PDE QUALIFYING EXAM-SPRING 2022

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1. Let $U \subset \mathbb{R}^n$ be an open set and let $u : U \to \mathbb{R}$ be a harmonic function in U such that $u(x_0) + u(y_0) = M$ for some $M \in \mathbb{R}$, and $x_0, y_0 \in U$. Show that there exists infinitely many pairs $(x, y) \in U \times U$ such that $u(x) + u(y) = M$.

Solution. Let $E \subset U$ be a connected open set containing x_0. By the maximal principal, we have $\max_{\bar{E}} u = \max_{\partial E} u \geq u(x_0)$ and $\min_{\bar{E}} u = \min_{\partial E} u \leq u(x_0)$. The equality cannot be achieved unless u is a constant function, in which case the claim is trivial. Thus, there exists some $x_1, x_2 \in \bar{E} \subset U$ such that $u(x_1) < u(x_0) < u(x_2)$. Similarly, there exists some $y_1, y_2 \in U$ such that $u(y_1) < u(y_0) < u(y_2)$. As a result, we have that

$$u(x_1) + u(y_1) < M_0 < u(x_2) + u(y_2).$$

The intermediate value theorem then implies that there exists infinitely many pairs $(x, y) \in U \times U$ such that $u(x) + u(y) = M$. $\square$

2. Let $U \subset \mathbb{R}^n$ be open and bounded and let $U_T := U \times (0, T)$. Let $u : \bar{U_T} \to \mathbb{R}$ be continuous, C^1 with respect to time variable and C^2 with respect to spatial variable, in U_T. Suppose that u satisfies

$$u_t - \Delta u \leq 0 \quad \text{in } U_T.$$

Show that

$$\max_{\bar{U_T}} u = \max_{\partial U \times [0,T] \cup U \times \{0\}} u.$$

Solution. Similar proof to Theorem 3 (mean value property for the hear equation) in Evans shows that

$$u(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t;r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} dyds.$$

Now, suppose that there exists some point $(x_0, t_0) \in U_T$ such that $u(x_0, t_0) = M = \max_{\bar{U_T}} u$. Take $r > 0$ sufficiently small so that $E(x_0, t_0; r) \subset U_T$ so that we obtain

$$\begin{aligned} M = u(x_0, t_0) &\leq \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} u(y,s) \frac{|x_0-y|^2}{(t-s)^2} dyds \\ &\leq \frac{M}{4r^n} \iint_{E(x_0,t_0;r)} \frac{|x_0-y|^2}{(t-s)^2} dyds = M, \end{aligned}$$

where we used $\iint_{E(x_0,t_0;r)} \frac{|x_0-y|^2}{(t-s)^2} dyds = 4r^n$. By continuity, u is a constant in $E(x_0, t_0; r)$. Therefore, we may extend this heat ball so that it reaches to the boundary $\bar{U_T}$. The claim is thus proven. $\square$

3. Let u be a classical solution of the problem

$$\begin{cases} u_{tt} = u_{xx} & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = g(x) & \text{for } x \in \mathbb{R}, \\ u_t(x,0) = h(x) & \text{for } x \in \mathbb{R}, \end{cases}$$

where $g, h : \mathbb{R} \to \mathbb{R}$ are smooth functions and have compact supports. Show that there exists $t_0 > 0$ such that

$$\int_{-\infty}^{\infty} u_t^2(x,t)dx = \int_{-\infty}^{\infty} u_x^2(x,t)dx$$

for all $t > t_0$.

Solution. By the d'Alembert formula, we have

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy.$$

Direct computation shows that

$$u_x = \frac{1}{2}(g'(x+t) + g'(x-t)) + h(x+t) - h(x-t)$$

and

$$u_t = \frac{1}{2}(g'(x+t) - g'(x-t)) + h(x+t) + h(x-t).$$

It is sufficient to prove that

$$\int_{-\infty}^{\infty} (u_t - u_x)(u_t + u_x)dx = 0,$$

which is equivalent to

$$I := \int_{-\infty}^{\infty} (g'(x+t) + 2h(x+t))(g'(x-t) - 2h(x-t))dx = 0.$$

There exists a constant $t_0 > 0$ such that g' and h are compactly supported in $[-t_0, t_0]$. It is clear that

$$\int_{-\infty}^{0} (g'(x+t) + 2h(x+t))(g'(x-t) - 2h(x-t))dx = 0$$

and

$$\int_{0}^{\infty} (g'(x+t) + 2h(x+t))(g'(x-t) - 2h(x-t))dx = 0$$

for $t > t_0$. The claim is then proved. $\square$