PDE QUALIFYING EXAM-FALL 2020

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1. Let $u(x, y)$ be a smooth function satisfying

$$\Delta u(x) = f(u(x)), \quad x \in B(0,1)$$
$$u(x) = 0, \quad x \in \partial B(0,1),$$

where $B(0,1)$ is the unit ball in $\mathbb{R}^n$ and f is continuous.

(a) Show that if $f(t)$ has the same sign of t, then u must be identically zero in $B(0,1)$.

(b) What can you say about the solution when $f(t) = t^4$.

Solution. (a) Suppose that there exists some $x_0 \in B(0,1)$ such that $u(x_0) \neq 0$. Without loss of generality, we assume that $u(x_0) > 0$ and hence $f(u(x_0)) > 0$. Let $U = \{x \in B(0,1) : u(x) > 0\}$ be an open set. It is clear that $U \neq \emptyset$ and $-\Delta u < 0$ on U. By the maximal principal of the subharmonic function, we have

$$\max_{\bar{U}} u = \max_{\partial U} u > 0.$$

However, the boundary ∂U either lies on $\partial B(0,1)$ or $u(x) = 0$ for $x \in \partial U$ (this can be justified by assuming otherwise and use the continuity argument). So we arrive at a contradiction from the maximal principal.

(b) When $f(t) = t^4$, the function u is subharmonic in $B(0,1)$. Maximal principle then implies that $\max_{B(0,1)} u \leq 0$. □

2. Suppose $u = u(t,x)$ is smooth and bounded, and solves the nonlinear heat equation

$$(\partial_t - \Delta)u = |\nabla u|^2, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$$
$$u(0,x) = f(x), \quad x \in \mathbb{R}^n.$$

(a) Prove that $v = e^u$ solves the linear heat equation $(\partial_t - \Delta)v = 0$.
(b) Find an explicit formula for u in terms of f.

Solution. (a) Direct computation shows that

$$\partial_t v - \Delta v = e^u \partial_t u - e^u \Delta u - e^u |\nabla u|^2 = 0.$$

As a consequence, $v = e^u$ solves the linear heat equation $(\partial_t - \Delta)v = 0$.

(b) Since $v = e^u$ solves

$$\partial_t v - \Delta v = 0$$
$$v(0,x) = e^{f(x)}.$$

By the fundamental solution of the heat equation, we have

$$v(t,x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} e^{f(y)} dy, \qquad x \in \mathbb{R}^n, t > 0.$$

Therefore,

$$u(t,x) = -\frac{n}{2} \log(4\pi t) + \log(\int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} e^{f(y)} dy)$$

is an explicit solution. □

3. Suppose $u = u(t,x)$ solves the following initial-boundary value problem

$$\begin{aligned}
(1+t)\partial_t^2 u - \Delta u + \partial_t u = 0, &\quad (t,x) \in \mathbb{R}_+ \times \Omega \\
u(t,x) = 0, &\quad (t,x) \in \mathbb{R}_+ \times \partial\Omega \\
u(0,x) = u_0(x), &\quad x \in \Omega \\
\partial_t u(0,x) = u_1(x), &\quad x \in \Omega
\end{aligned}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^3$, and $u_0(x)$ and $u_1(x)$ are smooth and compactly supported in Ω.

(a) Show that the L^2 norm of the solution u is bounded for all $0 < t < \infty$.
(b) What can you say about the L^2 norm of $\partial_t u$ as $t \to \infty$.

Solution. (a) Taking the inner product of the first equation with $\partial_t u$, we obtain

$$\int_\Omega (1+t)\partial_t^2 u \partial_t u - \int_\Omega \Delta u \partial_t u + \int_\Omega |\partial_t u|^2 = 0.$$

Using integration by parts and the Dirichlet boundary condition (the second equation), we arrive at

$$\int_\Omega (1+t)\frac{1}{2}\frac{d}{dt}|\partial_t u|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |\partial_t u|^2 = 0,$$

from where

$$\frac{d}{dt}\int_\Omega (1+t)|\partial_t u|^2 + \frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |\partial_t u|^2 = 0.$$

Integrating in time from 0 to τ, we get

$$\begin{aligned}
\int_\Omega (1+\tau)|\partial_t u(\tau)|^2 + \int_\Omega |\nabla u(\tau)|^2 &\le \int_\Omega |\partial_t u(0)|^2 + \int_\Omega |\nabla u(0)|^2 \\
&= \|u_1\|_{L^2}^2 + \|\nabla u_0\|_{L^2}^2.
\end{aligned} \tag{1}$$

Therefore, we infer that $\|\partial_t u(\tau)\|_{L^2}^2 \le C/(1+\tau)$, for some constant $C > 0$. Using the fundamental theorem of calculus, we have

$$u(T) = u(0) + \int_0^T \partial_t u(\tau) d\tau.$$

From the Jensen's inequality it follows that

$$\begin{aligned}
\|u(T)\|_{L^2}^2 &\le C\|u_0\|_{L^2}^2 + C\int_0^T \|\partial_t u(\tau)\|_{L^2}^2 d\tau \le C + \int_0^T \frac{C}{1+\tau} d\tau \\
&\le C + C\log(1+T).
\end{aligned}$$

Consequently, the L^2 norm of the solution u is bounded for all $0 < t < \infty$.

(b) From (1) we infer that $\|\partial_t u(\tau)\|_{L^2}^2 \le C/(1+\tau)$. Thus, the L^2 norm of $\partial_t u$ tends to 0 as $t \to \infty$. □