphi(t)\, dt$$

# 2013 Fall

1. (a) It does follow that $S_n/n \to X$. We first show that $X_n \to X$ in $L_1$. Note that $|X| \le 1$ a.s, because if $P(|X| > 1+\delta) = \varepsilon > 0$, then $P(|X_n - X| > \delta) \ge \varepsilon \not\to 0$. In particular, $|X_n - X| \le 2$. Thus, given any $\varepsilon \ge 0$,

$$\begin{aligned}\limsup_n E|X_n - X| &= \limsup_n E(|X_n - X| 1_{|X_n - X| < \varepsilon}) + E(|X_n - X| 1_{|X_n - X| > \varepsilon}) \\ &\le \limsup_n \;\; \varepsilon + 2P(|X_n - X| > \varepsilon) = \varepsilon\end{aligned}$$

This holds for all $\varepsilon$, proving $E|X_n - X| \to 0$. Let $|X_n - X|_1 = E|X_n - X|$, and given $\varepsilon$, choose $N$ so that $n > N$ implies $|X_n - X|_1 < \varepsilon$. Then, for $n > N$,

$$\begin{aligned}|S_n/n - X|_1 &\le \sum_1^\infty \tfrac{1}{n}|X_i - X|_1 \\ &= \frac{1}{n}\sum_1^N |X_i - X|_1 + \sum_{N+1}^n \tfrac{1}{n}|X_i - X|_1 \\ &\le \frac{1}{n}\sum_1^N |X_i - X|_1 + \sum_{N+1}^n \tfrac{1}{n}\cdot\varepsilon \\ &\le \frac{1}{n}\sum_1^N |X_i - X|_1 + \varepsilon \to \varepsilon \qquad \text{as } n \to \infty\end{aligned}$$

Taking the lim sup of the above inequality, the last sum converges to 0, proving $S_n/n \to X$ in $L_1$, and therefore in probability.

(b) Now the claim does not follow. Let

$$X_n = \begin{cases} 0 & \text{with probability } 1 - \frac{1}{n} \\ n & \text{with probability } \frac{1}{n} \end{cases}$$

so that $X_n \to 0$ in probability. However, we can show that $P(S_n/n \ge \frac{1}{2}) \ge \frac{1}{2}$ for all $n$. In order for $S_n/n$ to be bigger than $\frac{1}{2}$, it suffices for some $X_k$ to equal $k$, for $k \ge \frac{n}{2}$. Thus, noting that the below product is telescoping, we get

$$P(S_n/n \ge \tfrac{1}{2}) \ge P\left(\bigcup_{k=n/2}^n X_k = k\right) = 1 - \prod_{k=n/2}^n \frac{k-1}{k} = 1 - \frac{n/2 - 1}{n} \ge \frac{1}{2}$$

This shows $S_n/n \not\to 0$ in probability.

2. (a) This follows from $E(X) = \int_0^\infty P(X > x)\,dx$, and applying $\int_0^\infty$ to below:

$$P(X > \lceil x \rceil) \le P(X > x) \le P(X > \lfloor x \rfloor)$$

(b) Applying part (i) to $|X_n|/k$,

$$\sum P(|X_n| > kn) = \sum P(|X_n|/k > n) \ge E|X_n/k| = \infty$$

Using Borel-Cantelli, this says that for all $k$, $P(|X_n|/n > k \text{ i.o.}) = 1$. Thus, $P\left(\bigcap_{k\ge1}\{|X_n|/n > k \text{ i.o.}\}\right) = 1$, proving that $\limsup_n |X_n|/n = \infty$ a.s.

Note that

$$|S_n/n| = |X_n/n + \tfrac{n-1}{n}\tfrac{S_{n-1}}{n-1}| \ge |X_n/n| - |\tfrac{n-1}{n}| \cdot |\tfrac{S_{n-1}}{n-1}| \ge |X_n/n| - |\tfrac{S_{n-1}}{n-1}|$$

so

$$\limsup_n |\tfrac{S_n}{n}| + |\tfrac{S_{n-1}}{n-1}| \ge \limsup |X_n/n| = \infty \qquad a.s.$$

Thus, almost surely the sequence $|\frac{S_n}{n}| + |\frac{S_{n-1}}{n-1}|$ is unbounded, proving that $|S_n/n|$ is unbounded almost surely as well.

3. Note that $E(X_iY_i) = 0$, and $\text{Var }(X_iY_i) = E(X_i^2Y_i^2) = EX_i^2 = \text{Var } X_i^2 + (EX_i)^2 = \sigma^2 + \mu^2$. Thus, by CLT,

$$\frac{\sum X_kY_k}{\sqrt{n}} \implies N(0, \sigma^2 + \mu^2)$$

Furthermore, we have $\frac{1}{n}\sum X_k \to \mu$ a.s. by SLLN, so that

$$\frac{n}{\sum X_k} \to \frac{1}{\mu} \qquad a.s.$$

Using Slutsky's to multiply these two gives us

$$\frac{\sqrt{n}\sum X_kY_k}{\sum X_k} \to N(0, 1 + \tfrac{\sigma^2}{\mu^2})$$

## 2014 Spring

1. (a) We have Var $(S_n) = \sum \text{Var } X_i \leq nC$, so

$$E(S_n/n - \mu)^2 \leq \text{Var } (S_n/n) \leq \frac{Cn}{n^2} \to 0$$

proving convergence in $L_2$.

(b) For all $\varepsilon > 0$,

$$P(S_n/\mu - \mu > \varepsilon) = P((S_n/n - \mu)^2 > \varepsilon^2) \leq \frac{\text{Var } (S_n/n)}{\varepsilon^2} \to 0.$$

(c) There will be a subsequence $S_{n(k)}/n(k) \to \mu$ a.s. You won't have a.s. convergence in general, since you need independence, not just uncorrelation (I can't think of a specific counterexample though).

2. (a) Let $E_n$ be the event that he wins games $2n$ and $2n+1$. The $E_n$ are indpendent, and $\sum P(E_n) = \sum \frac{1}{\sqrt{2n(2n+1)}} = \infty$, so by second Borel Cantelli, $P(E_n$ i.o.). Since he gets a dollar each time $E_n$ occurs, his winnings will be infinite a.s.

(b) Let $F_n$ be the event he wins games $n, n+1$ and $n+2$. Then $P(F_n \text{ i.o.}) = 0$, since $\sum P(F_n) = \sum \frac{1}{\sqrt{n(n+1)(n+2)}} < \infty$. So, almost surely, he only gets finite monies.

3. Let

$$a_n = \frac{1}{2}\sum_1^n k^2 \qquad b_n = \sqrt{\sum_1^n \frac{k^4}{12}}$$

We'll use the Lindeberg-Feller CLT to show that $\frac{\sum X_k - a_n}{b_n} \to N(0,1)$.

Let $Y_{n,k} = (X_k - \frac{k^2}{2})/b_n$, so $EY_{n,k} = 0$. We have

$$\sum_1^n EY_{n,k}^2 = \sum_1^n \text{Var } (Y_{n,k}) = \frac{\sum_1^n \text{Var } X_k}{b_n^2} = \frac{\sum_1^n k^4/12}{b_n^2} = 1$$

Furthermore, for any $\varepsilon > 0$, consider

$$\sum_1^n EY_{n,k}^2 1_{\{|Y_{n,k}| > \varepsilon\}}$$

Note that $|Y_{n,k}| < \frac{n^2/2}{b_n} \to 0$ as $n \to \infty$. Thus, for large $n$, $Y_{n,k}^2 1_{|Y_{n,k}|>\varepsilon} = 0$ always, so $\lim_{n\to\infty}$ of the above sum is zero.

Thus, by the Lindberg Feller CLT, we have

$$\sum_1^n Y_{n,k} = \frac{\sum_1^n X_k - a_n}{b_n} \implies N(0,1)$$

# 2014 Fall

1. (a) ( $\Longrightarrow$ ) Assume that $P(E_n \text{ i.o.}) = 1$. Let $A$ be an event where $P(A) > 0$. Then

$$\begin{aligned} 1 &= P(E_n \text{ i.o.}) \\ &= P(\{E_n \text{ i.o.}\} \cap A) + P(\{E_n \text{ i.o.}\} \cap A^c) \\ &\leq P(\{E_n \text{ i.o.}\} \cap B) + P(A^c) \end{aligned}$$

so

$$P(\{E_n \text{ i.o.}\} \cap A) \geq 1 - P(A^c) = P(A) > 0.$$

Since the event $\{E_n \text{ i.o.}\} \cap A$ is the same as the event $\{E_n \cap A \text{ i.o.}\}$, the above shows that $P(E_n \cap A \text{ i.o.}) > 0$. By the (contrapositive of the) Borel-Cantelli lemma, this means that $\sum P(E_n \cap A) = \infty$.

($\Longleftarrow$) Assume that, whenever $P(A) > 0$, we have $\sum P(E_n \cap A) = \infty$. Let $A = \{E_n \text{ i.o.}\}^c$, and consider

$$\sum_{n \geq 1} P(E_n \cap A)$$

Notice that only finitely many of the above terms can be nonzero: if $\omega \in A$, then $\omega$ is in only finitely many $E_n$, so only finitely many $E_n \cap A$ are nonempty. Thus, the above sum is finite. Since such sums are always infintie when $P(A) > 0$, this means $P(A) = 0$, so that $P(A^c) = P(E_n \text{ i.o.}) = 1$.

(b) This is false. For the prabability space $(0, 1)$ with Lesbegue measure, let $E_n = (0, 1/n)$. Then $P(E_n \text{ i.o.}) = 0$, but $\sum P(E_n \cap (0, 1)) = \sum 1/n = \infty$.

2. Given $\varepsilon > 0$, choose $x$ so the distribution function of $X$ is continuous at $x$ and $P(X \leq x) < \varepsilon$. Then

$$P(X_n + Y_n \leq c) \leq P(\{X_n \leq x\} \cup \{Y_n \leq c - x\}) \leq P(X_n \leq x) + P(Y_n \leq c - x)$$

so

$$\limsup_n P(X_n + Y_n \leq c) \leq \limsup_n P(X_n \leq x) + P(Y_n \leq c - x) = \varepsilon + 0$$

Thus, for all $\varepsilon > 0$, $\limsup_n P(X_n + Y_n \leq c) \leq \varepsilon$, so $P(X_n + Y_n \leq c) \to 0$.

3. The answer is that $Y_n \to 0$ a.s. iff $a < e$.

   Note $Y_n \to 0$ a.s. $\iff$ $\log Y_n \to -\infty$ a.s. We have

$$E \log X_1 = \int_0^a \log x \cdot \frac{1}{a}\, dx = \log(a) - 1$$

   By SLLN,

$$\frac{\log Y_n}{n} = \frac{1}{n}\sum_1^n \log X_i \to \log(a) - 1 \qquad a.s.$$

   Thus, when $a < e$, we have $\frac{1}{n}\log Y_n$ a.s. converges to a negative constant, so $\log Y_n \to -\infty$ a.s. When $a > e$, the same reasoning shows $\log Y_n \not\to -\infty$. When $a = e$, CLT tells us that

$$\frac{\log Y_n}{\sigma\sqrt{n}} \implies N(0,1)$$

   where $\sigma^2 = \text{Var } \log X_1$. In particular, $P(\log Y_n > 0) = P(Y_n > 1) \to \frac{1}{2}$. Since $Y_n \to 0$ a.s. would imply $P(Y_n > 1) \to 0$, this means that $Y_n \not\to 0$ a.s.