PDE QUALIFYING EXAM-FALL 2017

LINFENG LI

1. Consider the Burger's equation

$$u_t + uu_x = 0, \quad x \in \mathbb{R},\ t > 0,$$
$$u(x,0) = g(x), \quad x \in \mathbb{R}.$$

where $g(x)$ is a given piecewise continuous function.

(a) Show that the characteristics are straight lines.
(b) Give an example of $g(x)$ so that the characteristics do not cover the entire (x,t) space.
(c) Give an example of $g(x)$ so that the characteristics intersect.

Solution.

(a) This equation is quasilinear and has the form

$$F(Du, u, x) = \mathbf{b}(x, u(x)) \cdot Du(x) = (1, u(x)) \cdot (u_t, u_x) = 0$$

In this case $D_pF = (1, z)$ and characteristics are

$$\dot{\mathbf{x}}(s) = (1, z),$$
$$\dot{z}(s) = D_pF \cdot \mathbf{p}(s) = 0,$$

For any $(x_0, 0) \in \mathbb{R} \times \{t = 0\}$, the characteristics emanating from it is $x = x_0 + g(x_0)t$, which is a straight line.

(b) For $g(x) = -x$ we have characteristics $x = x_0 - x_0 t$, which are straight lines passing through (1,0), they do not cover entire (t, x) space because they do not cover points such as $(1, 1)$.

(c) For $g(x) = \sin x$, choose $x_0 = \pi$ and $x_0 = \omega$, for some $\omega \in (\pi, 2\pi)$. Then we have two characteristics $x = \pi$ and $x = \omega + t \sin \omega$, by intermediate value theorem we can find some $t_0 > 0$ such that these two lines intersect.

□

2. Let $U \subset \mathbb{R}^n$ be an open set.

(a) Let $u \in C^2(U)$. Show that for any ball $\bar{B}(x_0, r) \subset U$ it holds

$$\frac{d}{dr} \fint_{\partial B(0,1)} u(x_0 + rz) dS(z) = \frac{r}{n} \fint_{B(0,1)} (\Delta u)(x_0 + rz) dz$$

Here we have used the notation $\fint_A f = \frac{1}{|A|} \int_A f$.

(b) Let $u \in C^2(U)$ be such that for any ball $\bar{B}(x_0, r) \subset U$ it holds

$$u(x_0) = \fint_{\partial B(x_0, r)} u dS.$$

Show that then $\Delta u = 0$ in U.

(c) Does the implication of part (b) still hold if you just assume $u \in C(U)$? Briefly explain your answer.

Solution.

(a)

$$\begin{aligned}
\frac{d}{dr} \fint_{\partial B(0,1)} u(x_0 + rz) dS(z) &= \fint_{\partial B(0,1)} Du(x_0 + rz) \cdot z dS(z) \\
&= \fint_{\partial B(x_0,r)} Du(y) \cdot \frac{y - x_0}{r} dS(y) \\
&= \fint_{\partial B(x_0,r)} Du(y) \cdot \nu dS(y) \\
&= \fint_{\partial B(x_0,r)} \frac{\partial u}{\partial \nu}(y) dS(y) \\
&= \fint_{\partial B(x_0,r)} \Delta u(y) dS(y) \\
&= \frac{r}{n} \fint_{B(0,1)} \Delta u(x_0 + rz) dz
\end{aligned}$$

(b) Assume there exist $x_0 \in U$ such that $\Delta u(x_0) > 0$. Since $u \in C^2(U)$, there exists $r > 0$ such that $\Delta u(x) > 0$ for all $x \in B(x_0, r)$. Define $\phi(r) = \fint_{\partial B(x_0,r)} u(y) dS(y)$, then from part (a) we know $\phi'(r) = \frac{r}{n} \fint_{B(x_0,r)} \Delta u(y) dy > 0$, but $u(x_0) = \phi(r)$ for all $r > 0$, therefore $\phi'(r) = 0$. Contradiction! By symmetry we can similarly prove there is no $x_0 \in U$ such that $\Delta u(x_0) < 0$.

(c) It still hold if we only assume $u \in C(U)$. In fact we can use standard mollifier to prove that $u \in C^\infty(U_\epsilon)$ for every ϵ, and then we proceed as in part (b).

□

3. Let U be the unit ball in R^n.

(a) For $u(x) = |x|^{-a}$ for $x \in U$, determine the values of a, n, p for which u belongs to the Sobolev space $W^{1,p}(U)$.

(b) Let $n \geq 2$. If $u(x) = \ln\ln(1 + \frac{1}{|x|})$ for $x \in U$, show that $u \in W^{1,n}(U)$ but not in $L^\infty(U)$.

Solution.

(a) $u(x) = |x|^{-a}$ is in $L^p(U)$ if and only if $ap \leq n$, $Du(x) = -a|x|^{-a-2}x$ is in $L^p(U)$ if and only if $(a+1)p \leq n$. Therefore $u \in W^{1,p}(U)$ if and only if $(a+1)p \leq n$.

(b) For $x \in B(0,1)$, we have

$$|\ln\ln(1 + \frac{1}{|x|})| \leq C(1 + \frac{1}{|x|}),$$

and by part (a)

$$\int_{B(0,1)} |u(x)|^n dx = \int_{B(0,1)} |\ln\ln(1 + \frac{1}{|x|})|^n dx \leq C \int_{B(0,1)} (1 + \frac{1}{|x|})^n dx$$
$$= C \int_0^1 \int_{\partial B(0,r)} (1 + \frac{1}{|x|})^n dS dr = C \int_0^1 (1 + \frac{1}{r})^n r^{n-1} dr < \infty$$

which shows $u \in L^n$. Furthermore,

$$Du(x) = \frac{1}{\ln(1 + \frac{1}{|x|})} \frac{1}{1 + \frac{1}{|x|}} (-1)|x|^{-3}x,$$

so

$$|Du(x)| = \frac{1}{\ln(1 + \frac{1}{|x|})} \frac{1}{|x| + 1} |x|^{-1} \leq C|x|^{-1},$$

By part (a) we know $Du \in L^n$, therefore $u \in W^{1,n}$. $u(x)$ is not in L^∞ since we can take $|x|$ sufficiently small so that $u(x)$ is arbitrarily large.

$\square$

4. (a) Let $U \subset \mathbb{R}^n$ be a bounded open set, let $T > 0$ be fixed and define $U_T = U \times (0, T)$. Assume $u \in C_1^2(\bar{U_T})$ solves the following initial boundary value problem

$$u_t - \Delta u = f \quad \text{in} \quad U_T, \tag{1}$$

$$\frac{\partial u}{\partial \nu} + u = h \quad \text{on} \quad \partial U \times (0, T), \tag{2}$$

$$u = g \quad \text{on} \quad U \times \{t = 0\}, \tag{3}$$

where f, g and h are given smooth functions, and ν is the outward pointing unit normal field of ∂U. Prove that there exists at most one such solution.

(b) Let $U \subset \mathbb{R}^2$ be a bounded open set and let $a > 0$, $b, c \in \mathbb{R}$ be given constants. Show that any solution $u \in C^2(\bar{U})$ of

$$\Delta u - au + b\partial_x u + c\partial_y u = 0 \quad \text{in } U,$$

cannot attain a positive maximum or negative minimum inside U.

Solution.

(a) Suppose there are two solutions $v_1(t, x)$ and $v_2(t, x)$ satisfying (3)–(5), we define $u(t, x) = v_1(t, x) - v_2(t, x)$, then

$$u_t - \Delta u = 0 \quad \text{in} \quad U_T, \tag{4}$$

$$\frac{\partial u}{\partial \nu} + u = 0 \quad \text{on} \quad \partial U \times (0, T), \tag{5}$$

$$u = 0 \quad \text{on} \quad U \times \{t = 0\}, \tag{6}$$

multiply (4) by u and integrate over U, we have $\langle u_t, t \rangle - \langle \Delta u, u \rangle = 0$. Integrate by part to the second term and use the boundary condition (5), we arrive at

$$\frac{1}{2} \frac{d}{dt} \int_U |u|^2 = - \int_{\partial U} |u|^2 - \int_U |Du|^2,$$

integrating from 0 to t, we have

$$\frac{1}{2} \int_U |u(t)|^2 - \frac{1}{2} \int_U |u(0)|^2 = - \int_0^t \int_{\partial U} |u(s)|^2 dS ds - \int_0^t \int_U |Du(s)|^2 dx ds$$

By the initial condition (6), the second term on the left side of above equality vanishes, and thus we have $u = 0$ on ∂U and $|Du| = 0$ since $u \in C_1^2(\bar{U_T})$, and we have $v_1 \equiv v_2$.

(b) Suppose there is a solution u that attains a positive maximum at some $x_0 \in U$, then $u(x_0) > 0$, $Du(x_0) = 0$ and $\Delta u(x_0) \leq 0$, but we have

$$\Delta u(x_0) = au(x_0)$$

Contradiction! By symmetry we can prove there is no negative minimum inside U.

□