PDE QUALIFYING EXAM-SPRING 2020

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1. Let $u(x, y)$ be a harmonic function on $\mathbb{R}^2$, and suppose that
$$\iint_{\mathbb{R}^2} |\nabla u(x, y)|^2 dxdy < \infty.$$
Show that u is a constant function.

Solution. Since $\nabla u(x, y)$ is also harmonic, the mean value property holds for $\nabla u(x, y)$. Namely, we have
$$\nabla u(x) = \frac{\iint_{B(x,r)} \nabla u(t, s)dtds}{m(B(x, r))}, \qquad r > 0, \qquad x \in \mathbb{R}^2,$$
where $m(\cdot)$ denotes the two-dimensional Lebesgue measure. We appeal to the Cauchy-Schwarz inequality, obtaining
$$\begin{aligned} |\nabla u(x)| &= \frac{|\iint_{B(x,r)} \nabla u(t, s)dtds|}{m(B(x, r))} \\ &\leq m(B(x, r))^{-1} (\iint_{B(x,r)} |\nabla u(t, x)|^2 dtds)^{\frac{1}{2}} (\iint_{B(x,r)} 1dtds)^{\frac{1}{2}} \\ &\leq m(B(x, r))^{-\frac{1}{2}} C, \end{aligned}$$
for some constant $C > 0$, since $\iint_{\mathbb{R}^2} |\nabla u(x, y)|^2 dxdy < \infty$. Sending $r \to \infty$, we infer that $\nabla u(x) = 0$ for all $x \in \mathbb{R}^2$. Therefore, u is a constant function. $\square$

2. Solve the following Cauchy problem

$$\partial_t u - x\partial_x u + u - 1 = 0, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}$$
$$u(0,x) = \cos x, \qquad x \in \mathbb{R}.$$

Discuss the behavior of $|\partial_x u(t,x)|$ for $t \to \infty$.

Solution. This is a quasilinear type PDE. Using the characteristic method, we have $Du(x,t) = (u_x, u_t)$ and $\mathbf{b} = (-x, 1)$. Moreover, we have $c = z^2 - 1$. Then the system becomes

$$\begin{cases} \dot{x} = -x, \ \dot{t} = 1, \\ \dot{z} = 1 - z^2. \end{cases}$$

Consequently,

$$\begin{cases} x(s) = e^{-s} + x_0 - 1, \ t(s) = s, \\ z(s) = \frac{1+Ce^{-2s}}{1-Ce^{-2s}}. \end{cases}$$

Plug in $z(0) = (1+C)/(1-C) = \cos x_0$, we get $C = (\cos x_0 - 1)/(\cos x_0 + 1)$. Thus, we arrive at

$$z(s) = \frac{e^{2s}(\cos x_0 + 1) + \cos x_0 - 1}{e^{2s}(\cos x_0 + 1) - (\cos x_0 - 1)}.$$

Now, we take $(x,t) \in \mathbb{R} \times \mathbb{R}_+$. Then we have

$$\begin{cases} x = e^{-s} + x_0 - 1 \\ t = s, \end{cases}$$

from where $s = t$ and $x_0 = x - e^{-t} + 1$. Consequently, we arrive at

$$u(x,t) = z(s) = \frac{e^{2t}(\cos(x - e^{-t} + 1) + 1) + \cos(x - e^{-t} + 1) - 1}{e^{2t}(\cos(x - e^{-t} + 1) + 1) - \cos(x - e^{-t} + 1) + 1}.$$

(b) Taking the derivative of the above solution, we conclude that $|\partial_x u|$ behaves like

$$\frac{2\sin(x+1)}{\cos(x+1)}$$

for $t \to \infty$. □

3. Consider the bounded smooth solution u to the linear heat equation

$$\partial_t u = \Delta u, \qquad u(0,x) = f(x)$$

on $[0,\infty) \times \mathbb{R}^d$, with initial data $f \in L^2(\mathbb{R}^d)$.

(a) Prove that $\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}$ for $t \geq 0$.

(b) Prove that $\lim_{t\to+\infty} \|u(t)\|_{L^2(\mathbb{R}^d)} = 0$.

Solution. (a) Taking the inner product of equation and u, we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u|^2 dx - \int_{\mathbb{R}^d} u\Delta u dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} |u|^2 dx + \int_{\mathbb{R}^d} |\nabla u|^2 dx = 0,$$

where we integrated by parts in space. Integrating the above identity in time from 0 to t, we obtain

$$\int_{\mathbb{R}^d} |u(t)|^2 dx \leq \int_{\mathbb{R}^d} |u(0)|^2 dx = \int_{\mathbb{R}^d} f^2 dx.$$

Thus, we have $\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}$ for $t \geq 0$.

(b) I think the problem should be phrased as compactly supported solution in $\mathbb{R}^d$. From Poincare inequality we infer that

$$\|u\|_{L^2(\mathbb{R}^d)} \leq C\|\nabla u\|_{L^2(\mathbb{R}^d)},$$

for some constant $C > 0$. The energy identity in (a) then implies that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{C}\|u\|^2_{L^2(\mathbb{R}^d)} \leq 0.$$

By the Gronwall inequality, we get

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} \leq \|u(0)\|^2_{L^2(\mathbb{R}^d)} e^{-Ct},$$

for some constant $C > 0$. As a result, we have $\lim_{t\to+\infty} \|u(t)\|_{L^2(\mathbb{R}^d)} = 0$. □