PDE QUALIFYING EXAM-FALL 2022

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1. Consider a solution u to the nonlinear equation

$$\begin{cases} -\Delta u = \lambda u^2(1-u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ is a constant and Ω is a bounded domain with smooth boundary. Prove that $0 \leq u \leq 1$ in Ω.

Solution. Suppose that there exist some $x_0 \in \Omega$ such that $u(x_0) < 0$. Then $U_0 := \{x \in U : u(x) < 0\}$ is an nonempty open set. From the first equation we have that $-u$ is subharmonic in U_0 which leads to

$$\max_{\partial U_0}(-u) = \max_{\bar{U_0}}(-u) = 0.$$

Thus, we obtain $u \geq 0$ in $\bar{U_0}$ which contradicts the definition of U_0. Now, we assume there exists some $x_0 \in U$ such that $u(x_0) > 1$. Then $U_0 := \{x \in U : u(x) > 1\}$ is an nonempty open set. The first equation then implies that u is subharmonic in U_0 which leads to

$$\max_{\partial U_0} u = \max_{\bar{U_0}} u = 1.$$

However, the above equation implies that $u \leq 1$ in U_0, contradicting to the definition of U_0. In conclusion, we have $0 \leq u \leq 1$ in Ω. □

2. Let $u(x,t)$ be a smooth solution of the initial value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = g(x),\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}^3, \end{cases}$$

where $g, h \in C^\infty(\mathbb{R}^3)$ are supported in the ball $B(0,R)$.

(a) Use the Kirchoff formula to show that

$$|t \cdot u(x,t)| \le C\left(2 + \frac{1}{t}\right) |\partial B(x,t) \cap B(0,R)|$$

for all $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, where $|\cdot|$ denotes two-dimensional Lebesgue measure.

(b) Show that for some $t_0 > 0$ the measure $|\partial B(x,t) \cap B(0,R)|$ can be bounded by a constant independent of $t \ge t_0$, and conclude that

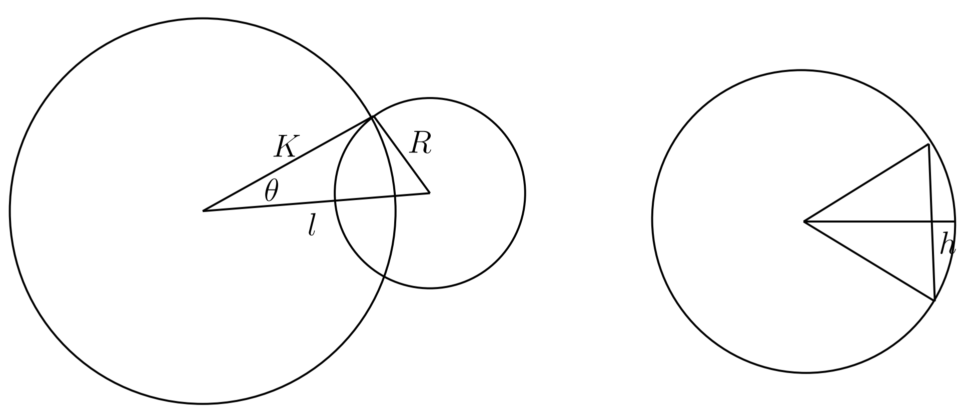
$$|u(x,t)| \le \frac{C}{t}$$

for all $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, where $C > 0$ is a constant.

Solution. (a) The Kirchoff formula implies that

$$\begin{aligned} |t \cdot u(x,t)| \le{} & \left|\fint_{\partial B(x,t)} t^2 h(y) dS(y)\right| + \left|\fint_{\partial B(x,t)} t g(y) dS(y)\right| \\ & + \fint_{\partial B(x,t)} t|Dg(y)||y-x| dS(y) \\ \le{} & C|\partial B(x,t) \cap B(0,R)| + \frac{C}{t}|\partial B(x,t) \cap B(0,R)| \\ & + C|\partial B(x,t) \cap B(0,R)| = C\left(2 + \frac{1}{t}\right)|\partial B(x,t) \cap B(0,R)|. \end{aligned}$$

(b)



The cosine formula implies that

$$R^2 = K^2 + l^2 - 2Kl\cos\theta.$$

The surface area equals

$$2\pi Kh = 2\pi K(K - K\cos\theta) = 2\pi K(K - \frac{K^2 + l^2 - R^2}{2l}).$$

Maximizing over $K - R \le l \le K + R$, where we take K sufficiently large, we get $l = \sqrt{K^2 - R^2}$. Thus, the surface area equals

$$\begin{aligned} 2\pi K(K - \frac{2K^2 - 2R^2}{2\sqrt{K^2 - R^2}}) &= 2\pi K(K - \sqrt{K^2 - R^2}) \\ &= \frac{2\pi K R^2}{K + \sqrt{K^2 - R^2}} = \frac{2\pi R^2}{1 + \sqrt{1 - (R/K)^2}} \le 2\pi R^2, \end{aligned}$$

for $K > 0$ sufficiently large. Thus, we have

$$|u(x,t)| \le C(\frac{1}{t} + \frac{1}{t^2}) \le \frac{C}{t},$$

for $t \ge t_0$. For $0 < t < t_0$, we have

$$|u(x,t)| \le \frac{C(1 + \frac{1}{t})tt_0}{t} \le \frac{C}{t},$$

where $C > 0$ is a constant depending on t_0. In conclusion, we have $|u(x,t)| \le C/t$ for all $(x,t) \in \mathbb{R}^2 \times (0,\infty)$. □

3. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $u(x,t)$ is a nonnegative function in $C^2(\bar{\Omega} \times [0,\infty))$, which solves the heat conduction equation with heat loss due to radiation

$$\begin{cases} (\partial_t - \Delta)u = -u^4 & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases}$$

Prove that we can find a constant C independent of the initial data $u(0)$, such that

$$E(1) := \int_\Omega u(x,1)^2 dx \leq C.$$

Solution. Taking the inner product of the equation with u, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \int_\Omega |\nabla u|^2 = -\int_\Omega |u|^5,$$

where we integrated by parts in space and used the Dirichlet boundary condition. By the Cauchy-Schwarz inequality, we have

$$\int_\Omega |u|^2 \leq C(\int_\Omega |u|^5)^{\frac{2}{5}},$$

where $C > 0$ is a constant depending on $|\Omega|$. Therefore, we arrive at

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq -C(\|u\|_{L^2}^2)^{\frac{5}{2}},$$

for some constant $C > 0$. Let $g(t) = \|u(t)\|_{L^2}^{-3}$. Then we have

$$g'(t) = -\frac{3}{2}(\|u(t)\|_{L^2})^{-5}\frac{d}{dt}\|u(t)\|_{L^2}^2 \geq C > 0.$$

Consequently, we get $g(1) \geq g(0) + C$, which leads to

$$(\|u(1)\|_{L^2}^2)^{\frac{3}{2}} \leq \frac{\|u(0)\|_{L^2}^3}{1 + C\|u(0)\|_{L^2}^3} \leq \frac{1}{C},$$

where $C > 0$ is independent of the initial data $u(0)$. The claim is thus proven. $\square$