# PDE QUALIFYING EXAM-SPRING 2019

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1. Let $\Omega$ be a bounded domain (open, connected) in $\mathbb{R}^n$. Suppose that $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is a solution of
$$\Delta u + \sum_{k=1}^{n} a_k(x)\partial_{x_k} u + c(x)u = 0$$
where $a_k$, $c \in C(\bar{\Omega})$ and $c < 0$ in $\Omega$. Show that if $u = 0$ on $\partial\Omega$, then $u = 0$ in all of $\Omega$.

**Solution.** Assume $u \not\equiv 0$ in all of $\Omega$, since $u = 0$ on $\partial\Omega$, there exist some point $x_0 \in \Omega$ such that $u(x_0)$ attains maximum or minimum in $\bar{\Omega}$. Without loss of generality, $u(x_0)$ is a maximum, then $Du(x_0) = 0$ and $D^2u(x_0)$ is negative definite. From the equation,
$$\Delta u(x_0) + c(x_0)u(x_0) = 0$$
but $\Delta u(x_0) \leq 0$ and $c(x) < 0$ and $u(x_0) > 0$, we have a contradiction. $\square$

2. Solve the equation

$$\partial_t u + x\partial_x u + u = 0$$

with initial data $u(0,x) = f(x)$, where $f$ is a compactly supported smooth function on $\mathbb{R}$. Sketch the graph of $u(t,x)$ as a function of $x$ for a non-trivial compactly supported initial datum of your choice for $t = 0$, $t = R$ and $t = -R$ for $R$ large (what happens to the support and amplitude)?

**Solution.** This PDE is linear and has the form

$$F(Du, u, x) = \mathbf{b}(x) \cdot Du(x) + c(x)u(x) = 0,$$

where $\mathbf{b}(x) = (1, x)$, $Du(x) = (\partial_t u, \partial_x u)$ and $c(x) \equiv 1$. Then the characteristics are

$$\dot{\mathbf{x}}(s) = (\dot{t}(s), \dot{x}(s)) = (1, x(s)),$$
$$\dot{z}(s) = -c(\mathbf{x}(s))z(s) = -z(s),$$

therefore, we have the characteristics emanating from $(0, x_0)$ is

$$\begin{cases} t(s) = s + t(0) = s \\ x(s) = x(0)e^s = x_0 e^s \end{cases}$$

on which we have $u(t(s), x(s)) = z(s) = z(0)e^{-s} = f(x_0)e^{-s}$.

Consider the bump function

$$f(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & \text{otherwise} \end{cases}$$

which is a smooth function with compact support. Then for $t = 0$, $u(0, x) = f(x)$ and for $t = R$ for $R$ large, we have

$$u(R, x) = \begin{cases} \exp(-R - \frac{1}{1-x^2 e^{-2R}}), & |x| < e^R \\ 0, & \text{otherwise} \end{cases}$$

which has bigger support than $|x| < 1$ and smaller amplitude. Similarly,

$$u(-R, x) = \begin{cases} \exp(R - \frac{1}{1-x^2 e^{2R}}), & |x| < e^{-R}, \\ 0, & \text{otherwise} \end{cases}$$

has smaller support and bigger amplitude than initial datum. $\square$

3. Suppose $u(t,x)$ is a function on $\mathbb{R}\times\mathbb{R}^3$ that satisfies the nonlinear wave equation

$$(\partial_t^2 - \Delta)u = u^3.$$

Assume that $u \in C^2(\mathbb{R}\times\mathbb{R}^3)$ and that $u(t,x)$ is compactly supported in $x$ for each $t$. Define the energy

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u(t,x)|^2 + (\partial_t u(t,x))^2)dx.$$

(a) Prove that

$$\partial_t E(t) = \int_{\mathbb{R}^3} u(t,x)^3 \partial_t u(t,x)dx, \text{ and that } \partial_t E(t) \le \|\partial_t u(t)\|_{L^2(\mathbb{R}^3)} \cdot \|u(t)\|^3_{L^6(\mathbb{R}^3)}.$$

(b) Prove that there exists a universal constant $C$ independent of $u$ such that

$$\partial_t E(t) \le CE(t)^2.$$

**Solution.**

(a) Since $u$ is compactly supported in $x$ for each $t$, then $u(t,x)$ vanishes outside some $B_R(0)$ for each $t$, and

$$\begin{aligned}\dot{E}(t) &= \int_{\mathbb{R}^3} \nabla u \cdot \partial_t \nabla u + \partial_t u \partial_t^2 u dx \\ &= \int_{\mathbb{R}^3} \partial_t u(-\Delta u + \partial_t^2 u)dx = \int_{\mathbb{R}^3} u(t,x)^3 \partial_t u(t,x)dx.\end{aligned}$$

By Cauchy-Schwartz inequality, we have

$$\dot{E}(t) \le \|\partial_t u(t)\|_{L^2(\mathbb{R}^3)} \cdot \|u(t)\|^3_{L^6(\mathbb{R}^3)}$$

(b) By Sobolev inequality, there exists constant $C$ such that

$$\|u\|_{L^6} \le C\|Du\|_{L^2},$$

so by discrete Young's inequality,

$$\begin{aligned}\dot{E}(t) &\le \|\partial_t u(t)\|_{L^2}\|u(t)\|^3_{L^6} \le C\|\partial_t u(t)\|_{L^2}\|Du(t)\|^3_{L^2} \\ &\le C(\frac{\|\partial_t u\|^4_{L^2}}{4} + \frac{\|Du(t)\|^4_{L^2}}{4/3}) \le CE(t)^2.\end{aligned}$$

$\square$