# PDE QUALIFYING EXAM-SPRING 2022

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1. Solve the following initial value problems and verify your solutions
   (a) $2u_x + 3u_t = u^2, \quad u(x,0) = h(x), \ t > 0, \quad x \in \mathbb{R}, \quad$ (here $h$ is given)
   (b) $u_t = x^2 u u_x, \quad u(x,0) = x, \ t > 0, \quad x \in \mathbb{R}$
   (c) $xu_x + yu_y + u_z = u, \quad u(x,y,0) = h(x,y), \quad z > 0, \quad (x,y) \in \mathbb{R}^2 \quad$ (here $h$ is given)
   (d) $u_x^2 + u_y^2 = u^2 \quad$ (here find the characteristic equations only)

**Solution.** (a) This is a quasilinear PDE. Using the method of characteristic, we have $Du = (u_t, u_x)$, $\mathbf{b} = (2,3)$, and $c = -z^2$. It follows that

$$\begin{cases} \dot{t}(s) = 2, \ \dot{x}(s) = 3, \\ \dot{z}(s) = z^2. \end{cases}$$

Solving the above system with the initial data, we obtain

$$\begin{cases} t(s) = 2s, \ x(s) = 3s + x_0 \\ z(s) = \frac{1}{\frac{1}{z_0} - s} = \frac{1}{\frac{1}{h(x_0)} - s}. \end{cases}$$

For $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, we get

$$\begin{cases} s = \frac{t}{2} \\ x_0 = x - \frac{3t}{2}, \end{cases}$$

from where

$$u(t,x) = z(s) = \frac{1}{\frac{1}{h(x - 3t/2)} - t/2} = \frac{h(x - \frac{3t}{2})}{1 - \frac{t}{2} h(x - \frac{3t}{2})}.$$

(b) Using the method of characteristic, we have $Du = (u_t, u_x)$, $\mathbf{b} = (1, -x^2 z)$, and $c = 0$. It follows that

$$\begin{cases} t(s) = s, \ x(s) = \frac{x_0}{x_0^2 s + 1}, \\ z(s) = x_0. \end{cases}$$

For $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, we get

$$\begin{cases} s = t \\ x_0 = \frac{1 \pm \sqrt{1 - 4xt}}{2xt} \end{cases}$$

from where

$$u(t,x) = z(s) = \frac{1 \pm \sqrt{1-4xt}}{2xt}.$$

(c) We have $Du = (u_x, u_y, u_z)$, $\mathbf{b} = (x, y, 1)$, and $c = -1$. It follows that

$$\begin{cases} x(s) = x_0 e^s, \ y(s) = y_0 e^s, \ z(s) = s \\ w(s) = h(x_0, y_0) e^s. \end{cases}$$

For $(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, we get

$$\begin{cases} s = z \\ x_0 = xe^{-z}, \ y_0 = ye^{-z}, \end{cases}$$

from where

$$u(x, y, z) = z(s) = e^z h(xe^{-z}, ye^{-z}).$$

(d) This is a fully nonlinear PDE. Here $F(p, z, x) = p_1^2 + p_2^2 - z^2$, so that the characteristic becomes

$$\begin{cases} \dot{p_1} = 2zp_1, \ \dot{p_2} = 2zp_2 \\ \dot{z} = (2p_1, 2p_2) \cdot (p_1, p_2) = 2p_1^2 + 2p_2^2 \\ \dot{x} = 2p_1, \ \dot{y} = 2p_2. \end{cases}$$

□

2. Let $B$ be the unit disc in $\mathbb{R}^2$, and $\partial B$ the unit circle. Let $f$ and $g$ be two *analytic* functions defined on $\partial B$.

   (a) Prove that for any point $x \in \partial B$, there exists a neighborhood $U$ of $x$ and a function $u$ harmonic in $U \cap B$, such that $u = f$, and the outward normal derivative $\partial_\nu u = g$, on $U \cap \partial B$.

   (b) Does there always exist a function $u$ harmonic in $B$, such that $u = f$ and $\partial_\nu u = g$ on $\partial B$? Why or why not?

**Solution.** (a) This is a direct consequence of Cauchy-Kovalevskaya extension theorem.

(b) Not always. Let $f \equiv 0$ and $g \equiv 1$. If $u$ is harmonic in $B$ with $u = 0$ on $\partial B$, then $u$ must equal to zero due to the maximal principal. However, this is contradicting with the normal derivative $\partial_\nu u = 1$. $\square$

3. Let $\theta(x,t)$ be a strictly positive smooth solution of the following heat equation

$$\theta_t - \nu\theta_{xx} = 0, \qquad x \in \mathbb{R},\ t > 0$$

where $\nu > 0$ is a positive constant.

(a) Show that $u = -\frac{2\nu\theta_x}{\theta}$ satisfies

$$u_t + uu_x - \nu u_{xx} = 0, \qquad x \in \mathbb{R},\ t > 0. \tag{1}$$

(b) For $u_0 \in C_c^2$, find a solution to (1) with initial data $u(x,0) = u_0(x)$ for which $\lim_{t\to\infty} u(x,t) = 0$.

**Solution.** (a) Direct computation shows that

$$\begin{aligned} u_t + uu_x - \nu u_{xx} = &\frac{-2\nu(\theta\theta_{xt} - \theta_t\theta_x)}{\theta^2} + \frac{4\nu^2\theta_x(\theta_{xx}\theta - \theta_x^2)}{\theta^3} \\ &+ \frac{2\nu^2(\theta_{xxx}\theta^3 + \theta_x\theta_{xx}\theta^2 - 4\theta^2\theta_x\theta_{xx} + 2\theta\theta_x^3)}{\theta^4} = 0. \end{aligned}$$

(b) For $u_0 \in C_c^2$, we solve the first order linear differential equation

$$\theta_0' + \frac{\theta_0 u_0}{2\nu} = 0.$$

Moreover, we require that $\theta_0$ is compactly supported. By the fundamental solution of the heat equation, we get

$$\theta(t,x) = \int_{\mathbb{R}} K(t, x-y)\theta_0(y)dt,$$

where $K(\cdot)$ is the one-dimensional heat kernel. It follows that

$$u = -\frac{2\nu\theta_x}{\theta}$$

solves (1) with the initial data $u_0(x)$. Direct computation shows that

$$u(t,x) = \frac{1}{t}\frac{\nu\int_{\mathbb{R}}(x-y)K(t,x-y)\theta_0(y)dy}{\int_{\mathbb{R}}K(t,x-y)\theta_0(y)dy}.$$

It is clear that the numerator of the second term is bounded from above, while the denominator is bounded from below, for all $t > 0$. Consequently, we have $\lim_{t\to\infty} u(t,x) = 0$. □