PDE QUALIFYING EXAM-FALL 2018

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1. Let Ω be open and bounded and let $g_j \in C(\partial\Omega)$ converge uniformly to $g \in C(\partial\Omega)$ (recall that this means that $\lim_{j\to\infty} \sup_{x\in\partial\Omega} |g_j(x) - g(x)| \to 0$). Let $u_j \in C^2(\Omega) \cap C(\bar{\Omega})$ be the solution of

$$\begin{aligned} \Delta u_j &= 0 \text{ in } \Omega, \\ u_j &= g_j \text{ on } \partial\Omega. \end{aligned}$$

Show that u_j converges uniformly to a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and that u solves

$$\begin{aligned} \Delta u &= 0 \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

Solution. First we claim u_j converges uniformly to some function $u \in C(\bar{\Omega})$. For m, n sufficiently large, let $v = u_m - u_n$ then we have

$$\begin{aligned} \Delta v &= 0 \text{ in } \Omega, \\ v &= g_m - g_n \text{ on } \partial\Omega. \end{aligned}$$

Since we have $\sup_{x\in\partial\Omega} |g_m(x) - g_n(x)| \to 0$ for m, n sufficiently large, and by maximal principal,

$$\max_{x\in\bar{\Omega}} v(x) = \max_{x\in\partial\Omega} g_m(x) - g_n(x) \to 0, \text{ as } m, n \to \infty$$

Similarly we have

$$\min_{x\in\bar{\Omega}} v(x) = \min_{x\in\partial\Omega} g_m(x) - g_n(x) \to 0, \text{ as } m, n \to \infty$$

Consequently, $v(x)$ converges uniformly to 0 in $\bar{\Omega}$ as $m, n \to \infty$, and we have $\{u_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $(C(\bar{\Omega}), \|\cdot\|_{C^0})$. Since $(C(\bar{\Omega}), \|\cdot\|_{C^0})$ is complete, there exists $u \in C(\bar{\Omega})$ such that u_m converges uniformly to u in $\bar{\Omega}$. Now we claim $u(x)$ satisfies the mean value property and therefore u is harmonic in Ω, it's clear that $u = g$ on $\partial\Omega$. For given $x \in \Omega$,

$$u_m(x) = \fint_{\partial B(x,r)} u_m(y)dS(y), \text{ for any } r > 0 \text{ with } B(x,r) \subset \Omega$$

For sufficiently large m, we have $u_m(x) \to u(x)$, and $\sup_{y\in\bar{\Omega}} |u_m(y) - u(y)| \to 0$, therefore $u(x) = \fint_{\partial B(x,r)} u(y)dS(y)$ for all $r > 0$ with $B(x,r) \subset \Omega$, therefore $u(x)$ satisfies mean value property so $\Delta u = 0$ in Ω. □

2. Let $u_0 \in C^2(B_1(0))$, $u_1 \in C^1(B_1(0))$, $f \in C((0,T) \times B_1(0))$. Show that the problem

$$\begin{aligned} \partial_{tt}u - \Delta u + u &= f \text{ in } (0,T) \times B_1(0), \\ u(0,x) &= u_0(x), \\ \partial_t u(0,x) &= u_1(x), \\ u(t,x) &= 0 \text{ on } (0,T) \times \partial B_1(0), \end{aligned}$$

has at most one solution $u \in C^2([0,T] \times \overline{B_1(0)})$.

Solution. Suppose there are two solutions, and we can do the subtraction and claim that the equation with $u \in C^2([0,T] \times \overline{B_1(0)})$

$$\partial_{tt}u - \Delta u + u = 0 \text{ in } (0,T) \times B_1(0), \tag{1}$$

$$u(0,x) = 0, \tag{2}$$

$$\partial_t u(0,x) = 0, \tag{3}$$

$$u(t,x) = 0 \text{ on } (0,T) \times \partial B_1(0), \tag{4}$$

has only trivial solution $u \equiv 0$. Define energy

$$E(t) = \|\partial_t u(t)\|_{L^2}^2 + \|Du\|_{L^2}^2 + \|u\|_{L^2}^2$$

Then we have

$$\begin{aligned} \dot{E}(t) &= 2\langle \partial_t u, \partial_{tt} u \rangle + 2\langle Du, D\partial_t u \rangle + 2\langle u, u_t \rangle \\ &= 2\langle \partial_t u, \partial_{tt} u - \Delta u + u_t \rangle + \int_{\partial B_1(0)} Du \partial_t u dS \\ &= 0 \end{aligned}$$

where the last equality follows from (1) and (4). Thus, $E(t)$ is conserved for this equation and we have $E(t) = E(0) = 0$, therefore $u(t) \equiv 0$ for any $t \geq 0$, and claim is proved. □

3. Consider the equation

$$\partial_t u + u\partial_x u = 0$$

on $(0,T)\times\mathbb{R}$. Show that a classical solution with initial data $u(0,x) = \frac{\pi}{2} - \arctan(x)$ can exist at most for a finite time.

Solution. This equation has the form

$$F(Du,u,x) = \mathbf{b}(x,u(x))\cdot Du(x) = (1,u(x))\cdot(u_t,u_x) = 0$$

We have $D_pF = (1,z)$ and characteristics are

$$\dot{\mathbf{x}}(\mathbf{s}) = (1,z),$$
$$\dot{z}(s) = D_pF\cdot\mathbf{p}(s)$$

For any given $(x_0,0)\in\mathbb{R}\times\{t=0\}$, the characteristics emanating from it is $x = x_0 + (\frac{\pi}{2} - \arctan x_0)t$, on which we have $u(t,x) = \frac{\pi}{2} - \arctan x_0$. Suppose there is global solution, since we can find x_0 and x_1 such that $x_0 \neq x_1$ and for some $t_0 > 0$ we have

$$x_0 + (\frac{\pi}{2} - \arctan x_0)t_0 = x_1 + (\frac{\pi}{2} - \arctan x_1)t_0$$
$$\iff \frac{\arctan x_0 - \arctan x_1}{x_0 - x_1} = \frac{1}{t_0}$$

which follows from mean value theorem. Then we have two characteristics intersecting at some $t = t_0$, with two different values $\frac{\pi}{2} - \arctan x_0$ and $\frac{\pi}{2} - \arctan x_1$, contradiction! Therefore, solution can only exist at most for a finite time. □