PDE QUALIFYING EXAM-SPRING 2022

LINFENG LI

1. Let $u \in C^2(U)$ be a subharmonic function (i.e., $\Delta u \geq 0$ in U).
   (a) Show that
$$u(x) \leq \fint_{\partial B(x,r)} u(y)dS(y)$$
   for every $x \in U$ and $r > 0$ such that $\overline{B(x,r)} \subset U$.
   (b) Show that if U is open with a C^1 boundary and $u \in C(\bar{U})$ then the maximum principle
$$\max_{\bar{U}} u = \max_{\partial U} u$$
   holds.
   (c) Show that if additionally $v \in C^2(U) \cap C(\bar{U})$ is superharmonic (i.e., $\Delta v \leq 0$ in U) and $u \leq v$ on ∂U then $u \leq v$ in U.

Solution. (a) Let
$$f(r) = \fint_{\partial B(x,r)} u(y)dS(y) = \fint_{\partial B(0,1)} u(x+rz)dS(z).$$
Taking derivative with respect to r, we obtain
$$f'(r) = \fint_{\partial B(0,1)} Du(x+rz) \cdot z dS(z),$$
from where
$$\begin{aligned} f'(r) &= \frac{1}{m(\partial B(0,1))} \int_{\partial B(0,1)} Du(y) \cdot \frac{y-x}{r} dS(y) \frac{m(\partial B(0,1))}{m(\partial B(0,r))} \\ &= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} dS(y) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y)dy \geq 0. \end{aligned}$$
In the last equality, we have used the Green's formula. Consequently, we infer that $f(r)$ is non-decreasing and thus
$$u(x) = \lim_{r \to 0} f(r) \leq \fint_{\partial B(x,r)} u(y)dS(y).$$

(b) Assume that the subharmonic function u attains a maximum at some $x_0 \in U$. There exists a constant $r > 0$ such that $B(x_0, r) \subset U$ and $u(x_0) \leq \fint_{\partial B(x_0,r)} u(y)dS(y)$. However, $u(x_0) \geq u(y)$ for any $y \in B(x_0, r)$ which implies that $u(x_0) = u(y)$ for

$y \in B(x_0, r)$. Using the above argument and choosing appropriate $r > 0$, we then obtain $\max_{\bar{U}} u = \max_{\partial U} u$.

(c) Let $w = u - v$. Then w is subharmonic since $\Delta w = \Delta u - \Delta v \geq 0$. By the maximal principal (b) we infer that

$$\max_{\bar{U}} w = \max_{\partial U} w \leq 0.$$

Consequently, we have that $u - v \leq 0$ in U. □

2. Find all classical solutions of the equation

$$u_t - \Delta u = t \sin 2x \sin 2y$$

in $U = (0,\pi)^2 \times \mathbb{R}_+$, such that

$$\begin{cases} u|_{\partial U} = 11 \\ u(x,y,0) = \sin x \sin 2y - 3 \sin 3x \sin y + 11 \quad \text{for } (x,y) \in U. \end{cases}$$

Solution. Set $v = u - 11$. The problem is reduced to solving the equation

$$v_t - \Delta v = t \sin 2x \sin 2y$$

in $U = (0,\pi)^2 \times \mathbb{R}_+$, such that

$$\begin{cases} v|_{\partial U} = 0 \\ v(x,y,0) = \sin x \sin 2y - 3 \sin 3x \sin y \quad \text{for } (x,y) \in U. \end{cases}$$

First we consider the homogeneous version of this initial value problem without the source term. We look for solution of the form $v(x,y,t) = f(t)g(x)h(y)$. Direct computation shows that

$$\frac{\partial_t f}{f} = \frac{\Delta g}{g} + \frac{\Delta h}{h} = \lambda.$$

It follows that

$$g(x) = C_1 \sin(kx) + C_2 \cos(kx)$$

and

$$h(y) = C_3 \sin(my) + C_4 \cos(my),$$

where $\lambda = -k^2 - m^2$. Moreover, we have $f(t) = C_{k,m} e^{-(k^2+m^2)t}$. Using the boundary condition, we infer that $C_2 = C_4 = 0$. Therefore, we conclude

$$v(x,y,t) = \sum_{k,m=0}^{\infty} \tilde{C_{k,m}} e^{-(k^2+m^2)t} \sin(kx)\sin(my).$$

The initial condition then implies that $\tilde{C_{1,2}} = 1$ and $\tilde{C_{3,1}} = -3$. Consequently, we arrive at

$$v(x,y,t) = e^{-5t} \sin(x)\sin(2y) - 3e^{-10t} \sin(3x)\sin(y).$$

Now we need to bring back the inhomogeneous term, which relies on the variation of parameter formula for first order ODE. □

3. Let $u(x,t)$ be a solution of the wave equation $u_{tt} = u_{xx}$ for $(x,t) \in \mathbb{R} \times (0,\infty)$ such that $u(x,0) = g(x)$, $u_t(x,0) = 0$, where $g \in C^2(\mathbb{R})$ has compact support. Show that

$$v(x,t) := \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t}} u(x,s) ds$$

solves the heat equation with initial condition $v(x,0) = g(x)$.

Solution. Using the d'Alembert formula, we get

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)).$$

Therefore,

$$\begin{aligned} v(x,t) &= \frac{1}{4\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t}} g(x+s) ds + \frac{1}{4\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t}} g(x-s) ds \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t}} g(x-s) ds, \end{aligned}$$

which solve the one-dimensional heat equation with initial data $v(x,0) = g(x)$. $\square$