MATH 507a QUALIFYING EXAM Friday, February 13, 2015. One hour and 50 minutes, starting at 4:30pm.

Ideas for solutions.

1. *Let* $X_n,\ n \ge 1$, *be independent random variables such that each* X_n *has Poisson distribution with mean* n^r *for some real number* r. *Determine the values of* r *for which*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} X_n}{\sum_{n=1}^{N} n^r} = 1$$

with probability one.

Solution. We need $\sum_k n^r = +\infty$, that is, $r \ge -1$. The direct way is to apply the strong law of large numbers that ensures, for independent X_k,

$$\lim_n \frac{\sum_{k=1}^{n} \big(X_k - \mathbb{E}(X_k)\big)}{\sum_{k=1}^{n} \mathbb{E}(X_k)} = 0$$

with probability one. Keep in mind that if $\sum_{k\ge 1} \mathbb{E}X_k$ is finite, then the limit is a random variable.

An alternative quick way is to embed the sum in a Poisson process.

2. *Let* $X_n,\ n \ge 1$, *be a sequence of iid random variables with a continuous distribution function. For* $m \ge 1$, *let* E_m *be the event that a record occurs at moment* m:

$$E_1 = \Omega,\ E_m = \{\omega : X_m(\omega) > X_k(\omega),\ 1 \le k < m\}, \quad m \ge 2.$$

Show that

a) $\mathbb{P}(X_m = X_n) = 0$ for $m \ne n$; [kind of obvious; a formal proof can integrate the joint distribution over the diagonal and use the Fubini theorem]

b) $\mathbb{P}(E_n) = 1/n$; [$\mathbb{P}(E_n) = (n-1)!/n! = 1/n$]

c) E_n and E_m are independent if $m \ne n$; [kind of obvious; can write a formal proof by conditioning]

d) With probability one, infinitely many records occur [b)+c)+Borel-Cantelli]

3. *Let* $X_n,\ n \ge 1$, *be iid random variables that are uniform on* $(0, 2.5)$, *and let* $Y_n = \prod_{k=1}^{n} X_k$. *True or false:* $\lim_{n\to\infty} Y_n = 0$ *with probability one. Explain your conclusion.*

Solution. True. Indeed, $\ln Y_n = \sum_k \ln X_k$; want the sum to diverge to $-\infty$; with iid X_k need $\mathbb{E}\ln X_k < 0$, and, for X uniform on $(0, a)$,

$$\mathbb{E}\ln X = \int_0^a \ln x dx = a\ln a - a < 0$$

if $a < e$; $2.5 < e = 2.718....$