PDE QUALIFYING EXAM-SPRING 2018

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1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded with normal vector field ν and let $u_0 \in C_b(\Omega)$ with $u_0 \geq 0$ be non-trivial. Show that the problem

$$\partial_t u - \Delta u = u^2 \quad \text{in} \quad \Omega \times (0, T), \tag{1}$$

$$\partial_\nu u = 0 \quad \text{on} \quad \partial\Omega \times (0, T), \tag{2}$$

$$u(x, 0) = u_0(x) \quad \text{for} \quad x \in \Omega, \tag{3}$$

exists for at most a finite time T.
Hint: Show that the mean $m(t) = \frac{1}{|\Omega|}\int_\Omega u(t,x)dx$ satisfies $\partial_t m(t) \geq m^2(t)$.

Solution. Let $m(t) = \frac{1}{|\Omega|}\int_\Omega u(t,x)dx$, using equation (1)–(2) and Holder's inequality

$$\begin{aligned}
\partial_t m(t) &= \frac{1}{|\Omega|}\int_\Omega \partial_t u dx = \frac{1}{|\Omega|}\int_\Omega \Delta u + u^2 dx \\
&= \frac{1}{|\Omega|}\int_{\partial\Omega} \partial_\nu u dS + \frac{1}{|\Omega|}\int_\Omega u^2 dx = \frac{1}{|\Omega|^2}\int_\Omega 1 dx \int_\Omega u^2 dx \\
&\geq \frac{1}{|\Omega|^2}(\int_\Omega u dx)^2 = m^2(t)
\end{aligned}$$

Then we can solve the ODE for $v(t)$ which satisfies

$$\begin{aligned}
&\partial_t v = v^2, \\
&v(0) = m(0) > 0.
\end{aligned}$$

The solution is $v(t) = \frac{1}{1/v(0) - t}$, which blows up in finite time. By the proof of Gronwall's lemma, we know $m(t) \geq v(t)$ for $t \geq 0$, therefore the solution exists at most a finite time. □

2. Let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded and let $R > 0$ such that $\Omega \subset B_R(0)$.

(a) Let $v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ with $\Delta v = 0$ in Ω. Show that

$$\max_{x \in \bar{\Omega}} v(x) = \max_{x \in \partial\Omega} v(x)$$

(b) Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be a solution of

$$\begin{aligned} -\Delta u &= 1 \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Show that

$$0 \le u(x) \le \frac{R^2 - |x|^2}{2n}$$

for all $x \in \bar{\Omega}$.

Solution.

(a) Suppose there exists a point $x_0 \in U$ with $u(x_0) = M = \max_{\bar{\Omega}} u$, since v is harmonic in Ω we can apply the mean value theorem for $B(x_0, r)$ within Ω

$$M = v(x_0) = \fint_{B(x_0,r)} v dy \le M,$$

equality holds if and only if $v \equiv M$ within $B(x_0, r)$. Hence the set $\{x \in \Omega | v(x) = M\}$ is open, it's also closed because it's the preimage of singleton by a continuous function v. It has to be either $\emptyset$ or Ω since Ω is connected, it's not empty because we assumed at least x_0 is in this set, then it has to be Ω. By continuity of v up to the boundary we have $\max_{x \in \partial\Omega} = M$. On the other hand, if there is no such x_0 then maximum is obtained on the boundary.

(b) Let $v = u + \frac{|x|^2}{2n}$, then $\Delta v = \Delta u + \frac{2n}{2n} = \Delta u + 1 = 0$ in Ω, with boundary condition $v \le 0 + \frac{R^2}{2n}$ on $\partial\Omega$. Part (a) applies, and we have

$$\max_{x \in \bar{\Omega}} v(x) = \max_{x \in \partial\Omega} v(x) \le \frac{R^2}{2n}$$

therefore, $\forall x \in \bar{\Omega}$, $u(x) + \frac{|x|^2}{2n} \le \frac{R^2}{2n}$. On the other hand, let

$$f(r) := \fint_{\partial B(x,r)} u(y) dS(y) = \fint_{\partial B(0,1)} u(x + rz) dS(z)$$

then we have

$$\begin{aligned} f'(r) &= \fint_{\partial B(0,1)} Du(x + rz) \cdot z dS(z) = \fint_{\partial B(0,1)} Du(y) \cdot \frac{y - x}{r} dS(z) \\ &= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu} dS(y) = \frac{r}{n} \fint_{B(x,r)} \Delta u(y) dy \le 0 \end{aligned}$$

which implies $f(r)$ is decreasing and we also have

$$u(x) = \lim_{s \to 0} f(s) = \lim_{s \to 0} \fint_{\partial B(x,s)} u(y) dS(y) \ge \fint_{\partial B(x,r)} u(y) dS(y)$$

therefore,

$$\int_0^r n\alpha(n) s^{n-1} u(x) ds \ge \int_0^r \int_{\partial B(x,s)} u(y) dS(y) ds = \int_{B(x,r)} u(y) dy$$

consequently,

$$u(x) \geq \fint_{B(x,r)} u(y)dy, \text{ for all } B(x,r) \subset \Omega$$

Now we assume there exists $x_0 \in \Omega$ such that $u(x_0) \leq u(x)$, $\forall x \in \bar{\Omega}$ and $u(x_0) < 0$, by the above inequality we have

$$u(x_0) \geq \fint_{B(x_0,r)} u(y)dy \geq \fint_{B(x_0,r)} u(x_0)dy = u(x_0)$$

equality holds when $u(x_0) = u(y)$ for all $y \in B(x_0, r)$, and we choose r such that $\partial\Omega \cap B(x_0, r) = \{x_1\}$, therefore by continuity $u(x_1) < 0$ and we have a contradiction. And we proved $u(x) \geq 0$ for all $x \in \Omega$, combining it with boundary condition we have $u(x) \geq 0$, for all $x\ in \bar{\Omega}$.

$\square$

3. Let u be a classical solution of the following initial boundary value problem:

$$u_t = u_{xx}, \quad \text{in} \quad (a,b) \times (0,T) \tag{4}$$
$$u(a,t) = u(b,t) = 0 \tag{5}$$
$$u(x,0) = u_0(x) \tag{6}$$

where u_0 is a continuous function.

(a) Show that the solutions are unique.

(b) Show that there exists a constant $\alpha > 0$ such that

$$\|u(\cdot,t)\|_{L^2}^2 \le e^{-\alpha t}\|u_0\|_{L^2}^2.$$

Solution.

(a) Suppose there are two solutions v_1 and v_2, let $u = v_1 - v_2$ and let $m(t) = \int_a^b |u(t,x)|^2 dx$, then $m'(t) = 2\int_a^b u u_{xx} dx = -\int_a^b |u_x|^2 dx \le 0$, $m(0) = 0$ but $m(t) \ge 0$, therefore $m(t) = 0$ for any t, and we have $v_1 \equiv v_2$.

(b) Multiply equation (4) by u and integrate over (a,b), we have

$$\frac{1}{2}\frac{d}{dt}\int_a^b |u|^2 dx = -\int_a^b |u_x|^2 dx$$

Since the domain is bounded and u vanishes on the boundary, we can apply Poincare's inequality, there exists constant C such that

$$\int_a^b |u|^2 dx \le C\int_a^b |u_x|^2 dx$$

Combining above two, we arrive at

$$m'(t) \le -Cm(t).$$

Using Gronwall's lemma, we have

$$m(t) \le m(0)e^{-Ct}$$

as desired.

□