PDE QUALIFYING EXAM-FALL 2019

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1. Suppose f is a compactly supported smooth function on $\mathbb{R}^3$. Prove that there is a unique smooth function u on $\mathbb{R}^3$, such that
$$-\Delta u = f, \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0.$$
For this u, find the value of
$$\lim_{|x|\to\infty} |x|u(x).$$

Solution. First of all, suppose there are two smooth functions u_1 and u_2 satisfying both conditions, let $v = u_1 - u_2$ then
$$-\Delta v = 0,$$
$$\lim_{|x|\to\infty} v(x) = 0.$$
We have that v is harmonic and bounded on R^3, by Liouville's theorem v is a constant function. $\lim_{|x|\to\infty} v(x) = 0$ implies $v \equiv 0$, uniqueness follows. To show existence, define
$$u(x) = \Phi * f = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy,$$
where $\Phi(x) = \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|}$ is locally integrable, since f is a compactly supported smooth function, we have that $\Phi * f$ is smooth. Now suppose f is compactly supported in $B_R(0)$, then for $|x| > 2R$ we have
$$\begin{aligned}
|u(x)| &\le C \int_{B_R(0)} \frac{|f(y)|}{|x-y|} dy \le C \int_{B_R(0)} \frac{|f(y)|}{|x|-|y|} dy \\
&\le CM \int_0^R \int_{\partial B(0,r)} \frac{1}{|x|-|y|} dS dr \\
&= CM \int_0^R \frac{1}{|x|-r} r^2 dr \le CM \int_0^R \frac{r^2}{|x|/2} dr \\
&= CM \frac{1}{|x|} \int_0^R r^2 dr \to 0 \text{ as } |x| \to \infty,
\end{aligned}$$
where constants above are absorbed into C. Furthermore,
$$\begin{aligned}
\Delta u(x) = &\int_{B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy + \int_{R^3 \setminus B(0,\epsilon)} \Phi(y) \Delta_x f(x-y) dy \\
:= &I_1 + I_2,
\end{aligned}$$
where
$$|I_1| \le \int_{B(0,\epsilon)} \|D^2 f\|_{L^\infty} |\Phi(y)| dy \le C\epsilon^2 \to 0 \text{ as } \epsilon \to 0. \tag{1}$$

Also,

$$I_2 = -\int_{\mathbb{R}^3\setminus B(0,\epsilon)} D\Phi(y) D_y f(x-y)dy + \int_{\partial B(0,\epsilon)} \Phi(y)\frac{\partial f}{\partial \nu}(x-y)dy := J_1 + J_2,$$

where

$$\begin{aligned} J_1 &= \int_{\mathbb{R}^3\setminus B(0,\epsilon)} \Delta\Phi(y) f(x-y)dy - \int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)dS(y) \\ &= -\int_{\partial B(0,\epsilon)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)dS(y) \\ &= -\frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(0,\epsilon)} f(x-y)dS(y) \\ &= -⨍_{\partial B(x,\epsilon)} f(y)dS(y) \to -f(x), \text{ as } \epsilon \to 0. \end{aligned} \tag{2}$$

and

$$|J_2| \leq \|Df\|_{L^\infty} \int_{\partial B(0,\epsilon)} |\Phi(y)| dS(y) \leq C\epsilon. \tag{3}$$

Combining (1)–(3), and letting $\epsilon \to 0$, we have that $-\Delta u = f$. In addition, for $|x|$ sufficiently large, $|\frac{x}{x-y}| f(y) \to f(y)$ pointwise since f is compactly supported smooth function, by dominated convergence theorem

$$\begin{aligned} |x|u(x) &= \frac{1}{n(n-2)\alpha(n)} \int_{B(0,R)} |\frac{x}{x-y}| f(y)dy \to \frac{1}{n(n-2)\alpha(n)} \int_{B(0,R)} f(y)dy \\ &= \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^3} f(y)dy, \text{ as } |x| \to \infty. \end{aligned}$$

□

2. Consider the following one-dimensional heat equation:

$$\begin{cases} \partial_t u = \partial_x^2 u, & (t,x) \in (0,+\infty) \times (0,1), \\ u = 0, & (t,x) \in (0,+\infty) \times \{0,1\}. \end{cases}$$

Find all solutions that have factorized form $u(t,x) = \alpha(t)\beta(x)$.

Solution. From the first equation, if we have the solution of the form $u(t,x) = \alpha(t)\beta(x)$, then we have

$$\alpha'(t)\beta(x) = \alpha(t)\beta''(x),$$

suppose $\alpha(t)$ and $\beta(x)$ are not zero, then

$$\frac{\alpha'(t)}{\alpha(t)} = \frac{\beta''(x)}{\beta(x)} = C$$

since the identity is independent of t and x. To solve the α, we have $\alpha(t) = \alpha(0)e^{Ct}$, and for β we need to solve second order ODE, and the solution $\beta'' = C\beta$:

case1: If $C > 0$, then $\beta(x) = c_1 e^{\sqrt{C}x} + c_2 e^{-\sqrt{C}x}$, plug in the boundary condition we have $u(t,x) \equiv 0$.

case2: If $C = 0$, then $\alpha(t) = 0$ and $\beta(x) = c_1 x + c_2$, plugging in boundary condition we have $\beta \equiv 0$ so $u \equiv 0$.

case3: If $C < 0$, then $\beta(x) = c_1 \cos(\sqrt{-C}x) + c_2 \sin(\sqrt{-C}x)$, plugging in boundary condition, we have

$$u(t,x) = Ce^{-k^2\pi^2 t} \sin(k\pi x), \text{ where } C \text{ is any real number and } k \text{ is any integer.}$$

□

3. Suppose u solves the following initial-boundary value problem:

$$\begin{cases} \partial_t^2 u = \partial_x^2 u - u^3, & (t,x) \in (0,+\infty) \times (0,1), \\ u = 0, & (t,x) \in (0,+\infty) \times \{0,1\}, \\ u(0,x) = u_0(x), & x \in (0,1), \\ \partial_t u(0,x) = u_1(x), & x \in (0,1), \end{cases}$$

where u_0, u_1 are smooth functions.

(a) Find an energy $E(t)$ which is independent of t.

(b) Show that u is bounded for all (t,x), namely $|u(t,x)| < C$ for some constant C for all $(t,x) \in (0,\infty) \times (0,1)$.

Solution.

(a) Let

$$E(t) = \frac{1}{2} \int_0^1 u^4(x) dx + \int_0^1 |\partial_x u(x)|^2 dx + \int_0^1 |\partial_t u(x)|^2 dx,$$

then we have

$$\begin{aligned} \dot{E}(t) &= 2 \int_0^1 u^3 \partial_t u dx + 2 \int_0^1 \partial_x u \partial_t \partial_x u dx + 2 \int_0^1 \partial_t u \partial_{tt} u dx \\ &= 2 \int_0^1 \partial_t u (u^3 - \partial_x^2 u + \partial_t^2 u) dx = 0 \end{aligned}$$

(b) From part (a) we have $E(t) = E(0) = \frac{1}{2} \int_0^1 |u_0(x)|^4 dx + \int_0^1 |\partial_x u_0(x)|^2 dx + \int_0^1 |u_1(x)|^2 dx \leq C$ since u_0 and u_1 are smooth functions on compact set $[0,1]$. For any $t > 0$, by fundamental theorem of calculus,

$$u(t,x) = u(t,0) + \int_0^x \partial_x u(t,y) dy,$$

therefore, using Cauchy-Schwartz inequality we have

$$|u(t,x)| \leq \int_0^1 |\partial_x u(t,y)| dy \leq (\int_0^1 1^2 dy)(\int_0^1 |\partial_x u(t,y)|^2 dy) \leq C,$$

where the last inequality follows from $E(t) \leq C$.

□