Ideas for solutions.

1. *Suppose $X_1, X_2, \dots$ are iid with values in a bounded interval $[a;b]$ with density $f(x)$.*
(a) Suppose the distribution is uniform in $[a;b]$. Show that $\liminf_n n(X_n - a) = 0$ a.s.
(b) Still supposing the distribution is uniform, what can you say about $\liminf_n n^2(X_n - a)$? [Answer: $+\infty$]
(c) Suppose instead that the endpoint $a = 0$ and the density satisfies $f(x) \sim cx^{-1/2}$, $x \to 0$, with $0 < c < \infty$. (Here $\sim$ means the ratio converges to 1.) Find the new values of the lim infs in (a) and (b) [Answer: both are zero.]

Solution. By Borel-Cantelli, with (pair-wise) independent continuous random variables $Y_n > 0$ and numbers $a_n > 0$, $\sum_n P(Y_n < Ca_n) < \infty$ for all $C > 0$ means $\liminf_n (Y_n/a_n) = +\infty$ and $\sum_n P(Y_n < Ca_n) = +\infty$ for all $C > 0$ means $\liminf_n (Y_n/a_n) = 0$.

Accordingly, in parts (a) and (b), with $Y_n = X_n - a$, $P(Y_n < Ca_n) = C/n$ in (a) and $P(Y_n < Ca_n) = C/n^2$ in (b). In (c), with $Y_n = X_n$, we have $P(Y_n < Ca_n) \sim C_1\sqrt{a_n}$.

2. *Let X be a non-negative random variable. Prove that for $b > 0$,*

$$\lim_{t\to+\infty} \frac{\log Ee^{-tX}}{t} = -b$$

if and only if

$$P(X \geq b) = 1.$$

Proposition 1. *Let ξ be a non-negative random variable. Then*
(1) $\mathbb{E}e^{-\lambda\xi} \geq p_0 > 0$ *if and only if* $\mathbb{P}(\xi = 0) \geq p_0$;
(2) $\mathbb{E}e^{-\lambda\xi} \leq e^{-\lambda\varepsilon_0}$ *if and only if* $\mathbb{P}(\xi \geq \varepsilon_0 > 0) = 1$.

Proof. If $\mathbb{P}(\xi = 0) \geq p_0 > 0$, then

$$\mathbb{E}e^{-\lambda\xi} \geq \mathbb{E}\Big(1(\xi = 0)\Big) = \mathbb{P}(\xi = 0) \geq p_0.$$

If $\mathbb{E}e^{-\lambda\xi} \geq p_0$, then, for every $n \geq 1$,

$$\begin{aligned} p_0 \leq \mathbb{P}(\xi = 0) \leq \mathbb{E}e^{-\lambda\xi} &= \mathbb{E}\left(e^{-\lambda\xi}1(\xi \leq 1/n)\right) + \mathbb{E}\left(e^{-\lambda\xi}1(\xi > 1/n)\right) \\ &\leq \mathbb{P}(\xi \leq 1/n) + e^{-\lambda/n}, \end{aligned} \tag{1}$$

that is,

$$p_0 \leq \mathbb{P}(\xi \leq 1/n) + e^{-\lambda/n},$$

or, after passing to the limit $\lambda \to +\infty$,

$$\mathbb{P}(\xi \leq 1/n) \geq p_0.$$

Then

$$\mathbb{P}(\xi = 0) = \mathbb{P}\left(\bigcap_n (\xi \leq 1/n)\right) \geq p_0.$$

If $\mathbb{P}(\xi \geq \varepsilon_0 > 0) = 1$, then

$$\mathbb{E}e^{-\lambda\xi} = \mathbb{E}\big(e^{-\lambda\xi}1(\xi \geq \varepsilon_0)\big) \leq e^{-\lambda\varepsilon_0}. \tag{2}$$

If $\mathbb{E}e^{-\lambda\xi} \leq e^{-\lambda\varepsilon_0}$, then, for every $\delta < \varepsilon_0$,

$$e^{-\lambda\varepsilon_0} \geq \mathbb{E}\left(e^{-\lambda\xi}1(\xi \leq \delta)\right) \geq e^{-\lambda\delta}\mathbb{P}(\xi \leq \delta),$$

that is

$$\mathbb{P}(\xi \leq \delta) \leq e^{-\lambda(\varepsilon_0 - \delta)},$$

or, after passing to the limit $\lambda \to +\infty$,

$$\mathbb{P}(\xi \le \delta) = 0$$

for every $\delta < \varepsilon_0$, meaning that $\mathbb{P}(\xi < \varepsilon_0) = 0$.

$\square$

Proposition 2. *Let ξ be a non-negative random variable. Then*

(1) $\lim_{\lambda\to+\infty} \mathbb{E}e^{-\lambda\xi} = p_0 > 0$ *is equivalent to* $\mathbb{P}(\xi = 0) = p_0$*;*

(2) $\lim_{\lambda\to+\infty} \frac{\ln \mathbb{E}e^{-\lambda\xi}}{\lambda} = -\varepsilon_0 < 0$ *is equivalent to* $\varepsilon_0 = \inf\big\{t > 0 : \mathbb{P}(\xi \ge t) = 1\big\}$.

Proof. If $\mathbb{P}(\xi = 0) = p_0 > 0$, then $\lim_{\lambda\to+\infty} \mathbb{E}e^{-\lambda\xi} = p_0$ after passing to the limit

$$\lim_{n\to+\infty} \lim_{\lambda\to+\infty}$$

in (1).

By the same argument, if $\lim_{\lambda\to+\infty} \mathbb{E}e^{-\lambda\xi} = p_0 > 0$, then $\mathbb{P}(\xi = 0) = p_0$.

If $\varepsilon_0 > 0$ is such that $\mathbb{P}(\xi \ge \varepsilon_0) = 1$, then (2) implies

$$\lim_{\lambda\to+\infty} \frac{\ln \mathbb{E}e^{-\lambda\xi}}{\lambda} \le -\varepsilon_0.$$

If in addition $\mathbb{P}(\varepsilon_0 \le \xi \le \varepsilon_0 + \delta) > 0$ for every $\delta > 0$, then

$$\mathbb{E}e^{-\lambda\xi} = \mathbb{E}\big(e^{-\lambda\xi}1(\xi \ge \varepsilon_0)\big) \ge e^{-\lambda(\varepsilon_0+\delta)}\mathbb{P}(\varepsilon_0 \le \xi \le \varepsilon_0 + \delta),$$

that is,

$$\lim_{\lambda\to+\infty} \frac{\ln \mathbb{E}e^{-\lambda\xi}}{\lambda} \ge -\varepsilon_0 - \delta.$$

Since $\delta > 0$ is arbitrary, we conclude that

$$\lim_{\lambda\to+\infty} \frac{\ln \mathbb{E}e^{-\lambda\xi}}{\lambda} = -\varepsilon_0. \tag{3}$$

If (3) holds, then take $\delta > 0$ and assume that $\mathbb{P}(\xi \le \varepsilon_0 - \delta) > 0$. Then

$$\mathbb{E}e^{-\lambda\xi} \ge \mathbb{E}\big(e^{-\lambda\xi}1(\xi \le \varepsilon_0 - \delta)\big) \ge e^{-\lambda(\varepsilon-\delta)}\mathbb{P}(\xi \le \varepsilon_0 - \delta).$$

Applying $\lambda^{-1}\ln(\cdot)$ to both sides and passing to the limit $\lambda \to +\infty$ leads to a contradiction $-\varepsilon_0 \ge -(\varepsilon_0 - \delta)$. Therefore, $\mathbb{P}(\xi < \varepsilon_0) = 0$. A similar contradiction results from the assumption that $\mathbb{P}(\xi < \varepsilon_0 + \delta) = 0$ for some $\delta > 0$.

$\square$

3. *Suppose that* $0 < \alpha \le 2$, *that* $X, X_1, X_2, \ldots$ *are independent and identically distributed, and that* $S_n = X_1, \cdots, X_n$. *Assume that* X *has characteristic function*

$$\phi(u) := \mathrm{E}e^{iuX} = e^{-|u|^\alpha}.$$

a) Is X symmetric (meaning X and $-X$ have the same distribution)? [Yes, because ϕ is real and so the pdf of X, which exists because ϕ is integrable, is an even function.]

b) Find a function $f(n)$, depending on n, such that $S_n/f(n)$ has the same distribution as X. [Look at the characteristic function of $S_n/f(n)$ to conclude that $f(n) = n^{1/\alpha}$.]

c) For what values of α does $S_n/\sqrt{n}$ converge in distribution to a Gaussian random variable with positive variance [Only for $\alpha = 2$.]

d) For what values of α does S_n/n converge to 0 in probability? [For $\alpha > 1$; the convergence is, in fact, with probability one.]

For parts c) and d) note, by looking at differentiabilty of ϕ at zero, that $\mathbb{E}|X| < \infty$ for $\alpha > 1$ and $\mathbb{E}X^2 < \infty$ only for $\alpha = 2$.