Numerical Analysis Preliminary Examination
Monday February 3, 2014

Work all problems and show all your work for full credit. This exam is closed book, closed notes, no calculator or electronic devices of any kind.

1. (a) Let $\{f_k\}_{k=1}^{n}$ be n linearly independent real valued functions in $L_2(a,b)$, and let Q be the $n \times n$ matrix with entries $Q_{i,j} = \int_a^b f_i(x) f_j(x) dx$. Show that Q is positive definite symmetric and therefore invertible.

   (b) Let g be a real valued functions in $L_2(a,b)$ and find the best (in $L_2(a,b)$) approximation to g in span$\{f_k\}_{k=1}^{n}$.

2. Let A be a 3×3 nonsingular matrix which can be reduced to the matrix

$$U = \begin{bmatrix} 1 & u_1 & u_2 \\ 0 & 1 & u_3 \\ 0 & 0 & 1 \end{bmatrix}$$

   using the following sequence of elementary row operations:

   (i) α_1 times Row 1 is added to Row 2.
   (ii) α_2 times Row 1 is added to Row 3.
   (iii) Row 2 is multiplied by $\frac{1}{\alpha_3}$.
   (iv) α_4 times Row 2 is added to Row 3.

   (a) Find an LU decomposition for the matrix A.

   (b) Let $b = [b_1 \quad b_2 \quad b_3]^T$ be an arbitrary vector in R^3 and let the vector $x = [x_1 \quad x_2 \quad x_3]^T$ in R^3 be the unique solution to the linear system $Ax = b$. Find an expression for x_3 in terms of the α_i's, the b_i's, and the u_i's, $i = 1,2,3$.

3. In this problem we consider the iterative solution of the linear system of equations $Ax = b$ with the following $(n-1) \times (n-1)$ matrices

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

(a) Show that the vectors $x^k = \left(\sin\frac{\pi k}{n},\ \sin\frac{2\pi k}{n},\ \cdots,\ \sin\frac{\pi(n-1)k}{n}\right)$, for $k = 1, \cdots, n-1$ are eigenvectors of B_J, the Jacobi iteration matrix corresponding to the matrix A given above.

(b) Determine whether or not the Jacobi's method would converge for all initial conditions x^0.

(c) Let L and U be, respectively, the lower and upper triangular matrices with zero diagonal elements such that $B_J = L + U$, and show that the matrix $\alpha L + \alpha^{-1} U$ has the same eigenvalues as B_J for all $\alpha \neq 0$.

(d) Show that an arbitrary nonzero eigenvalue, λ, of the iteration matrix

$$H(\omega) = (I - \omega L)^{-1}\big((1-\omega)I + \omega U\big)$$

for the Successive Over Relaxation (SOR) method satisfies the following equation

$$\lambda^2 - 2(1-\omega)\lambda - \mu^2\omega^2\lambda + (1-\omega)^2 = 0,$$

where μ is an eigenvalue of B_J (Hint: use the result of (c)).

(e) For $n = 4$, find the spectral radius of $H(1)$.

4. (a) Find the singular value decomposition (SVD) of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

(b) Let $\{\lambda_k\}$ and $\{\sigma_k\}$ be the sets of eigenvalues and singular values of $n \times n$ matrix A. Show that: $\min_k \sigma_k \le \min_k |\lambda_k|$ and $\max_k \sigma_k \ge \max_k |\lambda_k|$.

(c) Let A be a full column rank $m \times n$ matrix with singular value decomposition $A = U\Sigma V^*$, where V^* indicates the conjugate transpose of V.

(1) Compute the SVD of $A(A^*A)^{-1}A^*$ in terms of U, Σ, and V.

(2) Let $||\cdot|| = sup_{x\neq 0}\frac{||Ax||_2}{||x||_2}$ be the matrix norm induced by the vector 2-norm, and let σ_{max} be the largest singular value of A. Show that $||A|| = \sigma_{max}$.

Sp 2014:

1. a) By definition, we see that $Q_{ij} = Q_{ji}$ $\forall i, j$. So, Q is symmetric. Let $y \in \mathbb{R}^n$ be nonzero, and observe

$$y^T Q y = y^T \begin{bmatrix} \sum_{j=1}^n q_{1j} y_j \\ \vdots \\ \sum_{j=1}^n q_{nj} y_j \end{bmatrix} = \sum_{i=1}^n y_i \left(\sum_{j=1}^n q_{ij} y_j \right) = \sum_{i=1}^n \sum_{j=1}^n y_i q_{ij} y_j$$

$$= \sum_{i=1}^n \sum_{j=1}^n \int_a^b y_i f_i(x) f_j(x) y_j \, dx = \int_a^b \sum_{i=1}^n \sum_{j=1}^n (y_i f_i(x) y_j f_j(x)) dx$$

$$= \int_a^b \sum_{i=1}^n (y_i f_i(x)) \sum_{j=1}^n (y_j f_j(x)) \, dx = \int_a^b \left(\sum_{i=1}^n y_i f_i(x) \right)^2 dx \geq 0$$

(Also since each $f_i(x) \in L_2(a,b)$, the integral is finite).

The integral cannot be zero, because if this is the case, we have $\left(\sum_{i=1}^n y_i f_i(x) \right)^2 = 0$ (since it is positive) which implies $\sum_{i=1}^n y_i f_i(x) = 0$. By linear independence we get $y_i = 0$ $\forall i$, meaning $y = 0$, contradicting to the assumption. So, $y^T Q y > 0$ $\forall y \neq 0$. Thus Q is SPD, and so invertible.

b) Since $\{f_k\}_{k=1}^n$ is linearly independent, it is a basis for span$\{f_k\}_{k=1}^n$. Let $\tilde{g}$ denote the best approximation to g in span$\{f_k\}_{k=1}^n$. So,

$$\tilde{g}(x) = c_1 f_1(x) + \dots + c_n f_n(x) \qquad (*)$$

for constants $c_1, c_2, \dots, c_n$. We need to find those constants. Let we have n different interpolation points. Then we must have

$$g(x_i) = c_1 f_1(x_i) + \dots + c_n f_n(x_i) \qquad \forall i = 1, 2, \dots, n.$$

Then we get

$$\begin{bmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_n) \end{bmatrix}$$

Since f_i's are linearly indep and x_i's are distinct the matrix (call F) is invertible. So, we can decide c_i's uniquely and $(*)$ with these c_i's represents the best approximation in span $\{f_k\}_{k=1}^n$ to $g(x)$. Note that the "quality" of approximation depends also on the choice of interpolation nodes. ②

2. a) $\ell_{21} = -\alpha_1$ $\quad \ell_{31} = -\alpha_2$ $\quad \ell_{32} = -\dfrac{\alpha_4}{\alpha_3}$

So, $L = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha_1 & 1 & 0 \\ -\alpha_2 & -\alpha_4/\alpha_3 & 1 \end{bmatrix}$ $\quad U = \begin{bmatrix} 1 & u_1 & u_2 \\ 0 & 1 & u_3 \\ 0 & 0 & 1 \end{bmatrix}$

$Ax = b \iff LUx = b$ Let $y = Ux$ Then

$y_1 = b_1$

$y_2 = \alpha_1 y_1 + b_2 = \alpha_1 b_1 + b_2$

$y_3 = \alpha_2 y_1 + \dfrac{\alpha_4}{\alpha_3} y_2 + b_3 = \alpha_2 b_1 + \dfrac{\alpha_4 \alpha_1 b_1}{\alpha_3} + \dfrac{\alpha_4 b_2}{\alpha_3} + b_3$

Now, from $Ux = y$ we get $x_3 = \alpha_2 b_1 + \dfrac{\alpha_4 \alpha_1 b_1}{\alpha_3} + \dfrac{\alpha_4 b_2}{\alpha_3} + b_3$

3 $\quad A = \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{bmatrix}$

a) $D = \begin{bmatrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{bmatrix}$ $\quad L = \begin{bmatrix} 0 & & \\ -1 & \ddots & \\ & \ddots & \ddots \\ & & -1 & 0 \end{bmatrix}$ $\quad U = \begin{bmatrix} 0 & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & 0 \end{bmatrix}$ $\quad D^{-1} = \begin{bmatrix} 1/2 & & 0 \\ & \ddots & \\ 0 & & 1/2 \end{bmatrix}$

$L + U = \begin{bmatrix} 0 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 0 \end{bmatrix}$ $\quad B_j = -D^{-1}(L+U) = \begin{bmatrix} -1/2 & & \\ & \ddots & \\ & & -1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & & \\ 1/2 & 0 & \ddots & \\ & \ddots & \ddots & 1/2 \\ & & 1/2 & 0 \end{bmatrix}$

Claim: $B_j x^k = \lambda_k x^k$ for $k = 1, \dots, n-1$ and some λ_k

$$B_j x^k = \begin{bmatrix} \frac{1}{2}\sin\frac{2\pi k}{n} \\ \frac{1}{2}\left(\sin\frac{\pi k}{n} + \sin\frac{3\pi k}{n}\right) \\ \frac{1}{2}\left(\sin\frac{2\pi k}{n} + \sin\frac{4\pi k}{n}\right) \\ \vdots \\ \frac{1}{2}\left(\sin\frac{(n-3)\pi k}{n} + \sin\frac{(n-1)\pi k}{n}\right) \\ \frac{1}{2}\sin\frac{\pi(n-2)k}{n} \end{bmatrix}$$

* $\frac{1}{2}\sin\frac{2\pi k}{n} = \cos\frac{\pi k}{n}\sin\frac{\pi k}{n}$

* $\sin a + \sin b = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$

For $m = 1, 2, \dots, n-3$,

$\frac{1}{2}\left[\sin\frac{m\pi k}{n} + \sin\frac{(m+2)\pi k}{n}\right] = \cos\frac{\pi k}{n}\sin\frac{(m+1)\pi k}{n}$

* $\frac{1}{2}\sin\frac{\pi(n-2)k}{n} = \frac{1}{2}\sin\left[\frac{\pi(n-1)k}{n} - \frac{\pi k}{n}\right]$

$= \frac{1}{2}\left[\sin\left(\frac{\pi(n-1)k}{n}\right)\cos\frac{\pi k}{n} - \cos\left(\frac{\pi(n-1)k}{n}\right)\sin\left(\frac{\pi k}{n}\right)\right]$

$$= \frac{1}{2}\left[\sin\frac{(n-1)\pi k}{n}\cos\frac{\pi k}{n} + \cos\frac{\pi k}{n}\sin\frac{(n-1)\pi k}{n}\right] = \cos\frac{\pi k}{n}\sin\frac{(n-1)\pi k}{n}$$ ③

So, we see

$$B_j x^k = \cos\frac{\pi k}{n}\begin{bmatrix}\sin\frac{\pi k}{n}\\ \sin\frac{2\pi k}{n}\\ \vdots \\ \sin\frac{(n-1)\pi k}{n}\end{bmatrix} = \cos\frac{\pi k}{n}\, x^k$$

So, $\cos\frac{\pi k}{n}$ $k = 1, 2, \ldots, n-1$ are the corresponding eigenvalues.

b) $\rho(B_j) = \max_{k\in\{1,\ldots,n-1\}} |\lambda_k(B_j)| = \max_{k\in\{1,\ldots,n-1\}}\left|\cos\frac{\pi k}{n}\right| = \cos\frac{\pi}{n} < 1$

Since $\rho(B_j) < 1$ we say B_j is a convergent matrix That is $\lim_{k\to\infty} B_j^k = 0$ So, the Jacobi's method would converge for all initial conditions.

c) $$L = \begin{bmatrix} 0 & & & 0 \\ 1/2 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1/2 & 0\end{bmatrix} \qquad U = \begin{bmatrix} 0 & 1/2 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1/2 \\ 0 & & & 0\end{bmatrix}$$

Let $\alpha \neq 0$ and see

$$\alpha L + \frac{1}{\alpha}U = \begin{bmatrix} 0 & 1/2\alpha & & 0 \\ \alpha/2 & 0 & \ddots & \\ & \ddots & \ddots & 1/2\alpha \\ 0 & & \alpha/2 & 0\end{bmatrix}$$

let us prove it by induction on n, where n is the dimension of the matrix

For $n=1$, there is nothing to prove.

Let D_k represent the char. poly. of $\alpha L + \alpha^{-1}U$ for $k\times k$ matrix.

For $n=2$, $D_2 = \begin{vmatrix} -\lambda & 1/2\alpha \\ \alpha/2 & -\lambda\end{vmatrix} = \lambda^2 - \frac{1}{4}$ is the char. poly. of $\alpha L + \frac{1}{\alpha}U$

$$\begin{vmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda\end{vmatrix} = \lambda^2 - \frac{1}{4}$$ is the char. poly. of B_j

So, B_j and $\alpha L + \frac{1}{\alpha}U$ have the same eigenvalues when $n=2$.

Suppose they have same eigenvalues (so their char. poly. are the same) for all $n \leq k-1$ for any $k\in\mathbb{N}$.

Now, we want to show it for $n=k$

$$D_k = \begin{vmatrix} -\lambda & 1/2\alpha & & \\ \alpha/2 & -\lambda & \ddots & \\ & \ddots & \ddots & 1/2\alpha \\ & & \alpha/2 & -\lambda \end{vmatrix} = -\lambda D_{k-1} - \frac{1}{2\alpha}\,\frac{\alpha}{2} D_{k-2}$$

$$= -\lambda D_{k-1} - \frac{1}{4} D_{k-2}$$

On the other hand for $k \times k$ B_j^k,

$$|B_j^k - \lambda I| \quad \begin{vmatrix} -\lambda & 1/2 & & \\ 1/2 & -\lambda & \ddots & \\ & \ddots & \ddots & 1/2 \\ & & 1/2 & -\lambda \end{vmatrix} = -\lambda |B_j^{k-1} - \lambda I| - \frac{1}{2} \cdot \frac{1}{2} |B_j^{k-2} - \lambda I|$$

$$= -\lambda |B_j^{k-1} - \lambda I| - \frac{1}{4} |B_j^{k-2} - \lambda I|.$$

But, by the assumption $D_{k-1} = |B_j^{k-1} - \lambda I|$ and $D_{k-2} = |B_j^{k-2} - \lambda I|$.

So, $D_k = |B_j^k - \lambda I|$. Thus B_j^k and $\alpha L + \frac{1}{\alpha} U$ have the same eigenvalues

d) By result of part (c), we say that the matrices $\alpha L + \alpha^{-1} U$ have the same eigenvalues for any $\alpha \neq 0$. Then, let $\lambda \neq 0$ be an eigenvalue of $H(\omega)$. So, $\det(\lambda I - H(\omega)) = 0$. Since $\det(I - \omega L) = 1$, we have

$$0 = \det(I - \omega L)\det(\lambda I - H(\omega)) = \det(\lambda I - \lambda\omega L - (1-\omega)I - \omega U)$$

$$= \det((\lambda - (1-\omega))I - \lambda\omega L - \omega U) = \det\left[\frac{\lambda - 1 + \omega}{\omega} - \lambda L - U\right]$$

$$= \det\left[\frac{\lambda - 1 + \omega}{\omega\lambda^{-1/2}} - (\lambda^{1/2} L + \lambda^{-1/2} U)\right]$$

So, $\dfrac{\lambda - 1 + \omega}{\omega\lambda^{-1/2}}$ is an eigenvalue of $\lambda^{1/2} L + \lambda^{-1/2} U$. But by (c), $\dfrac{\lambda - 1 + \omega}{\omega\lambda^{-1/2}}$ is an eigenvalue of $L + U$. So, $\mu = \dfrac{\lambda - 1 + \omega}{\omega\lambda^{-1/2}}$, On the other hand, again by (c), if μ is an eigenvalue of B_j then $-\mu$ is also an eigenvalue of B_j. So,

$$\mu = \frac{\lambda - 1 + \omega}{\omega\lambda^{-1/2}} \quad \text{and} \quad \mu = -\frac{\lambda - 1 + \omega}{\omega\lambda^{-1/2}} \implies [\lambda - (1-\omega)] - \mu\omega\lambda^{1/2} = 0$$

$$[\lambda - (1-\omega)] + \mu\omega\lambda^{1/2} = 0$$

So multiplying $[\lambda - (1-\omega)]^2 - \mu^2\omega^2\lambda = 0 \implies \lambda^2 - 2(1-\omega)\lambda - \mu^2\omega^2\lambda + (1-\omega)^2 = 0.$

e) Let us compute the eigenvalues of $B_j = L + U$ where

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{vmatrix} -\mu & -1 & 0 & 0 \\ -1 & -\mu & -1 & 0 \\ 0 & -1 & -\mu & -1 \\ 0 & 0 & -1 & -\mu \end{vmatrix} = \mu^4 - 3\mu^2 - 1$$

Eigenvalues are $\pm \frac{3+\sqrt{5}}{2}$ and $\pm \frac{3-\sqrt{5}}{2}$ Eigenvalues of $H(1)$ satisfy

$$\lambda^2 - \mu^2 \lambda = 0 \quad \text{for} \quad \mu = \frac{3+\sqrt{5}}{2}, \ \mu = \frac{3-\sqrt{5}}{2}$$

$\rightarrow \lambda_1 = 0, \ \lambda_2 = \left(\frac{3+\sqrt{5}}{2}\right)^2, \ \lambda_3 = 0, \ \lambda_4 = \left(\frac{3-\sqrt{5}}{2}\right)^2$

So, $\rho(H(1)) = \left(\frac{3+\sqrt{5}}{2}\right)^2$

4. a)
$$A^T A = \begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 10 \end{bmatrix}$$

Eigenvalues of $A^T A$:

$$\begin{vmatrix} 9-\lambda & 3 \\ 3 & 10-\lambda \end{vmatrix} = \lambda^2 - 19\lambda + 81 = 0 \Rightarrow \lambda_{1,2} = \frac{19 \pm \sqrt{361-324}}{2}$$

$$= \frac{19 \pm \sqrt{37}}{2}$$

6

b) $\max_k \sigma_k \geq \max_k |\lambda_k|$:

assume v is the eigenvector corresp. to max. eigenvalue of A

$$\longrightarrow \quad (\max_k \sigma_k)^2 = \|A\|_2^2 = \max_{x \in \mathbb{R}^n} \frac{\|Ax\|_2^2}{\|x\|_2^2} \geq \frac{\|Av\|_2^2}{\|v\|_2^2}$$

$$= \frac{(\max_k |\lambda_k|)^2 \|v\|_2^2}{\|v\|_2^2} = (\max_k |\lambda_k|)^2$$

$$\Rightarrow \max_k \sigma_k \geq \max_k |\lambda_k|$$

$\min_k \sigma_k \leq \min_k |\lambda_k|$:

If A is singular, then we know $\min_k \sigma_k = 0 = \min_k |\lambda_k|$, and the inequality satisfied

Now, suppose A is nonsingular, in this case all eigenvalues and singular values are nonzero. then max singular value of A^{-1} is $\frac{1}{\min_k \sigma_k}$ and max eigenvalue of A^{-1} is $\frac{1}{\min_k |\lambda_k|}$. Then, by using first part

$$\frac{1}{\min_k \sigma_k} \geq \frac{1}{\min_k |\lambda_k|} \Rightarrow \min_k \sigma_k \leq \min_k |\lambda_k|.$$

c) Let A be a full column rank $m \times n$ matrix (so $m \geq n$), with $A = U \Sigma V^*$

1. $A^* = V \Sigma^* U^*$ and $A^* A = V \Sigma^* \Sigma V^*$. Then

$$(A^*A)^{-1} = V (\Sigma^* \Sigma)^{-1} V^*$$

$$A(A^*A)^{-1}A^* = U \Sigma V^* V (\Sigma^* \Sigma)^{-1} V^* V \Sigma^* U^*$$

$$= U \Sigma (\Sigma^* \Sigma)^{-1} \Sigma^* U^*$$

$$\Sigma^* \Sigma = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} \qquad \Sigma (\Sigma^* \Sigma)^{-1} \Sigma^* = \begin{bmatrix} I_{n \times n} & 0 \\ 0 & 0 \end{bmatrix}_{m \times m} =: I'$$

So, $A(A^*A)^{-1}A^* = U I' U^*$

2. Let $\sigma_{max} = \sigma_1$ wlog. Then, observe

$$\|A\|^2 = \sup_{x \neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2}$$

$$= \sup_{x \neq 0} \frac{\|U\Sigma V^* x\|_2^2}{\|x\|_2^2}$$

$$= \sup_{x \neq 0} \frac{\|\Sigma V^* x\|_2^2}{\|x\|_2^2}$$ by norm-invariance under unitary matrix multip.

$$= \sup_{y \neq 0} \frac{\|\Sigma y\|_2^2}{\|Vy\|_2^2}$$ Since V^* is unitary, it is nonsingular and for any $x \neq 0$ $\exists y \neq 0$ s.t. $V^*x = y$. So, we can replace V^*x by y and $V^*x = y \Rightarrow x = Vy$

$$= \sup_{y \neq 0} \frac{\|\Sigma y\|_2^2}{\|y\|_2^2}$$ by norm-invariance

$$= \sup_{y \neq 0} \frac{\sum_{i=1}^{n} \sigma_i^2 y_i^2}{\sum_{i=1}^{n} y_i^2}$$

Since $\sigma_1 = \sigma_{max}$, we see for any $y \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} \sigma_i^2 y_i^2 \leq \sum_{i=1}^{n} \sigma_1^2 y_i^2 = \sigma_1^2 \sum_{i=1}^{n} y_i^2 \Rightarrow \frac{\sum_{i=1}^{n} \sigma_i^2 y_i^2}{\sum_{i=1}^{n} y_i^2} \leq \sigma_1^2 \quad , \forall y \in \mathbb{R}^n$$

So, $$\|A\|^2 = \sup_{y \neq 0} \frac{\sum_{i=1}^{n} \sigma_i^2 y_i^2}{\sum_{i=1}^{n} y_i^2} \leq \sigma_1^2.$$

On the other hand, choosing $y = (1, 0, 0, \dots, 0)^T$, we see that

$$\frac{\sum_{i=1}^{n} \sigma_i^2 y_i^2}{\sum_{i=1}^{n} y_i^2} = \sigma_1^2$$

So, the supremum is achieved, implying that

$$\|A\|^2 = \sup \frac{\sum_{i=1}^{n} \sigma_i^2 y_i^2}{\sum_{i=1}^{n} y_i^2} = \sigma_1^2$$

Thus, $\|A\| = \sigma_{max}$.

Preliminary Exam in Numerical Analysis Fall 2013

Instructions

The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. **Linear systems**

   For $x \in R^n$ and $M \in R^{n \times n}$ let $\|x\|$ denote the norm of x, and let $\|M\|$ denote the corresponding induced matrix norm of M. Let $S \in R^{n \times n}$ be nonsingular and define a new norm on R^n by $\|x\|_S = \|Sx\|$.

(a) Show that $\| \quad \|_S$ is in fact a norm on R^n.

(b) Show that $\| \quad \|_S$ and $\| \quad \|$ are equivalent norms on R^n.

(c) Show that the induced norm of $M \in R^{n \times n}$ with respect to the $\| \quad \|_S$ norm is given by $\|M\|_S = \|SMS^{-1}\|$.

(d) Let $\kappa(M)$ denote the condition number of $M \in R^{n \times n}$ with respect to the $\| \quad \|$ norm, let $\kappa_S(M)$ denote the condition number of $M \in R^{n \times n}$ with respect to the $\| \quad \|_S$ norm and show that $\kappa_S(M) \le \kappa(S)^2 \kappa(M)$.

2. **Least squares**

   (a) Assume you observe four (x, y) data points: (0,1), (1, 1), (-1, -1), (2,0). You want to fit a parabola of the form $y = a + bx^2$ to these data points that is best in the least squares sense. Derive the normal equations for this problem and put them in matrix vector form (you do not need to solve the equations).

   (b) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ and consider the linear system $Ax = b$, for $b \in R^3$. Find the QR or SVD decomposition of A and the rank of A.

   (c) For a given $b \in R^3$, state the condition such that the equation in part (b) has a solution, and the condition such that the solution is unique.

   (d) Find the pseudoinverse of the matrix A given in part (b).

   (e) For $b = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$ find the solution x to the system given in part (b).

3. **Iterative Methods**

   Consider the stationary vector-matrix iteration given by

$$x_{k+1} = Mx_k + c \qquad (1)$$

   where $M \in C^{n \times n}$, $c \in C^n$, and , $x_0 \in C^n$ are given.

(a) Let $r(M)$ denote the spectral radius of the matrix M and show that if $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$, then $r(M) < 1$.

Now consider the linear system

$$Ax = b \tag{2}$$

where $A \in C^{n\times n}$ nonsingular and $b \in C^n$ are given.

(b) Derive the matrix $M \in C^{n\times n}$ and the vector $c \in C^n$ in (1) in the case of the Gauss-Seidel iteration for solving the linear system given in (2).

(c) Derive the matrix $M \in C^{n\times n}$ and the vector $c \in C^n$ in (1) in the case of the Successive Over Relaxation Method (SOR) with parameter θ for solving the linear system given in (2). (Hint: Use your answer in part (b) and write D as $D = \frac{1}{\theta}D + \left(1 - \frac{1}{\theta}\right)D$.)

(d) Show that if for the SOR method, $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$, then it is necessary that $\theta \in (0,2)$.

4. Computation of Eigenvalues and Eigenvectors

Let A be a nondefective $n \times n$ matrix with eigenvalues, $\lambda_1, \lambda_2, \dots, \lambda_n$, with $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$, and corresponding eigenvectors $u_1, u_2, \dots, u_n$. Let $x_0 \in C^n$ be such that $x_0 = \sum_{i=1}^{n} \alpha_i u_i$, with $\alpha_1 \neq 0$. Define the sequence of vectors $\{x_k\}_{k=1}^{\infty} \subseteq C^n$ recursively by $x_{k+1} = Ax_k$, $k = 0,1,2,\dots$

(a) Let $v \in C^n$ be any fixed vector that is not orthogonal to u_1. Show that $q_k = {v^T x_{k+1}}/{v^T x_k}$ converges to λ_1 as $k \to \infty$.

(b) Now suppose that $|\lambda_2| > |\lambda_3|$, $v \in C^n$ is orthogonal to u_1 but is not orthogonal to u_2 and $\alpha_2 \neq 0$. Show that $\lim_{k\to\infty} q_k = \lambda_2$.

(c) Now suppose $|\lambda_1| > |\lambda_2| > |\lambda_3| \geq |\lambda_4| \geq \dots \geq |\lambda_n|$, $v \in C^n$ is such that $\alpha_1 v^T u_1 \neq 0$. Show that for k sufficiently large, $q_k \approx \lambda_1 + C(\lambda_2/\lambda_1)^k$ for some constant C. (Hint: Show that $\lim_{k\to\infty}(q_k - \lambda_1)(\lambda_1/\lambda_2)^k = C$, for some constant C.)

Fa 2013

1. Linear Systems

a). For any $x \in \mathbb{R}^n$, clearly $\|x\|_S = \|Sx\| \geq 0$.

- $\|x\|_S = 0 \iff \|Sx\| = 0 \iff Sx = 0 \iff x = 0$ (since S is nonsingular)
- For any $c \in \mathbb{R}^n$, $\|cx\|_S = \|cSx\| = |c|\,\|Sx\| = |c|\,\|x\|_S$
- Let $x, y \in \mathbb{R}^n$ then, $\|x+y\|_S = \|S(x+y)\| = \|Sx + Sy\| \leq \|Sx\| + \|Sy\| = \|x\|_S + \|y\|_S$

b) By definition of matrix norm, we have

$$\|x\|_S = \|Sx\| \leq \|S\|.\|x\| \quad \text{for any } x \in \mathbb{R}^n$$

On the other hand, since S is invertible,

$$\|x\| = \|S^{-1}Sx\| \leq \|S^{-1}\|.\|Sx\| = \|S^{-1}\|.\|x\|_S \Rightarrow \frac{1}{\|S^{-1}\|}\|x\| \leq \|x\|_S$$, since S is invertible $\|S^{-1}\| \neq 0$

So, we get $\frac{1}{\|S^{-1}\|}\|x\| \leq \|x\|_S \leq \|S\|.\|x\|$ for any $x \in \mathbb{R}^n$. Thus, these two norms are equivalent.

c) $$\|M\|_S = \sup_{x \neq 0} \frac{\|Mx\|_S}{\|x\|_S} = \sup_{x \neq 0} \frac{\|SMx\|}{\|Sx\|} = \sup_{y \neq 0} \frac{\|SMS^{-1}y\|}{\|y\|} = \|SMS^{-1}\|$$

d) $K(M) = \|M\|.\|M^{-1}\|$, $K_S(M) = \|M\|_S.\|M^{-1}\|_S = \|SMS^{-1}\|\,\|SM^{-1}S^{-1}\|$

$$\Rightarrow K_S(M) \leq \|S\|\,\|M\|.\|S^{-1}\|.\|S\|.\|M^{-1}\|\,\|S^{-1}\| = (\|S\|.\|S^{-1}\|)^2\|M\|.\|M^{-1}\| = K(S)^2\,K(M)$$

2. Least Squares

a) $(0,1)$: $a = 1$

$(1,1)$: $a + b = 1$

$(-1,-1)$: $a + b = -1$

$(2,0)$: $a + 4b = 0$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

b) Let us find the QR factorization of A:

$$[a_1 \;\; a_2] = [q_1 \;\; q_2] \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} \Rightarrow \begin{aligned} a_1 &= r_{11} q_1 \\ a_2 &= r_{12} q_1 + r_{22} q_2 \end{aligned}$$

$$r_{11} = \|a_1\| = \sqrt{1+1} = \sqrt{2} \quad , \quad q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

(2)

$$r_{12} = q_1^T a_2 = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{\sqrt{2}}$$

$$r_{22}\, q_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 3/2 \\ 0 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix} \qquad r_{22} = \|r_{22} q_2\| = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \frac{\sqrt{6}}{2}$$

$$q_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

So,

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \qquad R = \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} \\ 0 & \sqrt{6}/2 \end{bmatrix}$$

Also, we observe that $\text{rank}(A) = 2$ (adding 2^{nd} and 3^{rd} rows, we get the 1^{st} row and 2^{nd} and 3^{rd} rows are linearly independent)

c) If the system is overdetermined, a leasts squares solution is always exists. If A is full rank, this solution is unique

d) $A^+ = (A^TA)^{-1}A^T$

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix} \qquad (A^TA)^{-1} = \frac{1}{3} \begin{bmatrix} 6 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2/3 \end{bmatrix}$$

$$A^+ = \begin{bmatrix} 2 & -1 \\ -1 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1/3 & 2/3 & -1/3 \end{bmatrix}$$

e)

$$x = A^+ b = \begin{bmatrix} 0 & -1 & 1 \\ 1/3 & 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

3 Iterative Methods

a) Taking limit of both sides of "$x_{k+1} = Mx_k + c$" as $k \to \infty$, we get $x^* = Mx^* + c$ and so,

$$x_{k+1} - x^* = M(x_k - x^*) = \dots = M^{k+1}(x_0 - x^*) \quad \text{for any choice of } x_0.$$

Then taking limit of "$x_{k+1} - x^* = M^{k+1}(x_0 - x^*)$" we get

$0 = (\lim_{k\to\infty} M^{k+1})(x_0 - x^*)$ for any x_0.

So, we must have $\lim_{k\to\infty} M^{k+1} = 0$.

Let X be a nonsingular matrix such that $M = XJX^{-1}$ where

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$$

and each J_p has the form

$$J_p = \begin{bmatrix} \lambda_p & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_p \end{bmatrix}$$

Then $J^k = X^{-1}M^kX$ and taking limit, we see that $\lim_{k\to\infty} J^k = 0$

Since

$$J^k = \begin{bmatrix} J_1^k & & \\ & \ddots & \\ & & J_p^k \end{bmatrix}$$

and each J_i^n consists of some powers of λ_i when n is sufficiently large. In this case, $\lim_{k\to\infty} J_i^k = 0 \iff |\lambda_i| < 1$. Thus, $\lim_{k\to\infty} J^k = 0 \iff$ $|\lambda_i| < 1, \forall i$. So, $r(M) < 1$.

b) $A = L + D + U$, where L is lower triangular, D is diagonal and U is upper triangular. Then,

$$Ax = b \iff (L+D+U)x = b \iff (L+D)x = -Ux + b$$

$$\iff x = -(L+D)^{-1}Ux + (L+D)^{-1}b$$

So, $x_{k+1} = -(L+D)^{-1}Ux_k + (L+D)^{-1}b$, where $B_{G-S} = -(L+D)^{-1}U$

c) Again let $A = L+D+U$ as above and observe

$$(L+D+U)x = b \iff (\theta D + L + D - \theta D + U)x = b$$

$$\iff (L+\theta D)x = -[(1-\theta)D + U]x + b$$

$$\iff x = -(L+\theta D)^{-1}[(1-\theta)D+U]x + b$$

$$\Rightarrow x_{k+1} = -(L+\theta D)^{-1}[(1-\theta)D+U]x_k + b$$

d) On last page

4. Computation of Eigenvalues and Eigenvectors

Suppose A has n eigenvalues $\lambda_1, \dots, \lambda_n$ such that $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n| \geq 0$

Let $u_1, \dots, u_n$ be corresponding eigenvectors.

$$x_0 = \sum_{i=1}^{n} \alpha_i u_i \quad \text{with} \quad \alpha_1 \neq 0$$

Then $x_{k+1} = Ax_k = \dots = A^{k+1} x_0 = \sum_{i=1}^{n} \alpha_i A^{k+1} u_i = \sum_{i=1}^{n} \alpha_i \lambda_i^{k+1} u_i$, and

a) $$q_k = \frac{v^T x_{k+1}}{v^T x_k} = \frac{\alpha_1 \lambda_1^{k+1} v^T u_1 + \alpha_2 \lambda_2^{k+1} v^T u_2 + \dots + \alpha_n \lambda_n^{k+1} v^T u_n}{\alpha_1 \lambda_1^k v^T u_1 + \alpha_2 \lambda_2^k v^T u_2 + \dots + \alpha_n \lambda_n^k v^T u_n}$$

Observe that $v^T u_1 \neq 0$ by assumption. Then,

$$q_k = \frac{(v^T x_{k+1})/\lambda_1^k}{(v^T x_k)/\lambda_1^k} = \frac{\alpha_1 \lambda_1 v^T u_1 + \alpha_2 \lambda_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v^T u_2 + \dots + \alpha_n \lambda_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v^T u_n}{\alpha_1 v^T u_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k v^T u_2 + \dots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k v^T u_n}$$

Then, we have $\left(\frac{\lambda_i}{\lambda_1}\right)^k \to 0$ as $k \to \infty$ $\forall i = 2, 3, \dots, n$. So, taking limit,

$$\lim_{k \to \infty} q_k = \frac{\alpha_1 \lambda_1 v^T u_1}{\alpha_1 v^T u_1} = \lambda_1 .$$

b) In this case, we have $v^T u_1 = 0$ and $\alpha_2 \neq 0$, $v^T u_2 \neq 0$. So,

$$q_k = \frac{\alpha_2 \lambda_2^{k+1} v^T u_2 + \alpha_3 \lambda_3^{k+1} v^T u_3 + \dots + \alpha_n \lambda_n^{k+1} v^T u_n}{\alpha_2 \lambda_2^k v^T u_2 + \alpha_3 \lambda_3^k v^T u_3 + \dots + \alpha_n \lambda_n^k v^T u_n}$$

Dividing by λ_2^k,

$$q_k = \frac{\alpha_2 \lambda_2 v^T u_2 + \alpha_3 \lambda_3 \left(\frac{\lambda_3}{\lambda_2}\right)^k v^T u_3 + \dots + \alpha_n \lambda_n \left(\frac{\lambda_n}{\lambda_2}\right)^k v^T u_n}{\alpha_2 v^T u_2 + \alpha_3 \left(\frac{\lambda_3}{\lambda_2}\right)^k v^T u_3 + \dots + \alpha_n \left(\frac{\lambda_n}{\lambda_2}\right)^k v^T u_n}$$

Now, since $|\lambda_2| > |\lambda_3| \geq |\lambda_4| \geq \dots \geq |\lambda_n|$, we have $\left(\frac{\lambda_i}{\lambda_2}\right)^k \to 0$ as $k \to \infty$ $\forall i = 3, 4, \dots, n$. So,

$$\lim_{k \to \infty} q_k = \frac{\alpha_2 \lambda_2 v^T u_2}{\alpha_2 v^T u_2} = \lambda_2$$

c) $$q_k - \lambda_1 = \frac{\alpha_2\left(\lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^k - \lambda_1\left(\frac{\lambda_2}{\lambda_1}\right)^k\right)v^T u_2 + \cdots + \alpha_n\left(\lambda_n\left(\frac{\lambda_n}{\lambda_1}\right)^k - \lambda_1\left(\frac{\lambda_n}{\lambda_1}\right)^k\right)u^T v_n}{\alpha_1 v^T u_1 + \alpha_2\left(\frac{\lambda_2}{\lambda_1}\right)^k v^T u_2 + \cdots + \alpha_n\left(\frac{\lambda_n}{\lambda_1}\right)^k v^T u_n}$$ 5

$$= \frac{\alpha_2(\lambda_2-\lambda_1)\left(\frac{\lambda_2}{\lambda_1}\right)^k v^T u_2 + \cdots + \alpha_n(\lambda_n-\lambda_1)\left(\frac{\lambda_n}{\lambda_1}\right)^k u^T v_n}{\alpha_1 v^T u_1 + \alpha_2\left(\frac{\lambda_2}{\lambda_1}\right)^k v^T u_2 + \cdots + \alpha_n\left(\frac{\lambda_n}{\lambda_1}\right)^k v^T u_n}$$

$$(q_k-\lambda_1)\left(\frac{\lambda_1}{\lambda_2}\right)^k = \frac{\alpha_2(\lambda_2-\lambda_1)v^T u_2 + \alpha_3(\lambda_3-\lambda_1)\left(\frac{\lambda_3}{\lambda_2}\right)^k v^T u_2 + \cdots + \alpha_n(\lambda_n-\lambda_1)\left(\frac{\lambda_n}{\lambda_1}\right)^k u^T v_n}{\alpha_1 v^T u_1 + \alpha_2\left(\frac{\lambda_2}{\lambda_1}\right)^k v^T u_2 + \cdots + \alpha_n\left(\frac{\lambda_n}{\lambda_1}\right)^k v^T u_n}$$

Observe that $\lim_{k\to\infty}\left(\frac{\lambda_i}{\lambda_1}\right)^k = 0 \quad \forall i = 2,3,\dots,n$ and $\lim_{k\to\infty}\left(\frac{\lambda_i}{\lambda_2}\right)^k = 0$ $\forall i = 3,4,\dots,5$ So,

$$\lim_{k\to\infty}(q_k-\lambda_1)\left(\frac{\lambda_1}{\lambda_2}\right)^k = \frac{\alpha_2(\lambda_2-\lambda_1)v^T u_2}{\alpha_1 v^T u_1} =: C$$

Thus, for sufficiently large k, $q_k \approx \lambda_1 + C\left(\frac{\lambda_2}{\lambda_1}\right)^k$

---

3. d) We know $B_{SOR} = (D+\theta L)^{-1}[(1-\theta)D - \theta U]$ Then,

$\det[(D+\theta L)^{-1}] = \prod_{i=1}^n \frac{1}{a_{ii}}$, $\det[(1-\theta)D - \theta U] = (1-\theta)^n \prod_{i=1}^n a_{ii}$. So,

$\det(B_{SOR}) = (1-\theta)^n$. On the other hand let $\lambda_1,\dots,\lambda_n$ represent the eigenvalues of B_{SOR}. Then, $\lambda_1 \cdot \dots \cdot \lambda_n = (1-\theta)^n$ and, $|\lambda_1| \cdot \dots \cdot |\lambda_n| = |1-\theta|^n$. So, $|1-\theta|^n \le |\lambda|^n_{max} \Rightarrow |1-\theta| \le |\lambda|_{max}$

Since $\rho(B_{SOR}) < 1$ is necessary and sufficient condition for convergence, it is necessary to have $|1-\theta| < 1 \Leftrightarrow \theta \in (0,2)$

1a

3c

Numerical Analysis Screening Exam, Spring 2013

Problem 1. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix. At the end of the first step of Gaussian Elimination without partial pivoting, we have:

$$A_1 = \left(\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline 0 & & & \\ \vdots & & \hat{A} & \\ 0 & & & \end{array} \right)$$

a. Show that $\hat{A}$ is also a SPD.

b. Use the first conclusion to show the existence of the LU factorization and Cholesky factorization of any SPD.

Problem 2.

A matrix A with all non-zero diagonal elements can be written as $A = D_A(I-L-U)$ where D_A is a diagonal matrix with identical diagonal as A and matrices L and U are lower and upper triangular matrices with zero diagonal elements. The matrix A is said to be consistently ordered if the eigenvalues of matrix $\rho L+\rho^{-1}U$ are independent of $\rho \neq 0$. Consider a tri-diagonal matrix A of the form

$$A = \begin{pmatrix} \alpha & \beta & 0 & \cdots & 0 \\ \beta & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta \\ 0 & \cdots & 0 & \beta & \alpha \end{pmatrix}$$

with $|\alpha| \geq 2\beta > 0$.

a. Show that the matrix A is consistently ordered.

b. Show that if $\lambda \neq 0$ is an eigenvalue of the iteration matrix B_ω of the Successive Over Relaxation (SOR) method for matrix A

$$B_\omega = (I - \omega L)^{-1}((1-\omega)I + \omega U),$$

then $\mu = (\lambda + \omega - 1)(\omega\sqrt{\lambda})^{-1}$ is an eigenvalue of $L + U$.

Problem 3.

a. Assume that $v_1 = (1,1,1)^T$ is an eigenvector of a 3×3 matrix B. Find a real unitary matrix V such that the first column of the matrix $V^T BV$ contains all zeros except on the first row.

b. Consider a matrix A defined by

$$A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 1 \\ -2 & 0 & 3 \end{pmatrix}$$

verify that $v_1 = (1,1,1)^T$ is an eigenvector of A and the first column of the matrix $V^T AV$ contains all zeros except on the first row where V is the matrix you obtained in (a).

c. Assume that $V^T AV$ has the form

$$V^T AV = \begin{pmatrix} * & * & * \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$$

Find a Schur decomposition of the matrix A. That is, find a unitary matrix U such that $U^H AU = R$ where R is an upper triangular matrix and U^H is the conjugate transpose of U.

Problem 4.

Consider a n by m matrix A and a vector $b \in \mathbb{R}^n$. A minimum norm solution of the least squares problem is a vector $x \in \mathbb{R}^m$ with minimum Euclidian norm that minimizes $\|Ax - b\|$. Consider a vector x^* such that $\|Ax^* - b\| \le \|Ax - b\|$ for all $x \in \mathbb{R}^n$. Show that x^* is a minimum norm solution if and only if x^* is in the range of A^*.

Problem 1.

b) Since A is SPD, we know $a_{11} > 0$, in particular $a_{11} \neq 0$. So, we apply GEWP and obtain A_1. Since $\hat{A}$ is also SPD by (a), then $\hat{a}_{11} \neq 0$ and we can apply GEWP again to get $\hat{a}_{i2} = 0$ for $i = 3, 4, \dots, n$. Continuning on this way, we get an upper triangular matrix U with nonzero diagonal. Keeping multipliers, we can form the lower triangular matrix L and so we get

$$A = LU$$

Note that since A is SPD it is nonsingular so, L also must be nonsingular. We already told above the nonsingularity of U. So, this argument shows the existence of LU factorization of A.

Since A is symmetric, we have

$$LU = U^T L^T.$$

Since L, U (and so $L^T\ U^T$) are invertible, we have

$$(U^T)^{-1} L = L^T U^{-1}$$

where $(U^T)^{-1}$ and L are lower triangular (and so is $(U^T)^{-1}L$) and L^T and U^{-1} are upper triangular (and so is $L^T U^{-1}$) so, by equality above, we must have

$$L^T U^{-1} = D$$

where D is diagonal. Then $L^T = DU$, and $L = U^T D^T = U^T D$. So, we obtain

$$A = U^T D U = U^T D^{1/2} D^{1/2} U$$

Letting $M = U^T D^{1/2}$ which is lower triangular, we observe $M^T = D^{1/2} U$. So, we see

$$A = MM^T$$

where M is lower triangular, showing the existence of Cholesky fact.

Problem 2 $\quad A = D_A - D_A L - D_A U$

a)
$$A = \underbrace{\begin{bmatrix} \alpha & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{bmatrix}}_{= D_A} - \underbrace{\begin{bmatrix} 0 & & & \\ -\beta & 0 & & \\ & \ddots & \ddots & \\ & & -\beta & 0 \end{bmatrix}}_{= D_A L} - \underbrace{\begin{bmatrix} 0 & -\beta & & \\ & 0 & \ddots & \\ & & \ddots & -\beta \\ & & & 0 \end{bmatrix}}_{= D_A U}$$

$$L = \begin{bmatrix} 1/\alpha & & & \\ & 1/\alpha & & \\ & & \ddots & \\ & & & 1/\alpha \end{bmatrix} \begin{bmatrix} 0 & & & \\ -\beta & 0 & & \\ & \ddots & \ddots & \\ & & -\beta & 0 \end{bmatrix} = \begin{bmatrix} 0 & & & \\ -\beta/\alpha & 0 & & \\ & \ddots & \ddots & \\ & & -\beta/\alpha & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1/\alpha & & & \\ & 1/\alpha & & \\ & & \ddots & \\ & & & 1/\alpha \end{bmatrix} \begin{bmatrix} 0 & -\beta & & \\ & 0 & \ddots & \\ & & \ddots & -\beta \\ & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\beta/\alpha & & \\ & 0 & \ddots & \\ & & \ddots & -\beta/\alpha \\ & & & 0 \end{bmatrix}$$

Now, observe

$$\rho L + \rho^{-1} U = \begin{bmatrix} 0 & \frac{-\beta}{\alpha\rho} & & 0 \\ \frac{-\beta\rho}{\alpha} & 0 & \ddots & \\ & \ddots & \ddots & \frac{-\beta}{\alpha\rho} \\ 0 & & \frac{-\beta\rho}{\alpha} & 0 \end{bmatrix}$$

Let us denote $\det(\rho L + \rho^{-1} U - \lambda I)$ by D_n when $\rho L + \rho^{-1} U$ is $n \times n$. Then observe

$$D_n = -\lambda D_{n-1} + \frac{\beta}{\alpha\rho} \begin{vmatrix} \frac{-\beta\rho}{\alpha} & \frac{-\beta}{\alpha\rho} & & \\ 0 & -\lambda & \frac{-\beta}{\alpha\rho} & \\ & \frac{-\beta\rho}{\alpha} & \ddots & \ddots \\ & & \ddots & \ddots \end{vmatrix} = -\lambda D_{n-1} - \frac{\beta^2}{\alpha^2} D_{n-2}$$

So, we have $D_n = -\lambda D_{n-1} - \frac{\beta^2}{\alpha^2} D_{n-2}$ where $n \geq 3$. We can find $D_1 = -\lambda$ and $D_2 = \lambda^2 - \frac{\beta^2}{\alpha^2}$. Then to solve recurrence relation, the characteristic polynomial is $r^2 + \lambda r + \frac{\beta^2}{\alpha^2} = 0$. So,

$$r_1 = \frac{-\lambda + \sqrt{\lambda^2 - \frac{4\beta^2}{\alpha^2}}}{2}, \quad r_2 = \frac{-\lambda - \sqrt{\lambda^2 - \frac{4\beta^2}{\alpha^2}}}{2}$$

Then $D_n = c_1 \left(\frac{-\lambda + \sqrt{\lambda^2 - \frac{4\beta^2}{\alpha^2}}}{2} \right)^n + c_2 \left(\frac{-\lambda - \sqrt{\lambda^2 - \frac{4\beta^2}{\alpha^2}}}{2} \right)^n$.

Then by using D_1 and D_2,

$$\left.\begin{aligned} c_1 r_1 + c_2 r_2 &= -\lambda \\ c_1 r_1^2 + c_2 r_2^2 &= \lambda^2 - \frac{\beta^2}{\alpha^2} \end{aligned}\right\}$$

Solving these equations, we see that c_1 and c_2 does not depend on p

So, substituting c_1 and c_2, we see that any coefficient of D_n which is the char. polynomial of $pL + p^{-1}U$ does not depend on p. Thus, eigenvalues of $pL + p^{-1}U$ does not depend on p, proving A is consistently ordered.

b) Firstly, since A is consistenty ordered by part (a), we deduce that for any $p \neq 0$, all the matrices $pL + \frac{1}{p}U$ have the same eigenvalues.

Let $\lambda \neq 0$ be an eigenvalue of B_ω. Then $\det(\lambda I - B_\omega) = 0$. Since $\det(I - \omega L) = 1$, we obtain

$$0 = \det\left[(I - \omega L)(\lambda I - B_\omega)\right] = \det\left[\lambda I - \lambda\omega L - (1-\omega)I - \omega U\right)$$

$$= \det\left[(\lambda + \omega - 1)I - \lambda\omega L - \omega U\right] = \det\left[\frac{\lambda + \omega - 1}{\omega} - \lambda L - U\right]$$

$$= \det\left[\frac{\lambda + \omega - 1}{\omega \lambda^{1/2}} - (\lambda^{1/2} L + \lambda^{-1/2} U)\right]$$

So, $\frac{\lambda + \omega - 1}{\omega \lambda^{1/2}}$ is an eigenvalue of $\lambda^{1/2} L + \lambda^{-1/2} U$. Since A is consistently ordered $\frac{\lambda + \omega - 1}{\omega \lambda^{1/2}}$ is also an eigenvalue of $L + U$.

Problem 3

a) By assumption $Bv_1 = \lambda v_1 \Rightarrow$ $b_{11}+b_{12}+b_{13}=\lambda$

$b_{21}+b_{22}+b_{23}=\lambda \Rightarrow Bv_1 = \begin{bmatrix}\lambda\\\lambda\\\lambda\end{bmatrix}$

$b_{31}+b_{32}+b_{33}=\lambda$

Let $V=[v_1\ v_2\ v_3]$ and then $V^TBV = \begin{bmatrix}v_1^T\\v_2^T\\v_2^T\end{bmatrix} B\,[v_1\ v_2\ v_3]$ and

$(V^TBV)_{ij} = v_i^T B v_j$. So,

$(V^TBV)_{21} = v_2^T B v_1 = v_2^T\begin{bmatrix}\lambda\\\lambda\\\lambda\end{bmatrix}$ and $(V^TBV)_{31} = v_3^T\begin{bmatrix}\lambda\\\lambda\\\lambda\end{bmatrix}$

So, choosing $v_2 = \begin{bmatrix}1/\sqrt2\\-1/\sqrt2\\0\end{bmatrix}$ $v_3 = \begin{bmatrix}1/\sqrt6\\1/\sqrt6\\-2/\sqrt6\end{bmatrix}$ we see that

$(V^TBV)_{21} = (V^TBV)_{31} = 0$ and $(V_TBV)_{11} = \lambda$, and also

$$V = \begin{bmatrix}1/\sqrt3 & 1/\sqrt2 & 1/\sqrt6\\ 1/\sqrt3 & -1/\sqrt2 & 1/\sqrt6\\ 1/\sqrt3 & 0 & -2/\sqrt6\end{bmatrix}$$

b) $Av_1 = \begin{bmatrix}1&-2&2\\-1&1&1\\-2&0&3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix}$ $\Rightarrow v_1$ is an eigenvalue

So, the eigenvalue is 1

$$(V^TAV)_{11} = [1/\sqrt3\ \ 1/\sqrt3\ \ 1/\sqrt3]\begin{bmatrix}1&-2&2\\-1&1&1\\-2&0&3\end{bmatrix}\begin{bmatrix}1/\sqrt3\\1/\sqrt3\\1/\sqrt3\end{bmatrix} = [1/\sqrt3\ \ 1/\sqrt3\ \ 1/\sqrt3]\begin{bmatrix}1/\sqrt3\\1/\sqrt3\\1/\sqrt3\end{bmatrix} = 1$$

$$(V^TAV)_{21} = [1/\sqrt2\ \ -1/\sqrt2\ \ 0]\begin{bmatrix}1&-2&2\\-1&1&1\\-2&0&3\end{bmatrix}\begin{bmatrix}1/\sqrt3\\1/\sqrt3\\1/\sqrt3\end{bmatrix} = [1/\sqrt2\ \ -1/\sqrt2\ \ 0]\begin{bmatrix}1/\sqrt3\\1/\sqrt3\\1/\sqrt3\end{bmatrix} = 0$$

$$(V^TAV)_{31} = [1/\sqrt6\ \ 1/\sqrt6\ \ -2/\sqrt6]\begin{bmatrix}1&-2&2\\-1&1&1\\-2&0&3\end{bmatrix}\begin{bmatrix}1/\sqrt3\\1/\sqrt3\\1/\sqrt3\end{bmatrix} = [1/\sqrt6\ \ 1/\sqrt6\ \ -2/\sqrt6]\begin{bmatrix}1/\sqrt3\\1/\sqrt3\\1/\sqrt3\end{bmatrix} = 0$$

Problem 4:

($\Rightarrow$) Suppose x^* is a minimum-norm solution. Then we know that $x^* = A^+ b$ where $A^+ = V\Sigma^{-1}U^T$ in case of $A = U\Sigma V^T$.

Then, (supposing there are r nonzero singular values of A for generality)

$x^* = V\Sigma^{-1}U^T b$ $\quad$ ($(U^Tb)_i$ – i^{th} entry of U^Tb) (V_i – ith column of V)

$$= V\Sigma^{-1}\begin{bmatrix}(U^Tb)_1\\ \vdots \\ (U^Tb)_n\end{bmatrix} = V\begin{bmatrix}\frac{1}{\sigma_1}(U^Tb)_1\\ \vdots \\ \frac{1}{\sigma_r}(U^Tb)_r \\ 0\end{bmatrix} = \frac{1}{\sigma_1}(U^Tb)_1 V_1 + \dots + \frac{1}{\sigma_1}(U^Tb)_r V_r \qquad (*)$$

On the other hand, any vector $d \in R(A^T)$ can be written as $d = A^Tc$ for some $c \in \mathbb{R}^{n\times 1}$. Then, similarly,

$$d = A^Tc = (V\Sigma^T U^T)c = \sigma_1 (U^Tc)_1 V_1 + \dots + \sigma_r (U^Tc)_r V_r$$

So, d is a linear combination of first r columns of V. Since we observe from $(*)$ that x^* is a linear combination of first r columns of V, we deduce $x^* \in R(A^T)$.

2nd proof: Suppose x^* is a min-norm solution. Let $x^* = x_1 + x_2$ where $x_1 \in R(A^T)$, $x_2 \in R(A^T)^\perp = N(A)$. Then observe

$b = Ax^* = Ax_1 + Ax_2 = Ax_1$, $\Rightarrow x_1$ is also a least-squares solution.

We have $\|x^*\|^2 = \|x_1\|^2 + \|x_2\|^2$ and by assumption $\|x^*\| \le \|x_1\|$. Combining,

$\|x_1\|^2 + \|x_2\|^2 \le \|x_1\|^2 \Rightarrow \|x_2\|^2 \le 0 \Rightarrow \|x_2\| = 0 \Rightarrow x_2 = 0$. So, $x^* = x_1 \in R(A^T)$

($\Leftarrow$) Claim: If $y_1, y_2 \in R(A^T)$ are two least squares solutions to $Ax = b$ then $y_1 = y_2$.

$A^T(Ay_1 - Ay_2) = 0 \Rightarrow A(y_1 - y_2) \in N(A^T) \cap R(A) = \{0\}$ $\Rightarrow y_1 - y_2 \in N(A)$

$\rightarrow Ay_1 = b$ and $Ay_2 = b$ $\cancel{\Rightarrow A(y_1-y_2) = 0 \Rightarrow y_1 - y_2 \in N(A)}$

$\rightarrow y_1 - y_2 \in R(A^T)$ (since $R(A^T)$ is a subspace) $\Rightarrow y_1 - y_2 \in R(A^T) = N(A)^\perp$

Thus, $y_1 - y_2 \in N(A) \cap N(A)^\perp = \{0\}$, and so $y_1 = y_2$

Let x be an arbitrary least-squares solution. Write $x = x_1 + x_2$ where $x_1 \in R(A^T)$, $x_2 \in R(A^T)^\perp = N(A)$. Then, $b = Ax = Ax_1 + Ax_2 = Ax_1$. So, $x_1 \in R(A^T)$ is also a least squares solution. But by claim above $x_1 = x^*$.

So, $x = x^* + x_2$ and $\|x\|^2 = \|x^*\|^2 + \|x_2\|^2 \geq \|x^*\|^2 \Rightarrow \|x\| \geq \|x^*\|$.

Eigenvalue (b)

Numerical Analysis Screening Exam, Fall 2012

Direct Methods for Linear Equations.

a. Let $A \in \mathbb{R}^{n\times n}$ be a symmetric positive definite (SPD) matrix. There exists a nonsingular lower triangle matrix L satisfying $A = L \cdot L^t$. Is this factorization unique? If not, propose a condition on L to make the factorization unique.

b. Compute the above factorization for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 13 & 8 \\ 1 & 8 & 14 \end{pmatrix}.$$

Iterative Methods for Linear Equations.

Consider the iterative method:

$$Nx_{k+1} = Px_k + b, k = 0, 1, \cdots,$$

where N, P are $n \times n$ matrices with $detN \neq 0$; and x_0, b are arbitraray $n-$dim vectors. Then the above iterates satisfy the system of equations

$$x_{k+1} = Mx_k + N^{-1}b, k = 0, 1, \cdots \tag{1}$$

where $M = N^{-1}P$. Now define $N_\alpha = (1+\alpha)N, P_\alpha = P + \alpha N$ for some real $\alpha \neq -1$ and consider the related iterative method

$$x_{k+1} = M_\alpha x_k + N_\alpha^{-1}b, \quad k = 0, 1, \cdots, \tag{2}$$

where $M_\alpha = N_\alpha^{-1}P_\alpha$.

a. Let the eigenvalues of M be denoted by: $\lambda_1, \lambda_2, \cdots, \lambda_n$. Show that the eigenvalues $\mu_{\alpha,k}$ of M_α are given by:

$$\mu_{\alpha,k} = \frac{\lambda_k + \alpha}{1+\alpha}, \quad k = 1, 2, \cdots, n$$

b. Assume the eigenvalues of M are real and satisfy: $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n < 1$. Show that the iterations in eq. (2) converge as $k \to \infty$ for any α such that $\alpha > \dfrac{1+\lambda_1}{2} > -1$.

Eigenvalue Problem.

a. Let λ be an eigenvalue of a $n \times n$ matrix A. Show that $f(\lambda)$ is an eigenvalue of $f(A)$ for any polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$.

b. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a symmetric matrix satisfying:

$$a_{1i} \neq 0, \qquad \sum_{j=1}^{n} a_{ij} = 0, \qquad a_{ii} = \sum_{j\neq i} |a_{ij}|, \qquad i = 1, \cdots, n$$

Show all eigenvalues of A are non-negative and determine the dimension of eigenspace corresponding to the smallest eigenvalue of A.

Least Square Problem.

a. Let A be an $m \times n$ real matrix with the following singular value decomposition: $A = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T & V_2^T \end{pmatrix}^T$, where $U = (U_1\ U_2)$ and $V = (V_1\ V_2)$ are orthogonal matrices, U_1 and V_1 have $r = rank(A)$ columns, and Σ is invertible.

For any vector $b \in \mathbb{R}^n$, show that the minimum norm, least squares problem:

$$\min_{x \in S} \|x\|_2, \qquad S = \{x \in \mathbb{R}^n \mid \|Ax - b\|_2 = min\}$$

always has a unique solution, which can be written as $x = V_1 \Sigma^{-1} U_1^T b$.

b. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Using part a) above, find the minimum norm, least squares solution to the problem:

$$\min_{x \in S} \|x\|_2, \qquad S = \{x \in \mathbb{R}^n \mid \|Ax - b\|_2 = min\}$$

Hint: You can assume that the U in the SVD of A must be of the form $U = \begin{pmatrix} a & a \\ a & -a \end{pmatrix}$ for some real $a > 0$.

Fall 2012

Direct Methods for Linear Equations

a) $A \in \mathbb{R}^{n\times n}$ - SPD

$\to 0 \neq \det(A) = \det(L)\det(L^T) = \det(L)^2 \Rightarrow \det(L) \neq 0 \Rightarrow L$ is nonsingular

So, $l_{ii} \neq 0 \quad \forall i = 1,2,\dots n$. We have here

$$a_{ij} = \sum_{k=1}^{\min(i,j)} l_{ik}\, l_{jk} \qquad (\star)$$

So, using formula $(\star)$ (direct calculation of L) the only point we need to decide the sign of l_{ii}'s $\forall i$. So, choosing them as positive, we solve the $(\star)$ for l_{ij}'s uniquely. Thus if we assume the diagonal of L is positive, then the factorization $A = LL^T$ is unique

b) $$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 13 & 8 \\ 1 & 8 & 14 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$\to l_{11}^2 = 1 \Rightarrow l_{11} = 1 \qquad \to l_{21}^2 + l_{22}^2 = 13 \Rightarrow l_{22} = 3$

$l_{11} l_{21} = 2 \Rightarrow l_{21} = 2 \qquad l_{21} l_{31} + l_{22} l_{32} = 8 \Rightarrow l_{32} = 2$

$l_{11} l_{31} = 1 \Rightarrow l_{31} = 1$

$\to l_{31}^2 + l_{32}^2 + l_{33}^2 = 14 \Rightarrow l_{33}^2 = 3$

So,
$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

Iterative Methods for Linear Equations

$$N x_{k+1} = P x_k + b \quad , k = 0,1,\dots \qquad \det(N) \neq 0 \quad , M, N \in \mathbb{R}^{n\times n}$$

$$\Rightarrow x_{k+1} = N^{-1}P x_k + N^{-1} b =: M x_k + N^{-1} b , \quad k = 0,1,2,\dots$$

$* M = N^{-1}P \quad$ Define $N_\alpha = (1+\alpha)N \quad , P_\alpha = P + \alpha N \quad , \alpha \neq -1$, real

$$x_{k+1} = M_\alpha x_k + N_\alpha^{-1} b \quad , k = 0,1,\ldots$$

$$M_\alpha = N_\alpha^{-1} P_\alpha$$

$$\rightarrow M_\alpha = \frac{1}{1+\alpha} N^{-1}(P+\alpha N) = \frac{1}{1+\alpha} N^{-1}P + \frac{\alpha}{1+\alpha} I = \frac{1}{1+\alpha}(M+\alpha I) \quad (*)$$

a) Eigenvalues of M : $\lambda_1, \ldots, \lambda_n$ By $(*)$,

$$M_\alpha = \frac{1}{1+\alpha}(M+\alpha I)$$

$$\det(M+\alpha I - \lambda I) = \det(M - (\lambda-\alpha)I)$$

So, eigenvalues of $M+\alpha I$ are $\lambda_1+\alpha, \lambda_2+\alpha, \ldots + \lambda_n+\alpha$ Multiplying all entries of a matrix by same number, we multiply its eigenvalues by same number, i.e,

Eigenvalues of $M_\alpha = \frac{1}{1+\alpha}(M+\alpha I)$ are $\frac{\lambda_1+\alpha}{1+\alpha}, \frac{\lambda_2+\alpha}{1+\alpha}, \ldots \frac{\lambda_n+\alpha}{1+\alpha}$

b) $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_n < 1 \Rightarrow \lambda_1+\alpha \le \lambda_2+\alpha \le \ldots \le \lambda_n+\alpha < 1+\alpha$

If $\alpha > -1$ (i.e. $1+\alpha > 0$), then

$$\frac{\lambda_1+\alpha}{1+\alpha} \le \frac{\lambda_2+\alpha}{1+\alpha} \le \cdots \le \frac{\lambda_n+\alpha}{1+\alpha} < \frac{1+\alpha}{1+\alpha} = 1$$

Here, for conv. we need $\frac{\lambda_1+\alpha}{1+\alpha} > -1 \iff \lambda_1+\alpha > -1-\alpha$

$\iff \lambda_1 + 1 > -2\alpha$

$\iff \frac{\lambda_1+1}{2} > -\alpha$

If $\alpha < -1$ then,

$$\frac{\lambda_1+\alpha}{1+\alpha} \ge \frac{\lambda_2+\alpha}{1+\alpha} \ge \cdots \ge \frac{\lambda_n+\alpha}{1+\alpha} > -1$$

Here, for conv. we need $\frac{\lambda_1+\alpha}{1+\alpha} < 1 \iff \lambda_1+\alpha > 1+\alpha$

$\iff \lambda_1 + 1 > 2$

$\iff \frac{\lambda_1+1}{2} > 1$

$\iff \lambda_1 > 1 \Rightarrow$ a contradiction

Since also $\lambda_1 < 1$ implies $\frac{\lambda_1+1}{2} < 1$, we say that for $\alpha > -1$, the iteration converges for any α such that

$$-\alpha < \frac{\lambda_1+1}{2} < 1$$

Eigenvalue Problem

a) Suppose v is the corresponding eigenvector. Then,

$$f(A)v = \left(\sum_{k=0}^{n} a_k A^k\right)v = \sum_{k=0}^{n} a_k (A^k v) = \sum_{k=0}^{n} a_k(\lambda^k v) = \left(\sum_{k=0}^{n} a_k \lambda^k\right)v$$

$$= f(\lambda)v$$

(Remember $a_k A^0 v = a_k I v = a_k v = a_k 1.v = a_k \lambda^0 v$)

Also, $A^k v = A^{k-1}(Av) = A^{k-1}\lambda v = \lambda A^{k-2}(Av) = \lambda^2 A^{k-2} v = \dots = \lambda^k v$

b) $A \in \mathbb{R}^{n \times n}$ symmetric with

$a_{1i} \neq 0$ (first row & column not 0) $\qquad \sum_{j=1}^{n} a_{ij} = 0$ (sum of each row is 0) $\qquad a_{ii} = \sum_{j \neq i} |a_{ij}| \quad i = 1, 2, \dots, n$

Gerschgorin Thm $\quad |\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \Rightarrow -\sum_{j \neq i} |a_{ij}| \leq \lambda - a_{ii} \leq \sum_{j \neq i} |a_{ij}|$

$$\Leftrightarrow \underbrace{a_{ii} - \sum_{j \neq i} |a_{ij}|}_{= 0 \text{ by assump.}} \leq \lambda \leq \sum_{j \neq i} |a_{ij}| + a_{ii} \qquad \forall i \qquad (*)$$

Since all eigenvalues are in $\bigcup_{i=1}^{n} D_i$ where $D_i = \{x : |x - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}$

and because of $(*)$, we see that all eigenvalues are nonnegative.

On the other hand, since $a_{ii} = -\sum_{j \neq i} a_{ij}$ and $a_{ii} = \sum_{j \neq i} |a_{ij}|$, we get (without a_{ii})

$$-a_{i1} - a_{i2} - \dots - a_{in} = |a_{i1}| + |a_{i2}| + \dots + |a_{in}| = a_{ii} \qquad (**)$$

So, we see $a_{ij} \leq 0 \quad \forall j \neq i$.

Let $e = [1 \dots 1]$, then $Ae = \begin{bmatrix} \sum a_{1j} \\ \vdots \\ \sum a_{nj} \end{bmatrix} = 0 \Rightarrow A$ is nonsingular

$\Rightarrow 0$ is an eigenvalue.

Since all eigenvalues are nonnegative, 0 is the smallest eigenvalue. $e = [1, \dots, 1]$ is the corresponding eigenvector.

Because of $(**)$ there is no other vector $v \neq 0$ such that $Av = 0$.

So, the dimension of eigenspace corresponding to $\lambda = 0$ is 1.

Least Squares Problem:

$$A = [U_1 \; U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \overset{V_1 \; V_2 \; ?}{[V_1^T \; V_2^T]^T} \qquad U = [U_1 \; U_2], \; V = [V_1 \; V_2] \text{ are orthogonal matrizes}$$

U_1 is $m \times r$ and V_1 is $n \times r$ where $r = \text{rank}(A)$.
Σ is invertible

We have $[U_1 \ U_2] \Sigma' V^T x = b$ where $\Sigma' = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$

Then since $U^T U = I$,

$$\begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} [U_1 \ U_2] \Sigma' V^T x = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} b \Rightarrow \Sigma' V^T x = \begin{bmatrix} U_1^T b \\ U_2^T b \end{bmatrix} =: \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Letting $V^T x = y$, the system will be

$$\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} y = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \rightarrow r \times r$$

Then, since Σ is invertible, we can solve $\Sigma y_1 = c_1$ $(y_1 \in \mathbb{R}^{r \times 1})$ uniquely and show it by $y_1 = \Sigma^{-1} c_1 = \Sigma^{-1} U_1^T b$. For the remaining part of y we can assign any value to them $\left(y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \right)$

So, $x = Vy = [V_1 \ V_2] \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = V_1 y_1 + V_2 y_2 = V_1 \Sigma^{-1} U_1^T b + V_2 y_2$ is a solution to the least-squares problem. But since y_2 is free, we can choose $y_2 = 0$ to minimize its norm. Thus $x = V_1 \Sigma^{-1} U_1^T b$ is the unique (by construction) minimum-norm solution.

b)

$$A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$|A^T A - \lambda I| = (2-\lambda)^2 - 4 = 0 \Rightarrow \lambda^2 - 4\lambda + 4 - 4 = 0 \Rightarrow \lambda = 0 \quad \lambda = 4$$

$\Rightarrow \sigma_1 = 2 \quad \sigma_2 = 0 \qquad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad \underline{r = 1}$

Corresp eigenvect:

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 - v_2 = 0 \rightarrow v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow v_1 + v_2 = 0 \rightarrow v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$u_i = \sigma_i^{-1} A v_i \quad i = 1, 2, \dots, r$

$\Rightarrow u_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\qquad U = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

Let $u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then, the mini norm solution is

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} 3 = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$$

Problem 2a - iterative improvement

Preliminary/Qualifying Exam in Numerical Analysis (Math 502a) Spring 2012

Instructions

The exam consists of four problems, each having multiple parts. You should attempt to solve all four problems.

1. **Linear systems**

(a) What is the LU-decomposition of an n by n matrix A, and how is it related to Gaussian elimination? Does it always exist? If not, give sufficient condition for its existence.

(b) What is the relation of Cholesky factorization to Gaussian elimination? Give an example of a symmetric matrix for which Cholesky factorization does not exist.

(c) Let $\mathrm{C} = \mathrm{A} + i\mathrm{B}$ where A and B are real n by n matrices. Give necessary and sufficient conditions on A and B for C to be Hermitian, and give a nontrivial example of a 3 by 3 Hermitian matrix.

2. **Least squares**

(a) Give a simple example which shows that loss of information can occur in forming the normal equations. Discuss how the accuracy can be improved by using iterative improvement.

(b) Compute the pseudoinverse, x†, of a nonzero row or column vector, x, of length n. Let a = [1, 0] and let b = $[1, 1]^T$. Show that (ab)† ≠ b†a†.

3. **Iterative Methods**

Consider the stationary vector-matrix iteration given by

$$x_{k+1} = Mx_k + c \tag{1}$$

where $M \in C^{n\times n}$, $c \in C^n$, and , $x_0 \in C^n$ are given.

(a) If $x^* \in C^n$ is a fixed point of (1) and $\|M\| < 1$ where $\|\cdot\|$ is any compatible matrix norm induced by a vector norm, show that x^* is unique and that $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$.

(b) Let $r(M)$ denote the spectral radius of the matrix M and use the fact that $r(M) = inf\|M\|$, where the infimum is taken over all compatible matrix norms induced by vector norms, to show that $\lim_{k\to\infty} x_k = x^*$ for any $x_0 \in C^n$ if and only if $r(M) < 1$.

Now consider the linear system

$$Ax = b \tag{2}$$

where $A \in C^{n\times n}$ nonsingular and $b \in C^n$ are given.

(c) What are the matrix $M \in C^{n\times n}$ and the vector $c \in C^n$ in (1) in the case of the Jacobi iteration for solving the linear system given in (2).

(d) Use part (a) to show that if $A \in C^{n\times n}$ is strictly diagonally dominant then the Jacobi iteration will converge to the solution of the linear system (2).

(e) Use part (b) together with the Gershgorin Circle Theorem to show that if $A \in C^{n \times n}$ is strictly diagonally dominant then the Jacobi iteration will converge to the solution of the linear system (2).

4. Computation of Eigenvalues and Eigenvectors

Consider an $n \times n$ Hermitian matrix A and a unit vector q_1. For $k = 2, \cdots n$, let $p_k = Aq_{k-1}$ and set

$$q_k = \frac{h_k}{\|h_k\|_2}, \qquad h_k = p_k - \sum_{j=1}^{k-1} (q_j^H \cdot p_k)\, q_j.$$

where $\|\cdot\|_2$ is the Euclidian norm in C^n.

(a) Show that the vectors q_k, for $k = 1, \cdots n$, form an orthogonal set if none of the vectors h_k is the zero vector.

(b) Consider the matrix $Q^H AQ$. Use part (a) to show that it is a tridiagonal matrix (Hint: $[Q^H AQ]_{i,j} = q_i^H A q_j$).

(c) Suggest a possible approach that uses the result of part (b) to reduce the number of operations in the QR-algorithm for the computation of the eigenvalues of the matrix A.

b) want: $q_i^H A q_j = 0$ for $|i-j| \geq 2$

$q_i^H A q_j = q_i^H p_{j+1} = q_i^H \left(h_{j+1} + \sum_{n=1}^{j} (q_n^H p_{j+1}) q_n \right) = 0$ since $|i-j| \geq 2$

explain

$j < i$

$i - j \geq 2$

Sp 2012

1. Linear Systems

a) Let $A \in \mathbb{R}^{n \times n}$. If we can find matrices $L, U \in \mathbb{R}^{n \times n}$ with L is lower triangular (1's on the diagonal) and U is upper triangular such that $A = L.U$ then this decomposition is called LU decomposition of A.

Let A_k denote the k^{th} leading principle of A (i.e. the submatrix first k rows, first k columns). Then if each A_k is nonsingular for $k = 1, 2, 3, \dots, n-1$ then LU decomposition of A exists. In addition, if A is nonsingular, then this LU decomposition is unique.

b) For a real matrix A and symmetric matrix, if we can find real and lower triangular matrix H such that $A = HH^T$, this decomposition is Cholesky decomposition of A. We can directly compute this matrix H (provided it exists) or we can use Gaussian elimination (without pivoting). During this process, if all the pivots and $a_{nn}^{(n-1)}$ are positive (for existence and uniqueness) then letting

$$D = \begin{bmatrix} a_{11}^{(0)} & & 0 \\ & a_{22}^{(1)} \ddots & \\ 0 & & a_{nn}^{(n-1)} \end{bmatrix}$$

and replacing main diagonal of U by 1's we get $A = LDU$. Then since $A = A^T$, we have $A = U^T D L^T$ where U^T is a lower triangular matrix with 1's on main diagonal and DL^T is an upper triangular matrix, then by uniqueness of LU decomposition, we deduce $U^T = L$. Now, let

$$D^{1/2} = \begin{bmatrix} \sqrt{a_{11}^{(0)}} & & 0 \\ & \sqrt{a_{22}^{(1)}} \ddots & \\ 0 & & \sqrt{a_{nn}^{(n-1)}} \end{bmatrix}$$

Then $A = (LD^{1/2})(D^{1/2}U)$ and $(LD^{1/2})^T = D^{1/2}L^T = D^{1/2}U$. So, we get the Cholesky decomposition of A.

Let $A = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$. Suppose A has Cholesky decomposition, $A = HH^T$. Then

$$\begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} h_{11} & 0 \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} h_{11} & h_{21} \\ 0 & h_{22} \end{bmatrix}$$

Then by equality, we have $h_{11}^2 = -2$, which is impossible for a real matrix H. Thus, Cholesky decomposition of A does not exist.

c) In order C to be Hermition, we must have $(A+iB)^H = (A+iB)$ which is equivalent to $A^T - iB^T = A + iB$, and this is equivalent to $A = A^T$ and $B = -B^T$. So, if A is symmetric and B is skew-symmetric, then $C = A + iB$ becomes Hermition. Conversely if C is Hermition then A is symmetric and B are skew symmetric.

The example for a Hermitian matrix can be as follows

$$C = \begin{bmatrix} 1 & 1+i & -2+i \\ 1-i & 3 & -2i \\ -2-i & 2i & -1 \end{bmatrix}$$

2. Least Squares

a) Suppose we have the following overdetermined system

$$\underbrace{\begin{bmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}}_{:=A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

where $0 < \varepsilon < \sqrt{\varepsilon_{mach}}$. Since $Ax = b$ and $A^TAx = A^Tb$. So, to obtain the normal equation, we need to find

$$A^TA = \begin{bmatrix} 1 & \varepsilon & 0 \\ 1 & 0 & \varepsilon \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix} = \begin{bmatrix} 1+\varepsilon^2 & 1 \\ 1 & 1+\varepsilon^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

So, A^TA is nonsingular.

For iterative improvement for least-squares problems firstly observe that the following linear system holds:

$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$b: r - Ax$

$A^T r = 0$

Where $A \in \mathbb{R}^{m \times n}$, $I \in \mathbb{R}^{m \times m}$, $b \in \mathbb{R}^{m \times 1}$, x is the least-squares solution of $Ax = b$ and $r = b - Ax$.

→ Solve directly normal eqns: error ~ $(\text{cond}(A))^2$

→ iterative improvement: error ~ $\text{cond}(A)$

Then, set $r^{(0)} = 0$, $x^{(0)} = 0$, and

For $k = 1, 2, 3, \ldots$

Compute $$\begin{bmatrix} r_1^{(k)} \\ r_2^{(k)} \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} - \begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r^{(k-1)} \\ x^{(k-1)} \end{bmatrix} \quad \leftarrow \text{Check}$$

Solve $$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} c_1^{(k)} \\ c_2^{(k)} \end{bmatrix} = \begin{bmatrix} r_1^{(k)} \\ r_2^{(k)} \end{bmatrix} \qquad (1)$$

Update $$\begin{bmatrix} r^{(k)} \\ x^{(k)} \end{bmatrix} = \begin{bmatrix} r^{(k-1)} \\ x^{(k-1)} \end{bmatrix} + \begin{bmatrix} c_1^{(k)} \\ c_2^{(k)} \end{bmatrix}$$

By this algorithm, in each step we get more accurate solution since by solving the system (1) and adding solution to the previously computed solution, we obtain closer solutions to the actual solution.

b) Let $x = (x_1, x_2, \ldots, x_n)^T$. Then by definition, we have

$$x^+ = (x^T x)^{-1} x^T$$

Then $x^T x = \sum_{i=1}^{n} x_i^2$, i.e. $x^T x = \|x\|_2^2$

So, $$x^+ = \frac{1}{\|x\|_2^2} (x_1, x_2, \ldots, x_n) = \left(\frac{x_1}{\|x\|_2^2}, \frac{x_2}{\|x\|_2^2}, \ldots, \frac{x_n}{\|x\|_2^2} \right)$$

Let $a = [1, 0]$ and $b = [1, 1]^T$. Then

$$ab = [1, 0] \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \qquad \text{So,} \quad (ab)^+ = 1.$$

On the other hand,

$$b^+ = \frac{1}{\|b\|_2^2} [1, 1] = \frac{1}{2} [1, 1] = [1/2, 1/2]$$

$a^+ = a^T (aa^T)^{-1}$

$aa^T = [1\ 0]\begin{bmatrix}1\\0\end{bmatrix} = 1$

So, $a^+ = \begin{bmatrix}1\\0\end{bmatrix}\cdot 1 = \begin{bmatrix}1\\0\end{bmatrix}$

Then $b^+ a^+ = [1/2 \quad 1/2]\begin{bmatrix}1\\0\end{bmatrix} = 1/2$, and we see $(ab)^+ \neq b^+ a^+$

3. Iterative Methods

Consider the stationary vector-matrix iteration given by

$$x_{k+1} = Mx_k + c \qquad (1)$$

where $M \in \mathbb{C}^{n\times n}$, $c \in \mathbb{C}^n$ and $x_0 \in \mathbb{C}^n$ are given.

a) Suppose x^* is a fixed point of (1) and $\|M\|<1$ where $\|\cdot\|$ is any compatible matrix norm induced by a vector norm.

Suppose to the contrary that $\tilde{x}$ be another fixed point of (1). Then we have

$$\|x^* - \tilde{x}\| = \|Mx^* + c - M\tilde{x} - c\| = \|M(x^* - \tilde{x})\| \underset{\substack{\downarrow \\ \text{by compatibility} \\ \text{of the norm}}}{\leq} \|M\|\cdot\|x^* - \tilde{x}\| < \|x^* - \tilde{x}\|.$$

We get $\|x^* - \tilde{x}\| < \|x^* - \tilde{x}\|$, a contradiction. So, x^* is unique.

Now, observe that for every $k \in \mathbb{N}$

$$\|x_{k+1} - x^*\| = \|Mx_k + c - Mx^* - c\| = \|M(x_k - x^*)\| \leq \|M\|\cdot\|x_k - x^*\| < \|x_k - x^*\|$$

This shows $\{\|x_k - x^*\|\}$ is a decreasing sequence. Since it is bounded below by 0, we deduce $\lim_{k\to\infty} \|x_k - x^*\| = 0$. Then firstly by continuity of the norm and then by definition of the norm, we get $\lim_{k\to\infty}(x_k - x^*) = 0$ which means $\lim_{k\to\infty} x_k = x^*$.

infimum definition ?

b) Let $r(M)$ denote the spectral radius of the matrix M

Suppose $\lim_{k\to\infty} x_k = x^*$. Observe that

$$x_{k+1} - x_k = M(x_k - x_{k-1}) = M^2(x_{k-1} - x_{k-2}) = \dots = M^k(x_1 - x_0).$$

Taking limit of both sides as $k \to \infty$, the left hand side is 0 and we get $0 = \lim_{k\to\infty} M^k (x_1 - x_0)$. Since $x_1 - x_0 \neq 0$, we get $\lim_{k\to\infty} M^k = 0$. Let λ be an arbitrary eigenvalue of M and v be the corresponding eigenvector. Then we get $M^k v = \lambda^k v$ for any $k \in \mathbb{N}$. Take limit of both sides as $k \to \infty$,

$$0 = \left(\lim_{k\to\infty} M^k\right) v = \lim_{k\to\infty} (M^k v) = \lim_{k\to\infty} (\lambda^k v) = \left(\lim_{k\to\infty} \lambda^k\right) \cdot v$$

or by Jordan canonical thm

Since $v \neq 0$, $\lim_{k\to\infty} \lambda^k = 0$. But in this case, for convergence, $|\lambda| < 1$ (otherwise $\{\lambda^k\}_{k\geq 1}$ does not converge). Since λ is arbitrary, we deduce,

$$r(M) = \max_{1\leq i\leq n} |\lambda_i(M)| < 1.$$

Conversely, suppose that $r(M) < 1$. So, denoting largest eigenvalue (in magnitude) of M by λ, we get $|\lambda| < 1$. In this case $|\lambda|^2 < 1$ which is the largest eigenvalue of M^*M. But since we know $\|M\|_2 = \sqrt{r(M^*M)} < 1$. Since all norms on $\mathbb{C}^{n\times n}$ are equivalent (finite dimensional), we can use 2-norm. But this will be the same proof in (a), just replace $\|\cdot\|$ by $\|\cdot\|_2$. So, we get $\lim_{k\to\infty} x_k = x^*$ (since we know the existence of the fixed point, x^*).

c) For Jacobi method, we write $A = L + D + U$ where

$$L = \begin{bmatrix} 0 & 0 & & 0 \\ a_{21} & 0 & & \\ a_{31} & a_{32} & & 0 \\ a_{n1} & a_{n2} & \cdots & 0 \end{bmatrix}, \quad D = \begin{bmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & a_{nn} \end{bmatrix}, \quad U = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ & 0 & & & \vdots \\ 0 & & & & 0 \; 0 \end{bmatrix}$$

Then,

$$Ax = b \iff (L+D+U)x = b \iff Dx = -(L+U)x + b \iff x = -D^{-1}(L+U)x + D^{-1}b.$$

Letting $B_j = -D^{-1}(L+U)$ and $d_j = D^{-1}b$, we get the following iteration

$$x^{(k+1)} = B_j x^{(k)} + d_j.$$

So, $M = -D^{-1}(L+U)$ and $c = D^{-1}b$ for L, D, U above.

d) We have that

$$L+U=\begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & 0 & & \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \qquad D^{-1}=\begin{bmatrix} 1/a_{11} & & & \\ & 1/a_{22} & & 0 \\ & & \ddots & \\ 0 & & & 1/a_{nn} \end{bmatrix}$$

So,

$$-D^{-1}(L+U)=\begin{bmatrix} 0 & -a_{12}/a_{11} & -a_{13}/a_{11} & \cdots & -a_{1n}/a_{11} \\ -a_{21}/a_{22} & 0 & -a_{23}/a_{22} & \cdots & -a_{2n}/a_{22} \\ -a_{31}/a_{33} & -a_{32}/a_{33} & 0 & \cdots & -a_{3n}/a_{33} \\ \vdots & \vdots & \vdots & & \\ -a_{n1}/a_{nn} & -a_{n2}/a_{nn} & -a_{n3}/a_{nn} & \cdots & 0 \end{bmatrix} = M$$

Then, for any $i \in \{1,2,\dots,n\}$,

$$\sum_{j=1}^{n} |m_{ij}| = \frac{1}{|a_{ii}|} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \qquad (1)$$

But since A is strictly diagonally dominant, we say that for any $i \in \{1,\dots,n\}$,

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$$

So, we see from (1) that $\sum_{j=1}^{n} |m_{ij}| < 1$ for any $i \in \{1,\dots,n\}$. Thus, by definition $\|M\|_\infty < 1$. So, by part (a), Jacobi iteration will converge to the solution of the linear system.

d) Suppose A is strictly diagonally dominant. Then for any i,

$$\frac{1}{|a_{ii}|} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| < 1.$$

On the other hand by Gershgorin Theorem we have

$$|\lambda - m_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |m_{ij}| \quad \text{for } i=1,2,\dots,n.$$ * λ is any eigenvalue

But since $m_{ii}=0$ for all i and by (1) in part (c), we get

$$|\lambda| < \frac{1}{|a_{ii}|} \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| < 1$$

Thus, $|\lambda|<1$ where λ is an arbitrary eigenvalue of M. Then, by definition $r(M)<1$ and by part (b), we say that Jacobi iteration converges to a solution of the linear system.

4. Computation of Eigenvalues and Eigenvectors

a) Let us prove it by induction. Suppose none of h_k is zero.

→ Show that $q_2 \perp q_1$. Here, it is enough to show that $h_2 \perp q_1$. By definition,

$$h_2 = p_2 - (q_1^H p_2)\, q_1 = Aq_1 - (q_1^H A q_1)\, q_1$$

Then, $q_1^H h_2 = q_1^H A q_1 - q_1^H (q_1^H A q_1) q_1 = q_1^H A q_1 - (q_1^H A q_1) \underbrace{q_1^H q_1}_{=1} = 0$

→ Suppose that $q_k \perp q_i$ for $i=1,\dots,k-1$, for any $k\in\mathbb{N}$. We want to show $q_{k+1} \perp q_i$ for all $i=1,2,\dots,k$. Again, it is enough to show $h_{k+1} \perp q_i$ for all $i=1,\dots,k$. Observe now,

$$h_{k+1} = p_{k+1} - (q_1^H p_{k+1})\, q_1 - \cdots - (q_k^H p_{k+1})\, q_k$$

$$= Aq_k - (q_1^H A q_k)\, q_1 - \cdots - (q_i^H A q_k)\, q_i - \cdots (q_k^H A q_k)\, q_k \quad (i \text{ arbitrary}, 1\leq i\leq k)$$

Then for any i between 1 and k $(1\leq i\leq k)$,

$$q_i^H h_{k+1} = q_i^H A q_k - (q_1^H A q_k) \underbrace{q_i^H q_1}_{=0} - \cdots - (q_i^H A q_k) \underbrace{q_i^H q_i}_{=1} - \cdots - (q_k^H A q_k) \underbrace{q_i^H q_k}_{=0}$$

$$= q_i^H A q_k - q_i^H A q_k$$

$$= 0$$

Firstly observe the case $i-j \geq 2$:

b) $[Q^H A Q]_{ij} = q_i^H A q_j = q_i^H p_{j+1} = q_i^H \left(h_{j+1} + \sum_{n=1}^{j} (q_n^H p_{j+1}) q_n \right)$

$= q_i^H h_{j+1} + \sum_{n=1}^{j} (q_n^H p_{j+1})(q_i^H q_n)$

$= \|h_{j+1}\|_2 \underbrace{q_i^H q_{j+1}}_{=0} + (q_1^H p_{j+1}) \underbrace{(q_i^H q_1)}_{=0} + \underbrace{\cdots}_{=0} + (q_j^H p_{j+1}) \underbrace{(q_i^H q_j)}_{=0}$ since $i \geq j+2$

$= 0$

So,

$$Q^H A Q = \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ & * & * & \cdots & * \\ & & * & \ddots & \vdots \\ & 0 & & \ddots & \vdots \\ & & & * & * \end{bmatrix}$$

But since A is Hermitian, $(Q^H A Q)^H = Q^H A^H (Q^H)^H = Q^H A Q$, we see $(Q^H A Q)_{ij} = 0$ also for $i-j \leq -2$. Thus $Q^H A Q$ is tridiagonal

c) Since Q is orthogonal, $Q^H A Q$ and A have the same eigenvalues, applying QR algorithm to $Q^H A Q$ instead of A, we can decrease the cost of QR iteration significantly. (since $Q^H A Q$ is tridiagonal, but A may be full and dense.)

Numerical Analysis Screening Exam, Spring 2011

FIRST NAME:

LAST NAME:

STUDENT ID NUMBER:

SIGNATURE:

PROBLEM 1. (LINEAR EQUATIONS)

(a) Give a definition of matrix A being positive definite.

(b)-i State **any** theorem for solving $Ax = b$ with symmetric positive definite (SPD) matrices.

(b)-ii What are the computational advantages for solving a problem $Ax = b$ with SPD matrices. Be specific.

(c) Find the largest interval for α so that A is positive definite:

$$A = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}.$$

(Hint: use Gaussian elimination.)

Name:

PROBLEM 2. (EIGENVALUE PROBLEMS)

(a) Let $A = [a_{i,j}]$ be an $n \times n$ matrix. Prove Gerschgorin's Theorem which states the following: For $i = 1, \cdots, n$ let $R_i = \sum_{j=1, j\neq i}^{n} |a_{i,j}|$. Every eigenvalue of A falls within one of the closed discs in the complex plane with center at $a_{i,i}$ and radius R_i. (Hint. Let

$$Ax = \lambda x, \tag{1}$$

and assume the largest component of x in absolute value is x_k. Consider the k-equation of (1).)

(b) Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Show that A is positive definite. (Hint. Use Gerschgorin's Theorem to show that A is positive semi-definite. Then consider the equation $Ax = 0$, where the first component of x equals 1.)

(c) Let

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Find the eigenvalues of A and verify that Gerschgorin's Theorem holds.

Name:

PROBLEM 3. (ITERATIVE METHODS)

(a) Let L be a strictly lower triangular and U be a strictly upper triangular $n \times n$ matrices. For any positive number ω define

$$B_\omega = (1 - \omega L)^{-1} \left[(1-\omega)I + \omega U\right].$$

Show that $\rho(B_\omega) \geq |\omega - 1|$ where $\rho(B_\omega)$ is the spectral radius of B_ω. (Hint. Calculate $\det(B_\omega)$ directly and by the product of eigenvalues and compare.)

(b) Consider solving $Ax = b$ where A is an $n \times n$ matrix and $x, b \in \mathbb{R}^n$.

(i) Give the matrix form of the SOR iterative method.

(ii) Using the result in question (a) above, show that the SOR method with parameter ω can only converge for $0 < \omega < 2$.

(c) Consider the equation $Ax = b$, where $x \in \mathbb{R}^3$ and

$$A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Can the SOR method be applied to this equation? Justify your answer.

Names:

PROBLEM 4. (LEAST SQUARES PROBLEM)

(a) Let $A \in R^{m\times n}$. Give a detailed description of the SVD of A and present briefly an algorithm to solve an overdetermined system using SVD.

(b) What are the advantages and disadvantages of using SVD for overdetermined systems.

(c) Consider the overdetermined system $Au = b$ with

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Find the family of least squares solutions using **any** method that is easy for hand calculation, and finally find the minimum-norm solution to the least squares problems.

Sp 2011:

Linear Equations

a) A matrix $A \in \mathbb{R}^{n\times n}$ is said to be positive definite if $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$.

b) i) Given a linear system $\underbrace{}_{}$ $Ax=b$ where $A \in \mathbb{R}^{n\times n}$ is an SPD matrix, there exists a lower triangular matrix H with nonzero diagonal entries so that $A = HH^T$. Then solving the linear system $Ax=b$ reduces to solve two linear triangular systems $Hy=b$ for y and then $H^T x = y$ for x.

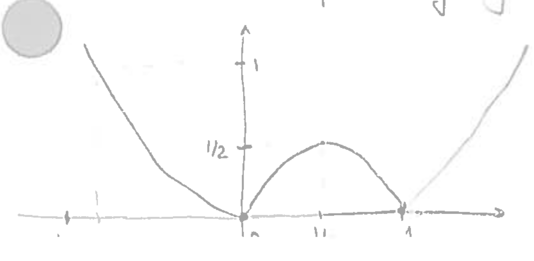
ii) If A is symmetric, we know than even Gaussian elimination without pivoting is stable and gives the LU decomposition of A Furthermore, since A is symmetric, this LU decomposition indeed the Cholesky decomposition of A because of A being symmetric and uniqueness of LU decomposition (when the main diagonal of L is assumed to be all 1's). Note that in this case, direct computation of Cholesky decomposition is cheaper than GEWP. So, we reduce the system $Ax=b$ into $HH^T x = b$ in a cheaper way, since solving triangular system is also cheaper.

c) We know that a necessary condition for a matrix to be SPD is that the largest element in magnitude of whole matrix is on the main diagonal. So, we must have $|\alpha| \le 1$.

On the other hand by GEWP, we obtain

$$A^{(1)} = \begin{pmatrix} 1 & \alpha & \alpha \\ 0 & 1-\alpha^2 & \alpha-\alpha^2 \\ 0 & \alpha-\alpha^2 & 1-\alpha^2 \end{pmatrix}.$$

By theorem, we know for SPD matrices if $|a_{ij}| \le 1$ then $|a_{ij}^{(k)}| \le 1$. Since $|\alpha| \le 1$, we must have $|\alpha - \alpha^2| \le 1$. Now, letting $f(\alpha) = |\alpha - \alpha^2|$, we have the following graph for $f(\alpha)$



Now, it is enough to find x. It is enough to solve $x - x^2 = -1$ or $x^2 - x - 1 = 0$. The roots are $\frac{1 \pm \sqrt{5}}{2}$. Here, the desired root is $\frac{1-\sqrt{5}}{2}$

So, the largest for α is $\left[\frac{1-\sqrt{5}}{2}, 1\right)$.

* This can be solvable by direct computation of eigenvalues $\lambda_{1,2} = 1-\alpha \quad \lambda_3 = 1+2\alpha$

Eigenvalue Problems:

a) Let λ be any eigenvalue and x be the corresponding eigenvector. Then for any $i \in \{1, \dots, n\}$, we have

$$\lambda x_i = \sum_{j=1}^{n} a_{ij} x_j \Rightarrow \lambda x_i - a_{ii} x_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j$$

Let x_k be the largest component of x, then we have

$$|\lambda - a_{kk}| |x_k| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{ij}| |x_j| \Rightarrow |\lambda - a_{kk}| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{ij}| \underbrace{\frac{|x_j|}{|x_k|}}_{\leq 1 \text{ for each } j} \leq \sum_{\substack{j=1 \\ j \neq k}} |a_{ij}| = R_k$$

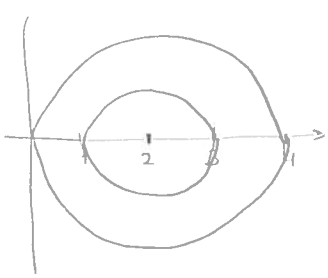
So, we showed that the arbitrary eigenvalue λ falls within R_k, i.e. $\lambda \in \{u : |u - a_{kk}| \leq R_k\}$.

b) For the given matrix A, we have

$R_1 = 1, \quad R_2 = 2, \quad R_3 = 2, \quad R_4 = 2, \quad R_5 = 1$

$a_{ii} = 2 \quad$ for $\quad i = 1, 2, 3, 4, 5$

So, all possible disks are $\{u : |u-2| \leq 1\}$ and $\{u : |u-2| \leq 2\}$ whose graph is as follows



We see that all eigenvalues of A are nonnegative (Note that the eigenvalues of A are real since A is symmetric).

Now, suppose for a moment that $\lambda = 0$ is an eigenvalue of A. Then we get $Ax=0$, for a nonzero vector $x \in \mathbb{R}^5$ Now consider the augmented matrix and apply GE,

$$\left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 5/4 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccccc|c} 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 3/2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 5/4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 6/5 & 0 \end{array}\right]$$

By back substition, we see that the only solution of $Ax=0$ is 0, but since an eigenvector cannot be 0 we deduce that $\lambda = 0$ cannot be an eigenvalue of A. Hence all of the eigenvalues of A must be positive

We know that A is SPD if and only if all of its eigenvalues are positive. Therefore, we proved A is SPD.

c)

$$0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2-\lambda \end{vmatrix}$$

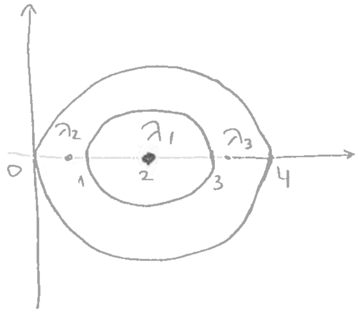
$$= (2-\lambda)\left[(2-\lambda)^2 - 1\right] - (2-\lambda)$$

$$= (2-\lambda)\left[(2-\lambda)^2 - 2\right]$$

$$= (2-\lambda)(2-\lambda-\sqrt{2})(2-\lambda+\sqrt{2})$$

$\Rightarrow \lambda_1 = 2$, $\lambda_2 = 2-\sqrt{2}$, $\lambda_2 = 2+\sqrt{2}$ are the eigenvalues of A.

The possible disks are $\{u : |u-2| \le 1\}$ and $\{u : |u-2| \le 2\}$.



So, it can be seen that the eigenvalues are in the union of disks.

Iterative Methods

a) Since L is strictly lower triangular, the matrix $I-\omega L$ has 1's on diagonal and so does $(I-\omega L)^{-1}$. So, $\det[(I-\omega L)^{-1}]=1$.

On the other hand since U is strictly upper triangular, ωU has 0's on diagonal and so, $(1-\omega)I+\omega U$ has $1-\omega$ on diagonal. So,
$\det[(1-\omega)I+\omega U]=(1-\omega)^n$ Hence,

$$\det(B_\omega)=\det[(I-\omega L)^{-1}]\det[(1-\omega)I+\omega U]$$
$$=(1-\omega)^n$$

On the other hand for any matrix A, letting $\lambda_1,\dots,\lambda_n$ being eigenvalues of A, we have

$$\det(A-\lambda I)=(\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda)$$

Setting $\lambda=0$, we get $\det(A)=\lambda_1\cdot\lambda_2\dots\lambda_n$.

Let $\lambda_1,\dots,\lambda_n$ be eigenvalues of B_ω. Then we get

$$(1-\omega)^n=\lambda_1\dots\lambda_n \Rightarrow |1-\omega|^n=|\lambda_1|\dots|\lambda_n|.$$

By definition, $\rho(B_\omega)=\max_{1\le i\le n}|\lambda_i|$. So,

$$|1-\omega|^n\le\rho(B_\omega)^n \Rightarrow \rho(B_\omega)\ge|\omega-1|.$$

b) i) For SOR method, we write $A=L+D+U$ where L is strictly lower triangular, D is diagonal, U is strictly upper triangular, and then

$$Ax=b \Leftrightarrow (L+D+U)x=b \Leftrightarrow (\omega L+\omega D+\omega U)x=\omega b$$
$$\Leftrightarrow (D+\omega L-D+\omega D+\omega U)x=\omega b \Leftrightarrow (D+\omega L)x=(D-\omega D-\omega U)x+\omega b$$
$$\Leftrightarrow x=(D+\omega L)^{-1}[(1-\omega)D-\omega U]x+(D+\omega L)^{-1}\omega b.$$

So, if $x^{(k+1)}=B_{SOR}x^{(k)}+d_{SOR}$, $k=1,2,\dots$ then $B_{SOR}=(D+\omega L)^{-1}[(1-\omega)D-\omega U]$ and

$d_{SOR} = (D+\omega L)^{-1} \omega b$.

i) If $D = \text{diag}(d_{11}, d_{22}, \dots, d_{nn})$ then as in part(a),

$$\det(B_{SOR}) = \frac{1}{d_{11} \cdot d_{22} \cdots d_{nn}} (1-\omega)^n \cdot d_{11} d_{22} \cdots d_{nn} = (1-\omega)^n.$$

Similarly we obtain $\rho(B_{SOR}) \geq |\omega - 1|$. In order to converge iteration $x^{(k+1)} = B_{SOR} x^{(k)} + d_{SOR}$, $k = 1, 2, \dots$, (the neccessary and sufficient condition is) we must have, $\rho(B_{SOR}) < 1$. So, we must have $|\omega - 1| < 1$ for convergence

That is, $0 < \omega < 2$ for convergence.

c) Here,

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then

$$D + \omega L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ \omega & 0 & 0 \\ \omega & -\omega & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \omega & -1 & 0 \\ \omega & -\omega & 2 \end{bmatrix}$$

But we see that $\det(D+\omega L) = 0$, i.e. $D+\omega L$ is singular. So, we cannot form the iteration matrix B_{SOR} and the vector d_{SOR}. Hence, we cannot apply SOR iteration to this system.

Least Squares Problem

a) Let $A \in \mathbb{R}^{m \times n}$. Let $m \geq n$. Because if $m < n$, then we can consider A^T. There exist orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that

$$A = U \Sigma V^T$$

where $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$. This decomposition is called as singular value decomposition.

When $m > n$ we have $Ax = b$ as an overdetermined system. If $U \Sigma V^T$ is reduced SVD of A, then (reduced SVD is cheaper)

$$Ax = b \iff U \Sigma V^T x = b \iff \Sigma V^T x = U^T b \overset{\substack{U^T b = \hat{b} \\ V^T x = y}}{\iff} \Sigma y = \hat{b}$$

So, an algorithm to solve the overdetermined system using SVD can be as follows

Input : The overdetermined system $Ax=b$

Output : The least-squares solution of $Ax=b$.

1. Compute the reduced SVD of A : $A=U\Sigma V^T$
2. Compute $\hat{b}$: $\hat{b}=U^Tb$
3. Solve the diagonal system $\Sigma y=\hat{b}$ for y.
4. Perform matrix-vector multiplication to get x : $x=Vy$.

b) By using SVD for overdetermined system, we only need to solve the system $\Sigma y=\hat{b}$ where Σ is diagonal, which is very easy to solve and we perform 2 matrix-vector multiplications. However, computing SVD is an expensive process. The cost of solving an overdetermined system by Gaussion elimination or QR factorization is cheaper. So, SVD is not used in practice to solve linear systems

On the other hand, if the matrix A is rank-deficient, then solving an overdetermined system by SVD is the best numerically reliable way. Furthermore, we know, in this case, by SVD, we can obtain the minimum norm solution.

c) The normal equations are $A^TAx=A^Tb$. So,

$$A^TA=\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\\1&1\end{bmatrix}=\begin{bmatrix}3&3\\3&3\end{bmatrix}, \quad A^Tb=\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}6\\6\end{bmatrix}$$

So, we have the following system,

$$\begin{bmatrix}3&3\\3&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}6\\6\end{bmatrix} \Rightarrow 3x_1+3x_2=6 \Rightarrow x_1+x_2=2.$$

Then setting $x_1=a$ we have $x_2=2-a$ and so, $x=[a \;\; 2-a]^T$. Note that x is a solution to the given system $\forall a\in\mathbb{R}$. But we want to find the minimum norm solution. Since $\|x\|_2=\sqrt{a^2+4-4a+a^2}=\sqrt{2a^2-4a+4}$, we get the result by minimizing $f(a):=2a^2-4a+4$. $f'(a)=4a-4$. So, the only critical point is $a=1$. Since $f''(a)=4>0$, $a=1$ is the local minimium of $f(a)$. Then we deduce $a=1$ is also the absolute minimium. Hence, the minimum norm solution to the given system is $x=[1 \;\; 1]^T$.

may see problem 4 again?
notnize

Numerical Analysis Screening Exam, Fall 2010

FIRST NAME:

LAST NAME:

STUDENT ID NUMBER:

SIGNATURE:

PROBLEM 1 Let A be an $m \times n$ matrix and b an $m \times 1$ vector. Consider the least squares problem (LS): find x that minimizes $\|Ax - b\|_2^2$.

(a) Give necessary and sufficient conditions for x to be a solution to (LS).

(b) When is this solution unique?

(c) When is the corresponding residual vector $r = Ax - b$ unique?

(d) Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find the solution x to (LS) that also minimizes $\|x\|_2^2$.

Name:

PROBLEM 2. A symmetric $n \times n$ matrix A is called Symmetric Positive Definite (SPD) iff $(Ax, x) > 0$ for all $x \neq 0$.

(a) Assume $n = 2$. Show that if A is SPD then A admits a Cholesky factorization, i.e $A = L \cdot L^T$ where L is a nonsingular lower triangular matrix.

(b) Let

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}.$$

Show that A is SPD and calculate the Cholesky factorization of A.

Name:

PROBLEM 3.

(a) Why does any eigenvalue solver have to be iterative?

(b) Present the QR-algorithm to solve eigenvalue problems in some detail and state the corresponding theorem for convergence.

(c) Let $A_k = \begin{pmatrix} c & s \\ s & 0 \end{pmatrix}$, where $c = \cos\theta$ and $s = \sin\theta$. Compute A_{k+1} using QR-iteration, and show that the off-diagonal elements of A_{k+1} are smaller than those in A_k.

Names:

PROBLEM 4. Consider a matrix A given by

$$A = \begin{pmatrix} 1.009 & -0.009 & -0.999 \\ 0.999 & 0.001 & -0.999 \\ 0.009 & -0.009 & 0.001 \end{pmatrix}$$

(a) Verify that $x = [2, 2, 1]^T$ is a solution of $Ax = [1.001, 1.001, 0.001]^T$. Consider the vectors $y = [2.2, 2.2, 1]^T$ and $z = [202, 202, 201]^T$. Verify that $Ax - Ay$ and $Ax - Az$ have the same infinity vector norm.

(b) Find an estimation of the condition number of A using the results of (a).

(c) Verify that

$$A^{-1} = \begin{pmatrix} -899 & 900 & 999 \\ -999 & 1000 & 999 \\ -900 & 900 & 1000 \end{pmatrix}.$$

Determine whether or not

$$B = \begin{pmatrix} -900 & 900 & 1000 \\ -1000 & 1000 & 1000 \\ -900 & 900 & 1000 \end{pmatrix}$$

is a sufficiently good approximation of the matrix A^{-1} to be used in an iterative method of the form $x_{k+1} = x_k - BAx_k + Bb$ for finding the solution of $Ax = b$.

(Hint: $B \cdot A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$)

Fall 2010

Problem 1) Let A be $m\times n$ matrix and b an $m\times 1$ vector. Consider the least squares problem (LS): find x that minimizes $\|Ax-b\|_2^2$.

a) Give necessary and sufficient conditions for x to be a solution to (LS).

b) When is the solution unique?

c) When is the corresponding residual vector $r=Ax-b$ unique?

d) Let

$$A=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}, \quad b=\begin{bmatrix}1\\0\end{bmatrix}$$

Find the solution x to (LS) that also minimizes $\|x\|_2^2$.

Solution: a) Suppose $m\geq n$. (If $m<n$, then since $A^T\in\mathbb{R}^{n\times m}$, we can state the following necessary and sufficient condition for A^T). We say that x is a solution of the LS problem $Ax=b$ if and only if it satisfies the following equation (which is called normal equations)

$$A^TAx=A^Tb.$$

b) A solution x to the LS problem $Ax=b$ is unique if and only if A has full rank. ?

c) The residual vector $r=Ax-b$ is unique if the solution x is unique

d) We know that underdetermined systems have either no solution or infinitely many solutions. If A has full rank, then the solution exists and the general solution can be written as

$$x=A^T(AA^T)^{-1}b+(I-A^T(AA^T)^{-1}A)y$$

where y is arbitrary. In order to get the minimum-norm solution, we need to set $y=0$

$$AA^T=\begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

Since A has full rank, we know AA^T is invertible and we have

$$(AA^T)^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

$$(AA^T)^{-1} b = \begin{bmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix}$$

$$A^T(AA^T)^{-1}b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2/3 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

So, $x = [1/3 \;\; 2/3 \;\; -1/3]^T$ is the minimum norm solution to the given system.

Problem 2) A symmetric $n \times n$ matrix A is called Symmetric Positive Definite (SPD) iff $(Ax, x) > 0$ for all $x \neq 0$.

a) Assume $n = 2$. Show that if A is SPD then A admits a Cholesky factorization, i.e. $A = LL^T$ where L is a nonsingular lower triangular matrix.

b) Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

Show that A is SPD and calculate the Cholesky factorization of A.

a) Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$. Since A is SPD, we know $a_{11} > 0$ and $a_{22} > 0$. Again, since A is SPD, we know all of its eigenvalues are positive. That is, $\lambda_1, \lambda_2 > 0$. Since

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{12} & a_{22} - \lambda \end{vmatrix} = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}^2,$$

we know that λ_1 and λ_2 are roots of this function. So,

$$\lambda_1 + \lambda_2 = a_{11} + a_{22} \quad \text{and} \quad \lambda_1 \cdot \lambda_2 = a_{11}a_{22} - a_{12}^2$$

Since $\lambda_1 . \lambda_2 > 0$, we get $a_{11}a_{22} - a_{12}^2 > 0$.

Now if $L=\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$, then we must have

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \Longleftrightarrow \begin{matrix} a^2 = a_{11} & ab = a_{12} \\ ab = a_{12} & b^2+c^2 = a_{22} \end{matrix}$$

Since $a_{11}>0$, then $a=\sqrt{a_{11}}$ (choose positive root for Cholesky factorization). So, we get $b = a_{12}/\sqrt{a_{11}}$. Then we get

$$b^2+c^2 = a_{22} \Longleftrightarrow \frac{a_{12}^2}{a_{11}} + c^2 = a_{22} \Longleftrightarrow c^2 = \frac{a_{11}a_{22}-a_{12}^2}{a_{11}}.$$

Since $a_{11}a_{22}-a_{12}^2>0$, we get $c=\sqrt{\dfrac{a_{11}a_{22}-a_{12}^2}{a_{11}}}$. So, if

$$L = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ \dfrac{a_{12}}{\sqrt{a_{11}}} & \sqrt{\dfrac{a_{11}a_{22}-a_{12}^2}{a_{11}}} \end{bmatrix}$$

then

$$LL^T = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ \dfrac{a_{12}}{\sqrt{a_{11}}} & \sqrt{\dfrac{a_{11}a_{22}-a_{12}^2}{a_{11}}} \end{bmatrix}\begin{bmatrix} \sqrt{a_{11}} & \dfrac{a_{12}}{\sqrt{a_{11}}} \\ 0 & \sqrt{\dfrac{a_{11}a_{22}-a_{12}^2}{a_{11}}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} = A$$

and we see that $\det(L) = \sqrt{a_{11}a_{22}-a_{12}^2} > 0$, so L is nonsingular and L is lower triangular. This shows that L admits a Cholesky factorization.

b) Let us find the eigenvalues of A.

$$0 = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 2-\lambda & 3 \\ 1 & 3 & 6-\lambda \end{vmatrix} = (1-\lambda)\left[(2-\lambda)(6-\lambda)-9\right] - (6-\lambda-3) + (3-2+\lambda)$$

$$= (1-\lambda)(\lambda^2-8\lambda+3) - 2(1-\lambda)$$

$$= (1-\lambda)(\lambda^2-8\lambda+1)$$

$\lambda_1 = 1 > 0$, $\quad \lambda_{2,3} = \dfrac{8 \pm \sqrt{64-4}}{2} > 0.$

Since all eigenvalues of A are positive, A is SPD.

Let

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

(By theorem, we know A admits a Cholesky factorization)

be such that $LL^T = A$.

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

* $l_{11}^2 = 1 \Rightarrow l_{11} = 1$

  $l_{21} l_{11} = 1 \Rightarrow l_{21} = 1$

  $l_{31} l_{11} = 1 \Rightarrow l_{31} = 1$

* $\underbrace{l_{21}^2}_{=1} + l_{22}^2 = 2 \Rightarrow l_{22} = 1$

  $\underbrace{l_{31} l_{21}}_{=1} + l_{32} \underbrace{l_{22}}_{=1} = 3 \Rightarrow l_{32} = 2$

* $\underbrace{l_{31}^2}_{=1} + \underbrace{l_{32}^2}_{=4} + l_{33}^2 = 6 \Rightarrow l_{33} = 1.$

So,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Problem 3)

a) Why does any eigenvalue solver have to be iterative?

b) Present the QR-algorithm to solve eigenvalue problems in some detail and state the corresponding theorem for convergence.

c) Let $A_k = \begin{bmatrix} c & s \\ s & 0 \end{bmatrix}$, where $c = \cos\theta$ and $s = \sin\theta$. Compute A_{k+1} using QR-iteration, and show that the off-diagonal elements of A_{k+1} are smaller than those in A_k.

Solution:

a) We know that eigenvalues of a matrix are the roots of characteristic polynomial of the matrix. We also know that there is no solution formula for the polynomials of degree higher than 4, and also a root of a polynomial of degree higher than 4 may not be expressed algebraically. So, the exact eigenvalues can be calculated in finite number of steps only for a very special class of matrices. For a general matrix (if it does not belong to that special class) it is impossible to compute eigenvalues in finite steps. Hence, any eigenvalue solver has to be iterative and root finding formulas are unstable

b) Input: An $n \times n$ matrix A

Output. A sequence of matrices $\{A_k\}$ containing the eigenvalues of A.

Set $A_0 = A$.

For $k = 1, 2, 3, \ldots$

→ Find QR factorization of A_{k-1}: $A_{k-1} = Q_{k-1} R_{k-1}$

→ Compute $A_k = R_{k-1} Q_{k-1}$.

We know that A_i is orthogonally similar to A_{i-1} for $i = 1, 2, \ldots$

→ $A_{i-1} = Q_{i-1} R_{i-1}$ and $A_i = R_{i-1} Q_{i-1}$.

Since Q_{i-1} is orthogonal, $Q_{i-1}^T A_{i-1} = R_{i-1}$ and so,

$A_i = Q_{i-1}^T A_{i-1} Q_{i-1}^T$

So, the eigenvalues of each A_i are the same. Under certain conditions, if $\{A_k\}$ converges to a triangular or quasi-triangular matrix, then we will be done.

Theorem: Let the eigenvalues $\lambda_1, \ldots, \lambda_n$ be such that

$|\lambda_1| > |\lambda_2| > \cdots |\lambda_n|$ and let the eigenvector matrix X of the left eigenvectors be such that its leading principal minors are nonzero. Then $\{A_k\}$ converges to an upper triangular matrix or to the real Schur form.

c) We know (from classical Gram-Schmidt)

$$a_1 = r_{11} q_1 , \quad r_{11} = \|a_1\|$$

$$a_2 = r_{12} q_1 + r_{22} q_2 , \quad r_{12} = q_1^T a_2 , \quad r_{22} = \|\hat{q}_2\| , \quad \hat{q}_2 = a_2 - r_{12} q_1$$

$$\longrightarrow r_{11} = \sqrt{c^2 + s^2} = 1 , \quad q_1 = \begin{bmatrix} c \\ s \end{bmatrix}$$

$$\longrightarrow r_{12} = \begin{bmatrix} c & s \end{bmatrix} \begin{bmatrix} s \\ 0 \end{bmatrix} = cs$$

$$\hat{q}_2 = \begin{bmatrix} s \\ 0 \end{bmatrix} - cs \begin{bmatrix} c \\ s \end{bmatrix} = \begin{bmatrix} s - c^2 s \\ -cs^2 \end{bmatrix} = \begin{bmatrix} s^3 \\ -cs^2 \end{bmatrix}$$

$$r_{22} = \sqrt{s^6 + c^2 s^4} = \sqrt{s^4(s^2 + c^2)} = s^2$$

$$q_2 = \begin{bmatrix} s \\ -c \end{bmatrix}$$

$$\text{So, } Q_k R_k = \begin{bmatrix} c & s \\ s & -c \end{bmatrix} \begin{bmatrix} 1 & cs \\ 0 & s^2 \end{bmatrix}$$

$$A_{k+1} = R_k Q_k = \begin{bmatrix} 1 & cs \\ 0 & s^2 \end{bmatrix} \begin{bmatrix} c & s \\ s & -c \end{bmatrix} = \begin{bmatrix} c(1+s^2) & s^3 \\ s^3 & -cs^2 \end{bmatrix}$$

The off-diagonal elements of A_k are the same: $\sin\theta$ and the off-diagonal elements of A_{k+1} are also the same: $\sin^3\theta$. If $\sin\theta = \pm 1$ then $\cos\theta = 0$ and $A_k = \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$ and since the colums of Q are already orthogonal, we will get $Q = A_k$ and $R = I$. So, let us ignore this case. Now, since $-1 < \sin\theta < 1$ we will have $|\sin^3\theta| \leq |\sin\theta|$. So, the off-diagonal entries got smaller in magnitude.

Problem 4) Consider a matrix A given by

$$A = \begin{bmatrix} 1.009 & -0.009 & -0.999 \\ 0.999 & 0.001 & -0.999 \\ 0.009 & -0.009 & 0.001 \end{bmatrix}$$

a) Verify that $x = [2\ 2\ 1]^T$ is a solution of $Ax = [1.001, 1.001, 0.001]^T$.
Consider the vectors $y = [2.2, 2.2, 1]^T$ and $z = [202, 202, 201]^T$.
Verify that $Ax - Ay$ and $Ax - Az$ has the same infinity vector norms.

b) Find an estimation of the condition number of A using the results of (a).

c) Verify that

$$A^{-1} = \begin{bmatrix} -899 & 900 & 999 \\ -999 & 1000 & 999 \\ -900 & 900 & 1000 \end{bmatrix}$$

Determine whether or not

$$B = \begin{bmatrix} -900 & 900 & 1000 \\ -1000 & 1000 & 1000 \\ -900 & 900 & 1000 \end{bmatrix}$$

is a sufficiently good approximation of the matrix A^{-1} to be used in an iterative method of the form $x_{k+1} = x_k - BAx_k + Bb$ for finding the solution of $Ax = b$

(Hint $B \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$)

Solution:

a)

$$Ax = \begin{bmatrix} 1.009 & -0.009 & -0.999 \\ 0.999 & 0.001 & -0.999 \\ 0.009 & -0.009 & 0.001 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - 0.999 \\ 2 - 0.999 \\ 0.001 \end{bmatrix} = \begin{bmatrix} 1.001 \\ 1.001 \\ 0.001 \end{bmatrix}$$

Now, observe that $x - y = [-0.2, -0.2, 0]^T$ and $x - z = [-200, -200, -200]$. Then,

$$Ax-Ay = A(x-y) = \begin{bmatrix} 1.009 & -0.009 & -0.999 \\ 0.999 & 0.001 & -0.999 \\ 0.009 & -0.009 & 0.001 \end{bmatrix} \begin{bmatrix} -0.2 \\ -0.2 \\ 0 \end{bmatrix} = -0.2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.2 \\ -0.2 \\ 0 \end{bmatrix}$$

So, $\|Ax-Ay\|_\infty = 0.2$

$$Ax-Az = A(x-z) = \begin{bmatrix} 1.009 & -0.009 & -0.999 \\ 0.999 & 0.001 & -0.999 \\ 0.009 & -0.009 & 0.001 \end{bmatrix} \begin{bmatrix} -200 \\ -200 \\ -200 \end{bmatrix} = -200 \begin{bmatrix} 0.001 \\ 0.001 \\ 0.001 \end{bmatrix} = \begin{bmatrix} -0.2 \\ -0.2 \\ -0.2 \end{bmatrix}$$

So, $\|Ax-Ay\|_\infty = 0.2$.

b) We know that

$$K(A) = \max_{\forall b_1, b_2} \frac{\|\delta x\| / \|x_1\|}{\|\delta b\| / \|b_1\|}$$

where $\delta x = x_1 - x_2$, $\delta b = b_1 - b_2$ with $Ax_1 = b_1$, $Ax_2 = b_2$.

From part (a), let $x_1 = [2,2,1]^T$, and $x_2 = [202, 202, 201]$. So,

$\delta x = [-200, -200, -200]^T$, $b_1 = [1.001, 1.001, 0.001]^T$ and

$\delta b = Ax_1 - Ax_2 = [-0.2, -0.2, 0.2]$

$\Rightarrow \|\delta x\|_1 = 600$, $\|x_1\|_1 = 5$, $\|\delta b\|_1 = 0.6$, $\|b\|_1 = 2.003$

$$\Rightarrow K_1(A) > \frac{|\delta x\| \ \|b_1\|}{\|\delta b\| \ \|x_1\|} = \frac{600 * (2.003)}{0.6 * 5} = 200 * (2.003) = 40.6$$

c)
$$C := A^{-1}A = \begin{bmatrix} -899 & 900 & 999 \\ -999 & 1000 & 999 \\ -900 & 900 & 1000 \end{bmatrix} \begin{bmatrix} 1.009 & -0.009 & -0.999 \\ 0.999 & 0.001 & -0.999 \\ 0.009 & -0.009 & 0.001 \end{bmatrix}$$

$$C_{11} = (-900+1)(1.009) + 900(0.999) + (1000-1)(0.009)$$

$$= -9*(100.9) + 1.009 + 9*(99.9) + 9 - 0.009$$

$$= 9(-100.9 + 99.9 + 1) + 1.009 - 0.009$$

$$= 1$$

$$C_{12} = (900-1)*(0.009) + 900*(0.001) - (1000-1)*(0.009)$$

$$= 8.1 - 0.009 + 0.9 - 9 + 0.009$$

$C_{13} = (900-1)(0.999) - 900*(0.999) + (1000-1)*(0.001)$

$= 900*(0.999) - 0.999 - 900*(0.999) + 1 - 0.001$

$= 0$

$C_{21} = -(1000-1)*(1.009) + 1000(0.999) + (1000-1).(0.009)$

$= -1009 + 1.009 + 999 + 9 - 0.009$

$= -10 + 1 + 9$

$= 0$

$C_{22} = (1000-1)*(0.009) + 1000*(0.001) + (1000-1)*(-0.009)$

$= 9 - 0.009 + 1 - 9 + 0.009$

$= 1$

$C_{23} = (1000-1)*(0.999) + 1000*(-0.999) + (1000-1)(0.001)$

$= 999 - 0.999 - 999 + 0.999$

$= 0$

$C_{31} = -900*(1.009) + 900*(0.999) + 1000*(0.009)$

$= -900(0.01) + 9$

$= -9 + 9$

$= 1$

$C_{32} = 900*(0.009) + 900*(0.001) + 1000*(-0.009)$

$= 8.1 + 0.9 - 9$

$= 0$

$C_{33} = 900*(0.999) - 900(0.999) + 1000*(0.001)$

$= 1$

$\|I - BA\| = \|(A^{-1} - B)A\| \leq \|A^{-1} - B\| . \|A\|$

$\Rightarrow \dfrac{\|I - BA\|}{\|A\|} \leq \|A^{-1} - B\|$

So, $A^{-1}A = I$.

We have $x_{k+1} = (I - BA)x_k + Bd$. If $\{x_k\}$ converges to some x then, in theory, since $x = x - BAx + Bb$, we deduce that x has to be the solution of $Ax = b$. But for convergence, the necessary and sufficient condition is $\rho(I - BA) < 1$.

By hint,

$$I-BA = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det((I-BA)-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ -1 & -\lambda & -1 \\ 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} 1-\lambda & 0 \\ -1 & -\lambda \end{vmatrix} = -\lambda(1-\lambda)(-\lambda)$$

So, eigenvalues of $I-BA$ are $0, 0$ and 1 which shows $\rho(I-BA) = 1$.
So, the given iteration does not converge.

Problem II

Numerical Analysis Screening Exam, Spring 2010

First Name:

Last Name:

I. Linear Equations

Consider the singular system

$$Bu = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} u = f.$$

(a) Find the range $\mathcal{R}(\mathcal{B})$ and null space $\mathcal{N}(\mathcal{B})$ of B.

(b) State the solvability condition for $Bu = f$.

(c) Find an example of f for which $Bu = f$ has *no* solutions.

(d) Find an example of f for which $Bu = f$ has *infinitely many* solutions, and find the explicit form of this solution (depending on a parameter c).

II. Least Squares Problem

Consider the matrix A and vector b given below

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -2 & 0 & -2 \\ 1 & 5 & 6 \end{pmatrix}, b = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

(a) Using the Householder transformation matrix method to find a QR-decomposition of the matrix A and a solution to the problem

$$\min \|Ax - b\|^2$$

(b) Find the minimum norm solution of the above least squares minimization problem.

III. Eigenvalue Problems

(a) First consider the following matrix

$$B = \begin{pmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}.$$

Find any real eigenvalue of B and any associated eigenvector.

(b) Now let A be any $n \times n$ matrix. Show that $\det(A) = \prod_{i=1}^{n} \lambda_i$ where $\lambda_1, \cdots, \lambda_n$ are eigenvalues of A. (Hint: Consider characteristic polynomial of A)

(c) Show that A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

IV. Iterative Methods

(a) Consider the iterative method:

$$x_{k+1} = Bx_k + c, \quad k = 1, 2, \cdots,$$

where B is a $n \times n$ matrix and x_0 and c are arbitrary vectors of $\mathbb{R}^n$. Define the spectral radius $\rho(B)$ of B and show that x_k converges for all initial vectors x_0 if $\rho(B) < 1$.

(b) Consider the Richardson iterative scheme

$$x_{k+1} = x_k + \omega(b - Ax_k), \quad k = 1, 2, \cdots,$$

where A is a $n \times n$ matrix and ω is a positive number. We assume that all the eigenvalues λ_i of A are real and satisfy $0 < \alpha \le \lambda_i \le \beta$ for $i = 1, \cdots, n$ and for some positive values α and β. Find the condition on the number ω in terms of α and β such that for any initial vector x_0, the sequence x_k converges to the solution of $Ax = b$.

(c) Assuming that $\alpha = \min_{1 \le i \le n} \lambda_i$ and $\beta = \max_{1 \le i \le n} \lambda_i$ in (b), what value of ω leads to the fastest convergence of the scheme

Sp2010:

1. Linear Equations:

Consider the singular system

$$Bu = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} u = f$$

a) Find the range $R(B)$ and null space $N(B)$ of B.

b) State the solvability condition for $Bu=f$.

c) Find an example of f for which $Bu=f$ has no solution

d) Find an example of f for which $Bu=f$ has infinitely many solutions and find the explicit form of this solution (depending on a parameter c)

Solution:

a) Let $B=[b_1\ b_2\ b_3]^T$. We see that $b_3=-b_1-b_2$ and b_1, b_2 are linearly independent. We know by definition, $R(B)=\text{span}\{b_1,b_2,b_3\}$, but because of linear dependence of columns of B, we have

$$R(B)=\text{span}\{b_1,b_2\} = \left\{ c_1\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2\begin{bmatrix}-1\\2\\-1\end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

By definition, $N(B)=\{u\in\mathbb{R}^3 : Bu=0\}$. So, consider

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = c,\ c\in\mathbb{R} \\ x_2 = c \\ x_1 = c \end{array}$$

So, $N(B)=\left\{ c\begin{bmatrix}1\\1\\1\end{bmatrix} : c\in\mathbb{R} \right\}$

b) The necessary and sufficient condition for the system $Bu=f$ has a solution (i.e. $Bu=f$ is solvable) is $f\in R(B)$.

c) By part (b) if we choose $f \notin R(B)$ then $Bu=f$ has no solution.

So, let $f=\begin{bmatrix}0\\0\\-1\end{bmatrix}$ f cannot be written as a linear combination of the basis vectors of $\mathcal{R}(B)$ which are $[1,-1,0]^T$, $[-1,2,-1]^T$. So, $Bu=[0,0,-1]$ has no solution

d) Any linear combination of $[1,-1,0]^T$ and $[-1,2,-1]^T$ will give infinitely many solutions to $Bu=f$. So, let $f=[0,1,-1]$. Then observe

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = c,\ c \in \mathbb{R} \\ x_2 = c+1 \\ x_1 = c+1 \end{array}$$

So, any $u \in A$ where

$$A = \left\{ c\begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}0\\1\\1\end{bmatrix} : c \in \mathbb{R} \right\}$$

will be a solution of $Bu=[0,1,-1]^T$.

2. Least Squares Problem

Consider the matrix A and vector b given below

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 0 & -2 \\ 1 & 5 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

a) Using the Householder transformation matrix method to find a QR decomposition of the matrix A and a solution to the problem

$$\min \|Ax-b\|_2^2$$

b) Find the minimum norm solution of the above least squares minimization problem.

Solution:

a) Let $A=[a_1\ a_2\ a_3]$. Then $u_1 = a_1 - \|a_1\|_2 e_1$. So,

$$u_1 = \begin{bmatrix}2\\-2\\1\end{bmatrix} - 3\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\-2\\1\end{bmatrix}, \quad u_1^Tu_1 = [-1\ -2\ 1]\begin{bmatrix}-1\\-2\\1\end{bmatrix} = 6$$

$$H_1 = I - 2\frac{u_1u_1^T}{u_1^Tu_1} = I - \frac{2}{6}\begin{bmatrix}-1\\-2\\1\end{bmatrix}[-1\ -2\ 1] = I - \frac{1}{3}\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$$

$$H_1 A = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -2 & 0 & -2 \\ 1 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 4 & 4 \\ 0 & 3 & 3 \end{bmatrix} = A_1$$

$$\hat{a}_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad u_2 = \hat{a}_2 - \|\hat{a}_2\|_2 e_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix} - 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \quad u_2^T u_2 = [-1 \ 3] \begin{bmatrix} -1 \\ 3 \end{bmatrix} = 10$$

$$\hat{H}_2 = I - 2 \frac{u_2 u_2^T}{u_2^T u_2} = I - \frac{2}{10} \begin{bmatrix} -1 \\ 3 \end{bmatrix} [-1 \ 3] = I - \frac{1}{5} \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{bmatrix}$$

$$H_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & 3/5 \\ 0 & 3/5 & -4/5 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 4 & 4 \\ 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} = R$$

$$Q = H_1 H_2 = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & 3/5 \\ 0 & 3/5 & -4/5 \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 & -2/3 \\ -2/3 & 2/15 & -11/15 \\ 1/3 & 14/15 & -2/15 \end{bmatrix}$$

Now, since $A = QR$, the system becomes $QRx = b$ and since Q is orthogonal, $Rx = Q^T b$.

$$Q^T b = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ -1/3 & 2/15 & 14/15 \\ -2/3 & -11/15 & -2/15 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 7/15 \\ -1/15 \end{bmatrix}$$

Now, we get

$$\begin{bmatrix} 3 & 1 & 4 \\ 0 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/3 \\ 7/15 \\ -1/15 \end{bmatrix}$$

Let $R_1 = \begin{bmatrix} 3 & 1 \\ 0 & 5 \end{bmatrix}$, $R_2 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $\hat{x}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\hat{x}_2 = [x_3]$, $c_1 = \begin{bmatrix} 5/3 \\ 7/15 \end{bmatrix}$, $c_2 = [-1/15]$.

We have $R_1 \hat{x}_1 + R_2 \hat{x}_2 = c_1$. So, $\hat{x}_1 = R_1^{-1}(c_1 - R_2 \hat{x}_2)$ is a solution for any $\hat{x}_2$

Let $\hat{x}_2 = 0$. Then since $R_1^{-1} c_1 = \begin{bmatrix} 1/3 & -1/15 \\ 0 & 1/5 \end{bmatrix} \begin{bmatrix} 5/3 \\ 7/15 \end{bmatrix} = \begin{bmatrix} 118/225 \\ 7/75 \end{bmatrix}$ is a solution to

the least square

b) Let $\hat{x}_3 = x$. Then

$$C_1 - R_2\hat{x}_3 = \begin{bmatrix} 5/3 \\ 7/15 \end{bmatrix} - \begin{bmatrix} 4x \\ 5x \end{bmatrix} = \begin{bmatrix} 5/3 - 4x \\ 7/15 - 5x \end{bmatrix}$$

Then $$\hat{x}_1 = \begin{bmatrix} 1/3 & -1/15 \\ 0 & 1/5 \end{bmatrix}\begin{bmatrix} 5/3 - 4x \\ 7/15 - 5x \end{bmatrix} = \begin{bmatrix} 118/225 - x \\ 7/75 - x \end{bmatrix}$$

Here, for any $x \in \mathbb{R}$, $\hat{x}_1$ is a solution. We want $\min_{x \in \mathbb{R}} \|\mathbf{x}\|$.

$$\|\mathbf{x}\|_2^2 = \left(\frac{118}{225} - x\right)^2 + \left(\frac{7}{75} - x\right)^2 + x^2 =: f(x)$$

$$\Rightarrow f'(x) = 2\left(\frac{118}{225} - x\right)(-1) + 2\left(\frac{7}{75} - x\right)(-1) + 2x$$

$$= 2x - \frac{236}{225} + \frac{14}{75} - 2x + 2x$$

$$= 2x - \frac{194}{225}$$

Since $f''(x) = 4 > 0$, $x = \frac{97}{225}$ is min of $f(x)$. So,

$$\hat{x}_1 = \begin{bmatrix} 21/225 \\ -76/225 \\ 97/225 \end{bmatrix}$$ is the minimum norm solution.

3 Eigenvalue Problem

a) First consider the following matrix

$$B = \begin{bmatrix} 0 & 0 & 6 \\ 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}$$

Find any real eigenvalue of B and any associated eigenvector.

b) Now let A be any $n \times n$ matrix. Show that $\det(A) = \prod_{i=1}^{n} \lambda_i$ where $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A. (Hint: Consider characteristic polynomial of A.)

c) Show that A is singular if and only if $\lambda = 0$ is an eigenvalue of A

Solution:

a) We know that eigenvalues are roots of $p(\lambda) = \det(B - \lambda I)$

$$\det(B-\lambda I) = \begin{vmatrix} -\lambda & 0 & 6 \\ 1/2 & -\lambda & 0 \\ 0 & 1/3 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 \\ 1/3 & -\lambda \end{vmatrix} + 6 \begin{vmatrix} 1/2 & -\lambda \\ 0 & 1/3 \end{vmatrix}$$

$$= -\lambda(\lambda^2 - 0) + 6(1/6 - 0)$$

$$= -\lambda^3 + 1$$

The only real solution of $p(\lambda) = -\lambda^3 + 1 = 0$ is $\lambda = 1$. So, $\lambda = 1$ is the only real eigenvalue of B.

Let v be the associated eigenvalue. Then $Bv = v \Rightarrow (B-I)v = 0$. Then, we get

$$\left[\begin{array}{ccc|c} -1 & 0 & 6 & 0 \\ 1/2 & -1 & 0 & 0 \\ 0 & 1/3 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 6 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1/3 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} -1 & 0 & 6 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} v_3 = c, \ c \in \mathbb{R} \\ v_2 = 3c \\ v_1 = 6c \end{array}$$

So, $v = c\,[6 \ 3 \ 1]^T$ is an eigenvalue for any $c \in \mathbb{R}$ with $c \neq 0$.

b) Since we have $p(\lambda) = \det(A - \lambda I)$ as the characteristic polynomial of A which is of degree n, we also know the eigenvalues $\lambda_1, \ldots, \lambda_n$ are the roots of $p(\lambda)$. So, we can write

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = \det(A - \lambda I) \qquad (1)$$

Since λ is an arbitrary parameter, letting $\lambda = 0$ in (1), we get

$$\det(A) = (-1)^n \cdot (-1)^n \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n} \lambda_i .$$

c) We know A is singular if and only if $\det(A) = 0$. So, observe

$$A \text{ is singular} \iff \det(A) = 0$$

$$\iff \prod_{i=1}^{n} \lambda_i = 0$$

$$\iff \text{any of } \lambda_i = 0$$

So, A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

4. Iterative Methods:

a) Consider the iterative method

$$x_{k+1} = Bx_k + c, \quad k = 1, 2, \dots,$$

where B is an $n \times n$ matrix and x_0 and c are arbitrary vectors of $\mathbb{R}^n$. Define the spectral radius $\rho(B)$ of B and show that x_k converges for all initial vectors x_0 if $\rho(B) < 1$.

b) Consider the Richardson iterative scheme

$$x_{k+1} = x_k + \omega(b - Ax_k), \quad k = 1, 2, \dots,$$

where A is an $n \times n$ matrix and ω is a positive number. We assume that all the eigenvalues of λ_i of A are real and satisfy $0 < \alpha \leq \lambda_i \leq \beta$ for $i = 1, \dots, n$ and for some positive values α and β. Find the condition on the number ω in terms of α and β such that for any initial vector x_0, the sequence x_k converges to the solution of $Ax = b$.

c) Assuming that $\alpha = \min\limits_{1 \leq i \leq n} \lambda_i$ and $\beta = \max\limits_{1 \leq i \leq n} \lambda_i$ in (b), what value of ω leads to the fastest convergence of the scheme?

Solution

a) We know $\rho(B) = \max\limits_{1 \leq i \leq 1} |\lambda_i(B)|$ where $\lambda_i(B)$ denotes ith eigenvalue of B. By assumption, we have $|\lambda_i(B)| < 1$ $\forall i$. Then, let λ be an arbitrary eigenvalue of B and v be the corresponding eigenvector. Then for any $k \in \mathbb{N}$, observe that

$$B^k v = B^{k-1}(Bv) = B^{k-1}(\lambda v) = \lambda B^{k-2}(Bv) = \lambda B^{k-2}(\lambda v) = \dots = \lambda^k v.$$

Then, we get

$$\left(\lim_{k \to \infty} B^k\right) v = \lim_{k \to \infty} (B^k v) = \lim_{k \to \infty} (\lambda^k v) = \underbrace{\left(\lim_{k \to \infty} \lambda^k\right)}_{= 0 \text{ since } |\lambda| < 1} v = 0.$$

Since $v \neq 0$, we get $\lim\limits_{k \to \infty} B^k = 0$. So, B is a convergent matrix. For any $k \in \mathbb{N}$, from $x_{k+1} = Bx_k + c$ and $x = Bx + c$, we get that

$$x_{k+1} - x = B(x_k - x), \quad k = 1, 2, \ldots$$

$$\Rightarrow x_{k+1} - x = B(x_k - x) = B[B(x_{k-1} - x)] = B^2(x_{k-1} - x) = \ldots = B^k(x_1 - x)$$

where x_1 is the initial vector. Then we have

$$\lim_{k\to\infty} x_{k+1} - x = \lim_{k\to\infty}(x_{k+1} - x) = \lim_{k\to\infty}[B^k(x_1 - x)] = \underbrace{(\lim_{k\to\infty} B^k)}_{=0}(x_1 - x_0) = 0$$

So, $\lim_{k\to\infty} x_{k+1} = x$ where x_1 is an arbitrary initial vector.

b) From the given iteration, we obtain

$$x_{k+1} = x_k + \omega b - \omega A x_k$$
$$= (I - \omega A)x_k + \omega b$$

Since we have $0 < \alpha \leq \lambda_i \leq \beta$ for the eigenvalues λ_i of A, we obtain that $1 - \omega\beta \leq \lambda_i(I - \omega A) \leq 1 - \omega\alpha$ where $\lambda_i(I - \omega A)$ is the i^{th} eigenvalue of $I - \omega A$.

We know by theorem, the necessary and sufficient condition for convergence of an iterative method of the form $x_{k+1} = Mx_k + p$, is $\rho(M) < 1$ with an arbitrary initial vector. So, we must have $-1 < \lambda_i(I - \omega A) < 1$ for all i.

So, we must have $1 - \omega\alpha < 1$ which is already satisfied since $\alpha > 0$, $\omega > 0$ and $-1 < 1 - \omega\beta$ which implies $\omega < \frac{2}{\beta}$

Hence, for convergence with any initial point of the given iteration we must have $0 < \omega < \frac{2}{\beta}$

c) By assumption $1 - \omega\beta$ is the smallest eigenvalue of $I - \omega A$ and $1 - \omega\alpha$ is the largest one. So, $1 - \omega\beta$ and $1 - \omega\alpha$ defines $\rho(I - \omega A)$.
We know that smaller $\rho(I - \omega A)$, faster the convergence of iteration.
Since all the eigenvalues are between $1 - \omega\beta$ and $1 + \omega\alpha$ if 0 is the

midpoint of $[1-\omega\beta, 1-\omega\alpha]$ then we will be done But this yields

$$1-\omega\beta+1-\omega\alpha=0 \Rightarrow 2=\omega(\alpha+\beta)$$

$$\Rightarrow \omega=\frac{2}{\alpha+\beta}.$$

So, if $\omega=\frac{2}{\alpha+\beta}$ then the convergence will be the fastest.

Problem 3 and 4

Numerical Analysis Screening Exam, Fall 2009

First Name:

Last Name:

PROBLEM 1 (LEAST SQUARES)

1. Define the Least Squares process using QR factorization for solving $Ax = b$, where $A \in \mathbb{R}^{m\times n}$.

2. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix}, b = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$.

   Using Householder QR factorization solve $Ax = b$.

PROBLEM 2 (EIGENVALUE PROBLEMS)

1. State the Schur Theorem.

2. Using the Gram-Schmidt process define the othonormal set $\{q_1, q_2\}$ for set of vectors $\{v_1, v_2\}$.

3. Find the Schur decomposition of matrix $A = \begin{pmatrix} 5 & 7 \\ -2 & -4 \end{pmatrix}$.

PROBLEM 3 (ITERATIVE METHODS)
Consider an iterative scheme of the form

$$(rI + H)x_{k+1} = (rI - H)x_k + b,$$

where H is a symmetric positive definite $n \times n$ matrix, $b \in \mathbb{R}^n$ and r is a positive constant.

(a) Rewrite the iteration in the form $x_{k+1} = Bx_k + c$.

(b) Show that the sequence x_k converges for any x_0.

(c) If the matrix is only non-negative definite, does the sequence still con-

PROBLEM 4 (LINEAR SYSTEMS)

(1) Let A be any $n \times n$ matrix and $\|\cdot\|$ be any norm on $\mathbb{R}^n$ (Euclidean n-dimensional space). If $\|I - A\| < 1$, then show that A is invertible and derive the estimate

$$\|A^{-1}\| < \frac{1}{1 - \|I - A\|}.$$

(2) An $n \times n$ matrix $A = [a_{i,j}]$ is strictly diagonally dominant if

$$|a_{i,i}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}|, \text{ for } i = 1, \cdots, n.$$

Show that any strictly diagonally dominant matrix A is invertible. (Hint: recall that $\|A\|_\infty = \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |a_{i,j}| \right\}$ and write $A = DB$ where D is the diagonal part of A and show that $\|I - B\|_\infty < 1$.)

verge for any b and x_0?

1

Fall 2009

Problem 1 (Least Squares):

1. Define the Least Squares process using QR factorization for solving $Ax=b$, where $A \in \mathbb{R}^{m\times n}$.

2. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 2 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$

Using Householder QR factorization solve $Ax=b$.

Solution:

1. Input: Overdetermined system $Ax=b$, $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ and $b \in \mathbb{R}^m$. (A is full rank)

Output: Least-squares solution x to $Ax=b$.

$\longrightarrow$ Find QR factorization of A (reduced QR with $Q \in \mathbb{R}^{m\times n}$, $R \in \mathbb{R}^{n\times n}$ is enough)

$\longrightarrow$ Compute $c = Q^T b$

$\longrightarrow$ Solve triangular system $Rx = c$.

If A is rank deficient, say $\text{rank}(A) = r$, $r < n$ then letting $A = QR$, observe that

$$Ax = b \iff QRx = b \iff Rx = Q^T b \iff Rx = c, \quad c = Q^T b$$

If $A = QR$ is full QR factorization, then

$$\begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \begin{matrix} R_1 \in \mathbb{R}^{r\times r} & x_1 \in \mathbb{R}^r & c_1 \in \mathbb{R}^r \\ R_2 \in \mathbb{R}^{r\times(n-r)} & x_2 \in \mathbb{R}^{n-r} & c_2 \in \mathbb{R}^{m-r} \end{matrix}$$

$\Rightarrow R_1 x_1 + R_2 x_2 = c_1$

$\Rightarrow x_1 = R_1^{-1}(c_1 - R_2 x_2)$ is the solution for any $x_2 \in \mathbb{R}^{n-r}$.

2. Let $A=[a_1\ a_2]$

$$u_1 = a_1 - \|a_1\|_2 e_1 = \begin{bmatrix}2\\1\\2\end{bmatrix} - 3\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\2\end{bmatrix}, \quad u_1^T u_1 = 6$$

$$H_1 = I - 2\frac{u_1 u_1^T}{u_1^T u_1} = I - \frac{1}{3}\begin{bmatrix}-1\\1\\2\end{bmatrix}\begin{bmatrix}-1 & 1 & 2\end{bmatrix} = I - \begin{bmatrix}1/3 & -1/3 & -2/3\\ -1/3 & 1/3 & 2/3\\ -2/3 & 2/3 & 4/3\end{bmatrix}$$

$$= \begin{bmatrix}2/3 & 1/3 & 2/3\\ 1/3 & 2/3 & -2/3\\ 2/3 & -2/3 & -1/3\end{bmatrix}$$

$$H_1 A = \begin{bmatrix}2/3 & 1/3 & 2/3\\ 1/3 & 2/3 & -2/3\\ 2/3 & -2/3 & -1/3\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & 0\\ 2 & 2\end{bmatrix} = \begin{bmatrix}3 & 2\\ 0 & -1\\ 0 & 0\end{bmatrix} = R$$

Here $Q = H_1$ Let $Q_1 = \begin{bmatrix}2/3 & 1/3\\ 1/3 & 2/3\\ 2/3 & -2/3\end{bmatrix}$ $R_1 = \begin{bmatrix}3 & 2\\ 0 & -1\end{bmatrix}$. So, Q_1R_1 is

the reduced QR factorization of A Then

$$c = Q_1^T b = \begin{bmatrix}2/3 & 1/3 & 2/3\\ 1/3 & 2/3 & -2/3\end{bmatrix}\begin{bmatrix}-1\\0\\2\end{bmatrix} = \begin{bmatrix}2/3\\ -5/3\end{bmatrix}$$

We need to solve

$$\begin{bmatrix}3 & 2\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2/3\\ -5/3\end{bmatrix} \Rightarrow \begin{array}{l} x_2 = 5/3 \\ 3x_1 = \frac{2}{3} - \frac{10}{3} = -\frac{8}{3} \end{array} \Rightarrow x_1 = -\frac{8}{9}$$

So, $x = [-8/9 \ \ 5/3]^T$ is the least-squares solution to the given system.

Problem 2 (Eigenvalue Problem)

1. State the Schur Theorem
2. Using the Gram-Schmidt process define the orthonormal set $\{q_1, q_2\}$ for the set of vectors $\{v_1, v_2\}$.
3. Find the Schur decomposition of the matrix $A = \begin{bmatrix}5 & 7\\ -2 & -4\end{bmatrix}$

Solution:

1. Let A be an $n \times n$ matrix. Then there exists a unitary matrix U such that

$$U^* A U = T$$

where T is a triangular matrix with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the diagonal entries.

2. Suppose $\{v_1, v_2\}$ are linearly independent. Otherwise it is impossible to obtain orthonormal set of vectors $\{q_1, q_2\}$ from $\{v_1, v_2\}$.

Step-1: $\hat{q}_1 = v_1$

$$q_1 = \frac{\hat{q}_1}{\|\hat{q}_1\|_2}$$

Step-2: $\alpha = q_1^T v_2$

$$\hat{q}_2 = v_2 - \alpha q_1$$

$$q_2 = \frac{\hat{q}_2}{\|\hat{q}_2\|_2}$$

3. Firstly, we need to compute eigenvalues and corresponding eigenvectors

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{vmatrix} = -20 - 5\lambda + 4\lambda + \lambda^2 + 14$$

$$= \lambda^2 - \lambda - 6$$

$\Rightarrow \lambda^2 - \lambda - 6 = 0 \Rightarrow (\lambda - 3)(\lambda + 2) = 0 \Rightarrow \lambda_1 = 3 \quad \lambda_2 = -2$

$\lambda_1 = 3$ $\quad (A - 3I)v = 0$

$$\left[\begin{array}{cc|c} 2 & 7 & 0 \\ -2 & -7 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} v_2 = 2c, \ c \in \mathbb{R} \ (c \neq 0) \\ v_1 = -7c \end{array}$$

$\Rightarrow v_1 = [-7, 2]^T$ is an eigenvalue

$\lambda_2 = -2$: $(A+2I)v = 0$

$$\left[\begin{array}{cc|c} 7 & 7 & 0 \\ -2 & -2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} v_2 = c \ , \ c \in \mathbb{R} \ (c \neq 0) \\ v_1 = -c \end{array}$$

$\Rightarrow v_2 = [-1, 1]^T$ is an eigenvalue.

Since v_1 and v_2 are linearly independent, we can find q_1, q_2.

$$\to q_1 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\to \alpha = q_1^T v_1 = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} -7 \\ 2 \end{bmatrix} = \frac{9}{\sqrt{2}}$$

$$\hat{q}_2 = v_1 - \alpha q_1 = \begin{bmatrix} -7 \\ 2 \end{bmatrix} - \frac{9}{\sqrt{2}}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -7+9/2 \\ 2-9/2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ -5/2 \end{bmatrix}$$

$$\|\hat{q}_2\| = \sqrt{\frac{25}{4} + \frac{25}{4}} = \frac{5}{\sqrt{2}}$$

$$q_2 = \frac{\sqrt{2}}{5}\begin{bmatrix} -5/2 \\ -5/2 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Now, let

$$U = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

Then consider

$$U^* A = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} -7/\sqrt{2} & -11/\sqrt{2} \\ -3/\sqrt{2} & -3/\sqrt{2} \end{bmatrix}$$

$$U^* A U = \begin{bmatrix} -7/\sqrt{2} & -11/\sqrt{2} \\ -3/\sqrt{2} & -3/\sqrt{2} \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2 & 9 \\ 0 & 3 \end{bmatrix}$$

Problem 3 (Iterative Methods)

Consider an iterative scheme of the form

$$(rI + H)x_{k+1} = (rI - H)x_k + b$$

where H is a symmetric positive definite $n \times n$ matrix, $b \in \mathbb{R}^n$ and r is a positive constant.

a) Rewrite the iteration in the form $x_{k+1} = Bx_k + c$.

b) Show that the sequence x_k converges for any x_0

c) If the matrix is only non-negative definite, does the sequence for any b and x_0?

Solution:

a) Since H is SPD, all of its eigenvalues are positive, $0 < \lambda_1 \leq \dots \leq \lambda_n$. Then, the eigenvalues of $rI+H$ are $0 < r+\lambda_1 \leq \dots \leq r+\lambda_n$ since $r>0$. So, $\det(rI+H) \neq 0$ (since the determinant of a matrix is the multiplication of its eigenvalues). Hence $rI+H$ is nonsingular and

$$x_{k+1} = (rI+H)^{-1}(rI-H)x_k + (rI+H)^{-1}b.$$

Denoting $B := (rI+H)^{-1}(rI-H)$ and $c = (rI+H)^{-1}b$, we write the given system as $x_{k+1} = Bx_k + c$.

b) Now, let $A := rI+H$. So, $rI = A-H$ and then

$$B = A^{-1}(A-H-H) = A^{-1}(A-2H) = I - 2A^{-1}H = I - 2(H^{-1}A)^{-1}$$
$$= I - 2\left[H^{-1}(rI+H)\right]^{-1} = I - 2(rH^{-1}+I)^{-1}.$$

$\rightarrow$ The eigenvalues of $rH^{-1}+I$: $\dfrac{\lambda_1}{r+\lambda_1} \leq \dots \leq \dfrac{\lambda_n}{r+\lambda_n} < 1$

$\rightarrow$ The eigenvalues of $\underbrace{I-2(rH^{-1}+I)^{-1}}_{B}$: $-1 < \dfrac{r-\lambda_n}{r+\lambda_n} \leq \dots \leq \dfrac{r-\lambda_1}{r+\lambda_1}$

But, since $\dfrac{r-\lambda_1}{r+\lambda_1} < \dfrac{r+\lambda_1}{r+\lambda_1} = 1$ $(\lambda_1 > 0)$, we deduce that $\rho(B) < 1$ which is a necessary and sufficient condition for convergence of the iteration $x_{k+1} = Bx_k + c$.

c) Let us take $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ Take $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So, $x^T A x = 0$ and for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have $x^T A x = x_1^2 \geq 0$ So, H is non-negative definite Now, for $r>0$, observe that

$$rI+H=\begin{bmatrix} r+1 & 0 \\ 0 & r \end{bmatrix}, \quad (rI+H)^{-1}=\begin{bmatrix} \frac{1}{r+1} & 0 \\ 0 & \frac{1}{r} \end{bmatrix}, \quad rI-H=\begin{bmatrix} r-1 & 0 \\ 0 & r \end{bmatrix}$$

So, $(rI+H)^{-1}(rI-H)=\begin{bmatrix} \frac{r-1}{r+1} & 0 \\ 0 & 1 \end{bmatrix}$ which is the iteration matrix.

Then $B^k=\begin{bmatrix} \left(\frac{r-1}{r+1}\right)^k & 0 \\ 0 & 1 \end{bmatrix}$, and we see that B is not a convergent matrix, and the iteration $x_{k+1}=Bx_k+c$ does not converge for any x_0.

The vector b does not affect the convergence of iteration.

Problem 4 (Linear Systems):

(1) Let A be any $n\times n$ matrix and $\|\cdot\|$ be any norm on $\mathbb{R}^n$ (Euclidean n-dimensional space). If $\|I-A\|<1$, then show that A is invertible and derive the estimate

$$\|A^{-1}\| < \frac{1}{1-\|I-A\|}.$$

(2) An $n\times n$ matrix $A=[a_{ij}]$ is strictly diagonally dominant if

$$|a_{i,i}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad \text{for } i=1,\dots,n$$

Show that any strictly diagonally dominant matrix A is invertible (Hint: Recall that $\|A\|_\infty = \max_{1\le i\le n}\left\{\sum_{j=1}^{n}|a_{ij}|\right\}$ and write $A=DB$ where D is the diagonal part of A and show that $\|I-B\|_\infty<1$).

Solution:

(1) Let $I-A=:B$ So, we have $\|B\|<1$ and we want to show $I-B$ is invertible. Since $\rho(B)\le\|B\|$, we say that $|\lambda|<1$ for any eigenvalue λ of A. Since the eigenvalues of $I-B$ are $1-\lambda_i$, we see that all of the eigenvalues of $I-B$ are nonzero and so, $\det(I-B)\neq 0$, i.e. $I-B$ is invertible.

On the other hand, $(I-B)(I+B+B^2+\cdots)=I$, since $\|B\|<1$, B is a convergent matrix. Also $\|\sum_{k=0}^{\infty} B^k\| \leq \sum_{k=0}^{\infty} \|B\|^k$ So, $\sum_{k=0}^{\infty} B^k$ converges to a matrix and $\sum_{k=0}^{\infty} B^k$ is inverse of $I-B$. So,

$$\|A^{-1}\| = \|(I-B)^{-1}\| = \|\sum_{k=0}^{\infty} B^k\| \leq \sum_{k=0}^{\infty} \|B\|^k = \frac{1}{1-\|B\|} = \frac{1}{1-\|I-A\|}.$$

Since $\|B\|<1$.

(2)

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{=A} = \underbrace{\begin{bmatrix} a_{11} & & & \\ & a_{22} & & 0 \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}}_{=D} \underbrace{\begin{bmatrix} 1 & a_{12}/a_{11} & \cdots & a_{1n}/a_{11} \\ a_{21}/a_{22} & 1 & \cdots & a_{2n}/a_{22} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}/a_{nn} & a_{n2}/a_{nn} & \cdots & 1 \end{bmatrix}}_{=B}$$

So,

$$(I-B)_{ii} = 0, \quad i=1,\dots,n$$

$$(I-B)_{ij} = -\frac{a_{ij}}{a_{ii}}, \quad i \neq j, \ \forall i,j$$

$$\Rightarrow \text{For each } i, \ \sum_{j=1}^{n} |(I-B)_{ij}| = \frac{\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|}{|a_{ii}|} < 1$$

$$\Rightarrow \|I-B\|_{\infty} < 1$$

So, B is invertible. Since $|a_{ii}| \neq 0$ for each i, we get $\det(D) \neq 0$

Finally, $\det(A) = \det(D)\det(B) \neq 0$, i.e. A is invertible.

Screening Exam in Numerical Analysis – Spring 2009

*Name*______________________________

1. Linear systems

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{pmatrix}$.

a. Compute LU decomposition of A, i.e. find such L and U that $A = LU$.

b. Show that A is a SPD matrix. Then compute Cholesky decomposition of A, i.e. find such L that $A = LL^T$.

2. Least Squares

Consider $Ax = b$, where $A \in R^{m\times n}$, $b \in R^m$. A minimum norm solution of the least squares problem is a vector $x \in R^n$ with minimum Euclidian norm that minimizes $\|Ax - b\|_2$.

a. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

Find range and null space of A^T, find least squares solution to $Ax = b$, and find minimum norm solution: $\min \|x\|_2$.

b. Show that a vector that minimizes $\|Ax - b\|_2$ is a minimum norm solution if and only if x is in the range of A^T.

3. Eigenvalue problems

a. Describe the QR iteration algorithm, present steps of efficient implementation, indicate why the method is numerically stable.

b. Verify that the eigenvalues are preserved in each step of shifted QR iteration algorithm.

c. What choice of the rotation angle θ will make A_0 tridiagonal?

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} = U^{-1}AU.$$

where $s = \sin\theta$, $c = \cos\theta$, $|\theta| \leq \pi/2$.

4. Iterative methods

$-2Ix_k$

a. Consider the iterative method $x_{k+1} = -2x_k + b$ to solve the linear system $3Ix = b$, where I is $n \times n$ identity matrix.

For what values of the initial vectors x_0 the iteration converges? What is the spectral radius of iteration matrix?

b. Let A be a $n \times n$ matrix such that $A = (1+\omega)P - (N + \omega P)$, with $P^{-1}N$ nonsingular and with real eigenvalues $1 > \lambda_1 \geq \ldots \geq \lambda_n$.

Find the values $\omega \in \mathbf{R}$ for which the following iterative method

$$(1+\omega)Px_{k+1} = (N + \omega P)x_k + b,$$

with $k \geq 0$, converges to the solution of $Ax = b$ for every initial vector x_0. Determine the values of ω for which the convergence rate is maximum.

Sp 2009

1) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{bmatrix}$

$$\begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 9 \end{matrix} \qquad \begin{matrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{matrix}$$

a) $\ell_{21} = \frac{1}{1} = 1 \qquad \ell_{31} = \frac{1}{1}$

$$A \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 4 \\ 0 & 4 & 13 \end{bmatrix} = A^{(1)} \qquad \ell_{32} = \frac{4}{4}$$

$$A^{(1)} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 4 \\ 0 & 0 & 9 \end{bmatrix} = U \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

b) $1 > 0$

$$\begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = 5 \cdot 1 = 4 > 0$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 5 & 5 \\ 1 & 5 & 14 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 5 \\ 5 & 14 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 5 \\ 1 & 14 \end{vmatrix} + 1 \begin{vmatrix} 1 & 5 \\ 1 & 5 \end{vmatrix} = (70-25) - (14-5) + 0$$

$$= 45 - 9$$

$$= 36 > 0$$

$\Rightarrow$ A is positive-definite.

$$A = \begin{bmatrix} h_{11} & 0 & 0 \\ h_{21} & h_{22} & 0 \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} h_{11} & h_{21} & h_{31} \\ 0 & h_{22} & h_{32} \\ 0 & 0 & h_{33} \end{bmatrix}$$

* $h_{11}^2 = 1 \Rightarrow h_{11} = 1$

$h_{21} h_{11} = 1 \Rightarrow h_{21} = 1$

$h_{31} h_{11} = 1 \Rightarrow h_{31} = 1$

* $\underbrace{h_{21}^2}_{=1} + h_{22}^2 = 5 \Rightarrow h_{22} = 2$

$\underbrace{h_{31} h_{21}}_{=1} + h_{32} \underbrace{h_{22}}_{2} = 5 \Rightarrow h_{32} = 2$

* $\underbrace{h_{31}^2}_{=1} + \underbrace{h_{32}^2}_{=4} + h_{33}^2 = 14 \Rightarrow h_{33} = 3$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{bmatrix}$$

... symmetric, so that $A = A^T$. What can be said about the eigenvalues and eigenvectors? ... your answer.

(b) Let $A = A^T$. Using 2D rotations, describe the Jacobi method to compute the eigenvalues and eigenvectors of A. Give 2 possible algorithms for applying the rotations ... annihilate elements. What is the rate of convergence to the eigenvalues?

(c) Give the main equations for the QR algorithm to compute the eigenvalues of a general matrix A by computing a sequence of matrices A_k. Prove that the A_k have the sam... eigenvalues as A. Now assume that sequence A_k converges to matrix U. What is the struc... U if A has all real eigenvalues? What is the structure of U if A has some complex eigenva...

(d) Describe the shifted QR algorithm. How does it improve convergence?

2

a) $A = [a_1\ a_2]$ Since $a_1 = a_2$, $R(A) = \{c\,[1\ 1\ 1]^T : c \in \mathbb{R}\}$

Let $Ax = 0$ then

$$\begin{bmatrix}1&1\\1&1\\1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \Rightarrow \begin{matrix}x_1 = c\\ x_2 = -c\end{matrix} \Rightarrow N(A) = \{c\,[1\ -1]^T \quad c \in \mathbb{R}\}$$

$$A^T = \begin{bmatrix}1&1&1\\1&1&1\end{bmatrix} = [a_1\ a_2\ a_3]$$ Since $a_1 = a_2 = a_3$, we say $R(A) = \{c\,[1\ 1]^T : c \in \mathbb{R}\}$

Let $A^Tx = 0$ i.e.

$$\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c}1&1&1&0\\1&1&1&0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&1&1&0\\0&0&0&0\end{array}\right] \quad \begin{matrix}x_3 = c_1, & c_1 \in \mathbb{R}\\ x_2 = c_2, & c_2 \in \mathbb{R}.\\ x_1 = -c_1 - c_2 & \end{matrix}$$

$$\Rightarrow N(A) = \left\{c_1\begin{bmatrix}-1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}-1\\1\\0\end{bmatrix} : c_1, c_2 \in \mathbb{R}\right\}$$

$\longrightarrow$ $Ax = b \Rightarrow A^TAx = A^Tb$

$$A^TA = \begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\\1&1\end{bmatrix} = \begin{bmatrix}3&3\\3&3\end{bmatrix}, \quad A^Tb = \begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}6\\6\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}3&3\\3&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}6\\6\end{bmatrix} \Rightarrow \begin{matrix}x_2 = c, \quad c \in \mathbb{R}\\ x_1 = 2 - c\end{matrix}$$

$\Rightarrow x = [2-c\ \ c]^T$ is a solution for any $c \in \mathbb{R}$.

$f(c) := (2-c)^2 + c^2$. Minimize $f(c)$ to get min. norm solt. ③

$f'(c) = 2(2-c)(-1) + 2c = 2(c-2) + 2c = 4c - 4$

Only C.P. is $\boxed{c=1}$

$f''(c) = 4 > 0 \Rightarrow c=1$ is local min $\Rightarrow$ abs. min

So, $x = [1 \ 1]^T$ is min. norm solt.

$Ax = b \Rightarrow \overset{n\times m}{A^T} \underbrace{\overset{m\times n}{A} \overset{n\times 1}{x}}_{=y} = \underbrace{A^T b}_{=c} \Rightarrow A^T y = c$

~~$\rightarrow \min \|y\|_2$ subject to $A^T y = c$~~

~~Let $y =$ be any solution of $A^T y = c$ (so y is a L/S solt.). Then~~

~~$y = y_1 + y_2$ where $y_1 \in R(A)$, $y_2 \in N(A^T)$ (or $y_1 \in R(A^T)$, $y_2 \in N(A)$)~~

$\rightarrow \min \|x\|_2$ subject to $A^T A x = A^T b$, write $x = x_1 + x_2$ such that $x_1 \in R(A^T)$ and $x_2 \in N(A)$. So, $Ax_2 = 0$. Then

$A^T b = A^T A x = A^T A(x_1 + x_2) = A^T(Ax_1 + \underbrace{Ax_2}_{=0}) = A^T A x_1$.

So, x_1 is also LS-solution. Since $x_1 \perp x_2$, we have

$$\|x_1\|_2^2 \le \|x_1\|_2^2 + \|x_2\|_2^2 = \|x\|_2^2$$

where x is any solution. This inequality becomes equality iff $x_2 = 0$.

So, we see that if $x \in R(A)$ then x is the min. norm solution to LS problem. If x^* is the minimum norm solution.

imensional vector. Consider the equation

$$\text{(LS)} \qquad Ax = b\,.$$

(a) Define the range of A and the rank of A. State a necessary and sufficient conditi the rank of the augmented matrix (A, b) for (LS) to have a solution. Express this in term the range of A. Use this condition to construct a simple case for m = 3, n= 2 where rank and there is no solution.

$\in R(A^T)$

$\|x_1\|^2 + \|x_2^2\| \le \|x\|^2$ $\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\\1&1\end{bmatrix} = \begin{bmatrix}3&3\\3&3\end{bmatrix}$

(b) Let $f(x) = \|Ax - b\|_2^2$. Define a *least squares solution*, x^*, of (LS) in terms of f(prove that x^* always exists. Write the normal equations that correspond to equation (LS ive the formula

$= \langle A^T Ax, x\rangle - 2\langle A^T b, x\rangle + \langle b, b\rangle$, $\begin{bmatrix}1&1&1\\1&2&3\end{bmatrix}\begin{bmatrix}1&1\\2&1\\3&1\end{bmatrix}$ $\begin{bmatrix}6&3\\14&6\end{bmatrix}$ $36-12=24$

re < , > denotes inner-product. Compute the gradient ´f(x) and use it to prove that x* sfies the normal equations. When is x* unique ?

$x_1 = x^*$? $\boxed{X = X_1 + X_2}$ $\in R(A^T)$ $N(A)$

(d) Define the pseudo-inverse, $A^\dagger$, of A. Show that $x^* = A^\dagger b$ is a least squares solu LS). What can be said about the norm of x* ?

$A^T A(x_1 - x^*) = 0$

(e) Define the singular values and singular vectors of A. State the result on the sing ies decomposition (SVD) of A and its relation to $A^\dagger$. Outline the steps of an algorithm to npute the (SVD). $x_1 \neq x^*$ $A^TAx^* = A^Tb$ $A^TAx_1 = A^Tb$ $A^TA\underbrace{(x^*-x_1)}_{\neq 0} = 0$ 4

b) Suppose x^* is a min norm solution. Write $x^* = x_1 + x_2$ where $x_1 \in R(A^T)$ and $x_2 \in N(A)$. So,

$$A^Tb = A^TAx^* = A^TA(x_1+x_2) = A^T(Ax_1 + \underbrace{Ax_2}_{=0}) = A^TAx_1$$

x_1 is also a solution. But we must have $\|x^*\| \le \|x_1\|$.

On the other hand, we have $\|x^*\|^2 = \|x_1\|^2 + \|x_2\|^2$ Combining, we get

$\|x_1\|^2 + \|x_2\|^2 \le \|x_1\|^2$. $\Rightarrow \|x_2\|^2 \le 0$ So, we must have $x_2 = 0$.

So, $x^* = x_1 \in R(A^T)$.

Conversely suppose x^* is a LS-sol'n to $Ax=b$ be such that $x^* \in R(A^T)$.

• Here firstly, let us prove that if $y_1, y_2 \in R(A^T)$ are two LS solutions to ~~problems~~ Ax- then $y_1 = y_2$. Suppose to the contrary that $y_1, y_2 \in R(A^T)$, $y_1 \neq y_2$ and LS sol to $Ax=b$.

$\rightarrow Ay_1 = b, Ay_2 = b \rightarrow A(y_1 - y_2) = 0 \rightarrow y_1 - y_2 \in N(A)$ | $N(A)^\perp = R(A^T)$

$\rightarrow y_1, y_2 \in R(A^T) \Rightarrow y_1 - y_2 \in R(A^T)$ $\Rightarrow y_1 - y_2 = 0 \Rightarrow y_1 = y_2$.

Now, let x be an arbitrary solt and write $x = x_1 + x_2$ where $x_1 \in R(A^T)$ and $x_2 \in N(A)$. Then we know x_1 also a solution and by the uniqueness argument above, $x_1 = x^*$. So, $x = x^* + x_2$ where x is an arbitrary solution. Then, since $R(A^T) = N(A)^{\perp}$, we have,

$$\|x\|_2^2 = \|x^*\|_2^2 + \|x_2\|^2 \geq \|x^*\|$$

So, x^* is min. norm. solt.

3) a) QR iteration algorithm is a method to obtain a sequence of matrices $\{A_k\}$ all having the same eigenvalues with the original matrix A. But here, we want the entries below diagonal to converge to 0 so that the diagonal entries will converge to eigenvalues.

The QR algorithm is as follows

$$\begin{cases} \rightarrow A^{(1)} = A \\ \rightarrow \text{For } k = 1, 2, 3, \dots \\ \quad \bullet \text{ Compute } A^{(k)} = Q^{(k)} R^{(k)} \\ \quad \bullet \text{ Compute } A^{(k+1)} = R^{(k)} Q^{(k)} \end{cases}$$

Here, if A full and dense then computing QR fact. requires $O(n^3)$ flops. So, instead of directly computing QR fact. of A, firstly reduce A to a Hessenberg matrix by orthogonal similarity and then apply QR algorithm to the obtained Hessenberg form. Note that the Hessenberg form will be preserved during QR algorithm. Since the cost of QR fact. of a Hessenberg matrix is $O(n^2)$, this way is less expensive, in practice.

We know that if T is computed real Schur form by QR iteration algorithm, then T is orthogonally similar to a nearby matrix $A+E$. That is, $Q^T(A+E)Q = T$, and $\|E\|$ is small. This explains the stability of QR iteration algorithm.

b) For any $k = 1, \dots$ we have

$$A_k = Q_k R_k \quad , \; Q_k \text{ is orthogonal}$$

$$A_{k+1} = R_k Q_k$$

Since Q_k is orthogonal, we have $R_k = Q_k^T A_k$. So,

$$A_{k+1} = Q_k^T A_k Q_k \quad , \forall k.$$

Then, $A_{k+1} - \lambda I = Q_k^T A_k Q_k - \lambda I$

$$= Q_k^T A_k Q_k - Q_k^T \lambda I Q_k$$

$$= Q_k^T (A_k - \lambda I) Q_k$$

$$\Rightarrow \det(A_{k+1} - \lambda I) = \det(Q_k^T) \det(A_k - \lambda I) \det(Q_k)$$

$$= \det(\underbrace{Q_k^T Q_k}_{= I}) \det(A_k - \lambda I)$$

$$= \det(A_k - \lambda I).$$

So, eigenvalues are preserved in each step.

c) $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ c-s & 2c-s & c-3s \\ s+c & 2s+c & s+3c \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ c-s & 2c-s & c-3s \\ s+c & 2s+c & s+3c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix} = \begin{bmatrix} * & * & s+c \\ * & * & * \\ s+c & * & * \end{bmatrix}$$

We want $\cos\theta + \sin\theta = 0$ where $|\theta| \le \frac{\pi}{2}$

So, we must have $\theta = -\frac{\pi}{4}$. So, $\cos(-\frac{\pi}{4}) = +\frac{1}{\sqrt{2}}$ and $\sin(-\frac{\pi}{4}) - \frac{1}{\sqrt{2}}$

4) a) $x_{k+1} = -2Ix_k + b$ to solve $3Ix = b$.

We know that the necessary and sufficient condition for an iteration of the form $x_{k+1} = Bx_k + b$ to converge is $|\rho(B)| < 1$.

(a) Show that f is bounded on (a, b).
(b) Show that fg is absolutely continuous on (a, b).

7

But since eigenvalues of $-2I$ are all -2, $\rho(-2I) \geq 1$ and we can say that this iteration does not converge for every x_0.

Furthermore, this iteration corresponds

$$x_{k+1}^{(i)} = -2x_k^{(i)} + b^{(i)}, \quad i=1,\dots,n \quad \text{where } x_k = (y_k^{(1)}, y_k^{(2)}, \dots, y_k^{(n)})^T.$$

We know that the actual solution is $x = [b_1/3, b_2/3, \dots, b_n/3]^T$. But for any $x_0 = [x_0^{(1)}, \dots, x_0^{(n)}]^T$, we will have

$$x_1^{(i)} = -2x_0^{(i)} + b^{(i)}, \quad x_2^{(i)} = +4x_0^{(i)} - b^{(i)}, \quad x_3^{(i)} = -8x_0^{(i)} + 3b^{(i)}, \dots \quad \forall i$$

We observe then the magnitude of each $x_k^{(i)}$ is increasing. So, this iteration does not converge for any x_0. conv. Only with actual solt.

b) $(1+\omega)Px_{k+1} = (N+\omega P)x_k + b$.

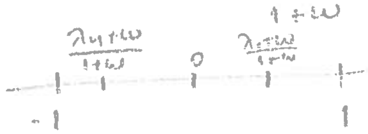
Since P is invertible, $x_{k+1} = \underbrace{\left[\frac{1}{1+\omega}P^{-1}N + \frac{\omega}{1+\omega}I\right]}_{:=Q_\omega} x_k + b$

Then eigenvalues of Q_ω will be

$$\frac{\lambda_n+\omega}{1+\omega} \leq \dots \leq \frac{\lambda_1+\omega}{1+\omega} < \frac{1+\omega}{1+\omega} = 1$$

Now if $-1 < \frac{\lambda_n+\omega}{1+\omega}$ then $\rho(Q_\omega) < 1$ and this iteration will converge $\forall x_0$. So, we must have $\omega > \frac{-1-\lambda_n}{2}$ to converge with every x_0.

We have the largest eigenvalue as $\frac{\lambda_1+\omega}{1+\omega}$ and the smallest eigenvalue as $\frac{\lambda_n+\omega}{1+\omega}$ and they are all in $(-1, 1)$. That is



We know that the smaller the spectral radius the faster convergence rate.

So, if 0 is exactly at the middle of $\frac{\lambda_n+\omega}{1+\omega}$ and $\frac{\lambda_1+\omega}{1+\omega}$, then the spectral radius will be the smallest possible. So,

$$\frac{\lambda_n+\omega}{1+\omega} + \frac{\lambda_1+\omega}{1+\omega} = 0 \implies \omega = \frac{-\lambda_1-\lambda_2}{2}$$ gives the fastest convergence.

4.2

Screening Exam in Numerical Analysis – Fall 2008

Linear Algebra

1. Perform LU factorization on Hilbert matrix $H_3 = [h_{ij}]_{1\le i,j\le 3}$, with elements
$$h_{ij} = \frac{1}{i+j-1}.$$

2. Let $A \in \mathbb{R}^{n\times n}$ have LU factorization, and $P \in \mathbb{R}^{n\times n}$ be given by $P = (e_n, e_{n-1}, \dots, e_1)$, where e_i is unit vector. Prove that PAP has UL factorization, that is, there exists upper triangular U and lower triangular L satisfying $PAP = UL$.

3. Let $B = [b_{ij}]_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Show that for any $1 \le i, j, k \le n$
$$b_{ij} + b_{jk} + b_{ki} \le b_{ii} + b_{jj} + b_{kk}.$$

Least squares

1. Let $A = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find orthonormal matrix $Q \in \mathbb{R}^{3\times 2}$ and upper-triangle matrix $R \in \mathbb{R}^{2\times 2}$, such that $A = QR$.

2. Let $b = (2, -1, 1)^T$. Find $x \in \mathbb{R}^{2\times 1}$, which minimizes $\|Ax - b\|_2$.

3. Prove Hadamard's determinant inequality:
If $A = (a_1, a_2, \dots, a_n) \in \mathbb{R}^{n\times n}$, then
$$|\det(A)| \le \Pi_{j=1}^{n} \|a_j\|_2,$$
with equality only if $A^T A$ is diagonal matrix or A has a zero column. (Hint: Consider QR factorization $A = QR$.)

Iterative Methods

1. Consider solving $Au = f$, where $A \in R^{n\times n}$ is consistently ordered.

   a. Give the matrix form of Jacobi, Gauss-Seidel and SOR iterations.

   b. If the eigenvalues of the Jacobi iteration matrix, Q_J are $\lambda_i(Q_J) = cos(\frac{\pi i}{n+1})$, i=1,...,n, what is the optimal over-relaxation parameter ω_{opt}?

2. Consider solving $Au = f$, where $A \in R^{n\times n}$ and $A = A^T$.

   a. Define the conjugate gradient method.

   b. Give the estimate of its rate of convergence.

   c. Compute estimate of the rate of convergence if the eigenvalues of A are $\lambda_i(A) = 2 + 2cos(\frac{\pi i}{n+1})$, i=1,...,n.

Eigenvalue Problems.

1. Show that if X is a unitary matrix, and the first column of X is an eigenvector of A associated with eigenvalue λ, then

$$X^*AX = \begin{bmatrix} \lambda & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

2. Consider the matrix

$$A = \begin{bmatrix} -2 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & -2 & 3 \end{bmatrix},$$

with an eigenvalue $\lambda = 2$ and corresponding eigenvector $x = [1, 2, 2]^T$. Construct a Householder matrix H such that

$$HAH^* = \begin{bmatrix} 2 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

Fall 2008

1) Linear Algebra

1.

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 1/12 & 4/45 \end{bmatrix} \qquad \ell_{21} = \frac{1}{2}, \ \ell_{31} = \frac{1}{3}$$

$$\rightarrow \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 0 & 1/180 \end{bmatrix} \qquad \ell_{32} = 1$$

So,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 0 & 1/180 \end{bmatrix}$$

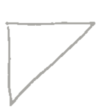
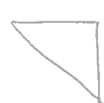
2. $A = LU$

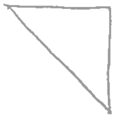
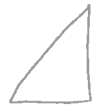
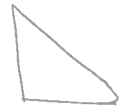
$$P = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

$$PAP = PLUP = \underbrace{PLP}\underbrace{PUP} \quad \text{since} \quad PP = I.$$

We know that multiplying a matrix by P from left changes the rows of matrix according to entries of P and multiplication from right changes columns of the matrix similarly.

So,

$L =$  $\Rightarrow PL =$  $\Rightarrow PLP =$  (upper triangular)

$U=$  $\Rightarrow PU=$  $\Rightarrow PUP=$  (lower triangular)

So, we can write $PAP = U'L'$ where $U' = PLP$, an upper triangular matrix and $L' = PUP$, a lower triangular matrix.

3. On p.7

2) Least Squares

1. $A = [a_1 \ a_2] = [q_1 \ q_2]\begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix} = QR$ $\qquad a_1 = r_{11} q_1$
$\qquad a_2 = r_{12} q_1 + r_{22} q_2, \quad r_{12} = q_1^T a_2$

$r_{11} = \|a_1\| = \sqrt{2+1+1} = 2$

$q_1 = [\sqrt{2}/2 \ \ 1/2 \ \ 1/2]^T$

$r_{12} = [\sqrt{2}/2 \ \ 1/2 \ \ 1/2]\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = 0$

$\hat{q}_2 = a_2 \quad , \ r_{22} = \sqrt{0+1+1} = \sqrt{2} \ , \quad q_2 = [0 \ \ -1/\sqrt{2} \ \ 1/\sqrt{2}]^T$

So, $\quad Q = \begin{bmatrix} \sqrt{2}/2 & 0 \\ 1/2 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} \end{bmatrix} \qquad R = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

2. $Ax = b \Leftrightarrow QRx = b \Leftrightarrow Rx = Q^T b$

$Q^T b = \begin{bmatrix} \sqrt{2}/2 & 1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$

$2x_1 = \sqrt{2} \Rightarrow x_1 = 1/\sqrt{2}$ and $\sqrt{2}x_2 = \sqrt{2} \Rightarrow x_2 = 1$

So, $x = [1/\sqrt{2} \ \ 1]^T$ minimizes $\|Ax-b\|_2$.

3. Let $A = QR$, where Q is orthonormal and R is upper triangular. Then

$a_i = r_{1i} q_1 + r_{2i} q_2 + \dots + r_{ii} q_i \quad , \; i = 1, 2, \dots, n.$

$\Rightarrow \|a_1\|_2^2 = r_{11}^2$

$\|a_2\|_2^2 = r_{12}^2 + r_{22}^2$

$\vdots$

$\|a_n\|_2^2 = r_{1n}^2 + r_{2n}^2 + \dots + r_{nn}^2$

$$\Rightarrow \prod_{j=1}^{n} \|a_j\|_2^2 = \prod_{j=1}^{n} \left(\sum_{i=1}^{j} r_{ij}^2 \right) \geq \prod_{j=1}^{n} r_{jj}^2 \Rightarrow \prod_{j=1}^{n} \|a_j\|_2 \geq \prod_{j=1}^{n} |r_{jj}| \quad (1)$$

On the other hand, since Q is orthonormal, $|\det(Q)| = |\det(Q^T)| = 1$ and since R is upper triangular, $|\det(R)| = \sum_{j=1}^{n} |r_{jj}|$ (2).

Since $|\det(A)| = |\det(Q^T)| \, |\det(R)|$, we get $|\det(A)| = |\det(R)|$. So, from (1) and (2),

$$|\det(A)| \leq \prod_{j=1}^{n} \|a_j\|_2 .$$

Since $A = QR$, $A^T A = R^T Q^T Q R = R^T R$. So, if $A^T A$ is diagonal then $R^T R$ is also diagonal. Since

$$R^T R = \begin{bmatrix} r_{11} & 0 & \cdots & 0 \\ r_{12} & r_{22} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ r_{1n} & r_{2n} & \cdots & r_{nn} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{bmatrix} = \begin{bmatrix} r_{11}^2 & r_{11} r_{12} & r_{11} r_{13} & \cdots \\ & r_{12}^2 + r_{22}^2 & & \\ & & \ddots & \end{bmatrix}$$

we must have R to be diagonal. Then from (1) we get

$$\prod_{j=1}^{n} \|a_j\|_2^2 = \prod_{j=1}^{n} r_{jj}^2 \Rightarrow \prod_{j=1}^{n} \|a_j\|_2 = \prod_{j=1}^{n} |r_{jj}| .$$

So, $$|\det(A)| = \sum_{j=1}^{n} |r_{jj}| = \prod_{j=1}^{n} \|a_j\|_2$$

Also if A has zero diagonal, then $\det(A) = 0$ and $\prod_{j=1}^{n} \|a_j\|_2 = 0$.

3) Iterative Methods :

1.a) $A = L + D + U$

→ **Jacobi :** $Ax = b \Rightarrow (L+D+U)x = b$

$$\Rightarrow Dx = -(L+U)x + b$$

$$\Rightarrow x = -D^{-1}(L+U)x + D^{-1}b$$

So, $x_{k+1} = B_j x_k + d$, where $B_j = -D^{-1}(L+U)$, $d = D^{-1}b$.

→ **Gauss-Seidel :** $Ax = b \Rightarrow (L+D+U)x = b$

$$\Rightarrow (L+D)x = -Ux + b$$

$$\Rightarrow x = -(L+D)^{-1}Ux + (L+D)^{-1}b$$

So, $x_{k+1} = B_G x_k + d$, where $B_G = -(L+D)^{-1}U$, $d = (L+D)^{-1}b$

→ **SOR :** $Ax = b \Rightarrow wAx = wb$

$$\Rightarrow w(L+D+U)x = wb$$

$$\Rightarrow (wL + wD + wU + D - D)x = wb$$

$$\Rightarrow (wL+D)x = [(1-w)D - wU]x + wb$$

$$\Rightarrow x = (wL+D)^{-1}[(1-w)D - wU]x + (wL+D)^{-1}wb$$

So, $x_{k+1} = B_{SOR} x_k + d$, where $B_{SOR} = (wL+D)^{-1}[(1-w)D - wU]$, $d = (wL+D)^{-1}wb$

b) So, we have $\rho(B_j) = \max\limits_{1 \le i \le n} |\lambda_i(B_j)| = \cos\left(\frac{\pi}{n+1}\right)$

Since we know

$$w_{opt} = \frac{2}{1 + \sqrt{1 - (\rho(B_j))^2}}$$

we get

$$w_{opt} = \frac{2}{1 + \sqrt{1 - \cos^2\left(\frac{\pi}{n+1}\right)}} = \frac{2}{1 + \sin\left(\frac{\pi}{n+1}\right)}$$

2. $Au = f$, $A \in \mathbb{R}^{n \times n}$, $A = A^T$.

a) For CG method A has to be SPD.

→ Choose an initial approximation and set $p_0 = r_0 = b - Ax_0$.

For $k = 0, 1, 2, \dots$

→ $w = Ap_k$

→ Compute step length: $\alpha_k = \dfrac{r_k^T r_k}{p_k^T w}$

→ Update iteration: $x_{k+1} = x_k + \alpha_k p_k$

→ Update residual $r_{k+1} = r_k - \alpha_k w$

→ Test for convergence: If $\|r_{k+1}\| \geq \varepsilon$, continue

→ Compute $\beta_k = \dfrac{r_{k+1}^T r_{k+1}}{r_k^T r_k}$

→ Update the direction vector: $p_{k+1} = r_{k+1} + \beta_k p_k$

b) → Let $e_k = x - x_k$, $k = 0, 1, 2, \dots$ Then

max ?

$$\frac{\|e_k\|_A}{\|e_0\|_A} \leq \min_{p_k} \max_{i=1,\dots,n} |p_k(\lambda_i)|$$

where min. is taken over all polynomials of degree $\leq k$ with $p_k(0) = 1$.

→ If A has k distinct eigenvalues, then CG converges at most k steps.

→ $$\frac{\|e_k\|_A}{\|e_0\|_A} \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k, \quad \kappa = \text{cond}_2(A) = \|A\|_2 \, \|A^{-1}\|_2 \underset{\text{SPD}}{=} \frac{\lambda_n \to \text{largest}}{\lambda_1 \to \text{smallest}}$$

c) $\lambda_i(A) = 2\left(1 + \cos\left(\frac{\pi i}{n+1}\right)\right)$, $i = 1, \dots, n$.

Then $\lambda_n = 2 + 2\cos\frac{\pi}{n+1}$ and $\lambda_1 = 2 - 2\cos\frac{\pi}{n+1}$

So, $\kappa = \dfrac{2 + 2\cos\frac{\pi}{n+1}}{2 - 2\cos\frac{\pi}{n+1}} = \dfrac{1 + \cos\frac{\pi}{n+1}}{1 - \cos\frac{\pi}{n+1}}$

On p 7

4) Eigenvalue Problems

1. $AX = A[v \ x_2 \ \dots \ x_n] = [Av \ \ Ax_2 \ \dots \ Ax_n] = [\lambda v \ \ Ax_2 \ \dots \ Ax_n]$

$$X^*AX = [\lambda X^*v \quad X^*Ax_2 \ \dots \ X^*Ax_n] \qquad (1)$$

$$X^*v = \begin{bmatrix} \bar{v}^T v \\ \bar{x}_2^T v \\ \vdots \\ \bar{x}_n^T v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ since } X \text{ is unitary}$$

So, from (1), we write

$$X^*AX = \begin{bmatrix} \lambda & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$$

2) By part (a), we have to construct a Householder matrix H s.t its first column is $[1 \ 2 \ 2]^T$. By symmetry $(H = H^T)$, the matrix will be

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & h_1 & h_2 \\ 2 & h_2 & h_3 \end{bmatrix} = I - \frac{2uu^T}{u^Tu} \text{ for some } u.$$

$$\Rightarrow \frac{2}{\underset{\|u\|^2}{u^Tu}} uu^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 2 & h_1 & h_2 \\ 2 & h_2 & h_3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -2 \\ -2 & 1-h_1 & -h_2 \\ -2 & -h_2 & 1-h_3 \end{bmatrix}$$

$$\Rightarrow uu^T = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} [u_1 \ u_2 \ u_3] = \underbrace{\begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & u_2^2 & u_2u_3 \\ u_1u_3 & u_2u_3 & u_3^2 \end{bmatrix} = \begin{bmatrix} 0 & -2\|u\|^2 & -2\|u\|^2 \\ -2\|u\|^2 & (1-h_1)\|u\|^2 & -h_2\|u\|^2 \\ -2\|u\|^2 & -h_2\|u\|^2 & (1-h_3)\|u\|^2 \end{bmatrix}}$$

$\Rightarrow u_1^2 = 0 \quad \Rightarrow \boxed{u_1 = 0}$

$u_1u_2 = -2\|u\|^2$ (u must be nonzero vector)

$\Rightarrow \boxed{u_1u_2 \neq 0}$

1.3) Wlog assume $i<j<k$. Then observe

$$\underbrace{\begin{bmatrix} 0 & \cdots & \overset{i\text{th}}{1} & \cdots & \overset{j\text{th}}{-1} & \cdots & 0 \end{bmatrix}}_{x^T} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & & & \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ b_{j1} & b_{j2} & \cdots & b_{jn} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix}}_{x} \begin{array}{l} \\ \\ \leftarrow i^{th} \\ \leftarrow j^{th} \\ \\ \\ \end{array} \Rightarrow x^TBx > 0 \text{ by assumption}$$

$$\Rightarrow x^TBx = [b_{i1}-b_{j1},\ b_{i2}-b_{j2},\ \ldots b_{ii}-b_{ji} \ldots b_{ij}-b_{jj} \ldots b_{in}-b_{jn}] \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= b_{ii} - b_{ji} - b_{ij} + b_{jj} > 0$$

Since $B = B^T$, $b_{ij} = b_{ji}$ and so, $b_{ii} + b_{jj} > 2b_{ij}$ (1)

By multiplying a suitable matrix, we can obtain

$$b_{jj} + b_{kk} > 2b_{jk} \quad (2) \quad \text{and} \quad b_{ii} + b_{kk} > 2b_{ki} \quad (3)$$

Summing (1), (2) and (3) we get,

$$2(b_{ii} + b_{jj} + b_{kk}) > 2(b_{ij} + b_{jk} + b_{ki})$$

$$\Rightarrow b_{ii} + b_{jj} + b_{kk} > b_{ij} + b_{ik} + b_{ki}$$

3.2) c)

$$\frac{\sqrt{K}-1}{\sqrt{K}+1} = \frac{K - 2\sqrt{K} + 1}{K-1} \underset{\substack{\uparrow \\ \text{set } \beta = \cos\frac{\pi}{n+1}}}{=} \frac{\frac{1+\beta}{1-\beta} - 2\sqrt{\frac{1+\beta}{1-\beta}} + 1}{\frac{1+\beta}{1-\beta} - 1} = \frac{\frac{2}{1-\beta} - 2\sqrt{\frac{1+\beta}{1-\beta}}}{\frac{2\beta}{1-\beta}}$$

$$= \frac{2 - 2\sqrt{1-\beta^2}}{\cancel{1-\beta}} \cdot \frac{\cancel{1-\beta}}{2\beta} = \frac{1-\sqrt{1-\beta^2}}{\beta} = \frac{1 - \sin\frac{\pi}{n+1}}{\cos\frac{\pi}{n+1}}$$

So,

$$\frac{\|x_k - x\|_A}{\|x_0 - x\|_A} \leq 2\left(\frac{1 - \sin\frac{\pi}{n+1}}{\cos\frac{\pi}{n+1}}\right)^k$$

Screening Exam on Numerical Analysis – Spring 2008

*Name*______________________

1. **Linear Equations**

a. Let $A \in \mathbb{R}^{n \times n}$. Show that the sum of eigenvalues of A is equal to the sum of the diagonal elements of A.

Hint: compare $det(A - \lambda I)$ and $p(\lambda) = (\lambda_1 - \lambda)...(\lambda_n - \lambda)$.

b. Consider solving $Ax = b$, where $A \in \mathbb{R}^{n \times n}$ is non-singular. Derive the estimate for the relative error in the solution x if the right hand side b is perturbed by δ.

c. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{pmatrix}, \epsilon > 0.$

How small ϵ should be for you to call the matrix ill-conditioned?

2. **Least Squares**

a. Prove the following Theorem:

If $A = QR$ with $Q^T Q = I$, then the least squares solution to $Ax = b$ is $x = R^{-1} Q^T b$,

where A is a $n \times m$ matrix.

b. Compute QR factorization of $A = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

3. **Eigenvalue Problems**

a. Let $B \in \mathbb{R}^{m \times n}$ and $C \in \mathbb{R}^{n \times m}$ be two given matrices. Prove that the nonzero eigenvalues of BC and CB are the same.

b. Suppose $n \geq 1$ and $T \in \mathbb{R}^{(2n+1)\times(2n+1)}$ is a tridiagonal matrix of the form

$$T = \begin{bmatrix} a_1 & b_2 & & & \\ b_2 & a_2 & b_3 & & \\ & b_3 & \ddots & \ddots & \\ & & \ddots & a_{2n} & b_{2n+1} \\ & & & b_{2n+1} & a_{2n+1} \end{bmatrix}$$

$$\begin{bmatrix} & & b_{2n+1} & a_{2n+1} \end{bmatrix}$$

$b_i \neq 0$ for all i but $b_{n+1} = 0$. Show that there exists at least one eigenvalue with multiplicity one. (Hint: Study the multiplicity of eigenvalues of $T_1 \in \mathbb{R}^{n \times n}$ and $T_2 \in \mathbb{R}^{(n+1)\times(n+1)}$, where $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$.)

c. Perform one step of a QR algorithm without shift on $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix}$

4. **Iterative Methods**

Consider $A \in \mathbb{R}^{n\times n}$ to be strictly positive definite, that is, $\xi^T A\xi > 0$ for all nonzero $\xi \in \mathbb{R}^n$. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are eigenvalues and spectral radius is $\rho = max_i|\lambda_i(A)|$.

a. Show that $\rho \leq \|A\|$ for any consistent matrix norm $\|\cdot\|$.

b. We use the Richardson iteration formula to solve $Ax = b$:

$$x^{(k+1)} = x^{(k)} + \omega(b - Ax^{(k)}), \quad k = 0, 1, 2, \ldots \tag{1}$$

Show that (1) is convergent for any $0 < \omega < 2/\rho$.

c. Compare two iterations of (1), by taking $\omega_1 = 1/\rho$ and $\omega_2 = 1/(\lambda_1 + \lambda_n)$. Which one has faster convergence rate? (Hint: compare spectral of the iteration matrix for these two cases).

1. Linear Equations

a) We know by Schur Theorem, there exists a unitary matrix U such that $U^*AU = T$ where T is upper triangular with $T = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then $A = UTU^*$ and

$$A - \lambda I = UTU^* - \lambda I = UTU^* - U(\lambda I)U^* = U(T - \lambda I)U^*$$

$$\Rightarrow \det(A - \lambda I) = \det(U)\det(T - \lambda I)\det(U)$$
$$= \underbrace{\det(UU^*)}_{=1}\det(T - \lambda I)$$
$$= \det(T - \lambda I).$$

On the other hand, observe that for any matrices B, C, $tr(BC) = tr(CB)$:

$$(BC)_{ii} = \sum_{k=1}^{n} b_{ik}c_{ki} \implies tr(BC) = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}c_{ki}$$

$$(CB)_{ii} = \sum_{k=1}^{n} c_{ik}b_{ki} \implies tr(CB) = \sum_{i=1}^{n}\sum_{k=1}^{n} c_{ik}b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} b_{ki}c_{ik}$$

$\Rightarrow tr(BC) = tr(CB)$.

So, $tr(A) = tr(UTU^*) = tr(U^*UT) = tr(T)$. (1)

Now since T is upper triangular, $\sum_{i=1}^{n} \lambda_i(T) = \sum_{i=1}^{n} t_{ii}$. (2)

Since $\det(A - \lambda I) = \det(T - \lambda I)$, A and T have the same eigenvalues. So,
$\sum_{i=1}^{n} \lambda_i(A) = \sum_{i=1}^{n} \lambda_i(T)$ (3)

Thus, (1), (2) and (3) concludes that $\sum_{i=1}^{n} \lambda_i(A) = tr(A)$.

b) If we perturb right-hand side by δb, there will be difference in the solution and let us denote it by δx. Then, we get

$$A(x + \delta x) = b + \delta b$$

where the original equation is $Ax = b$. So we get $A\delta x = \delta b$

$\Rightarrow \delta x = A^{-1}\delta b$ and $\|\delta x\| \leq \|A^{-1}\|.\|\delta b\|$

$\Rightarrow \|b\| \leq \|A\|\,\|x\| \Rightarrow \dfrac{1}{\|x\|} \leq \dfrac{\|A\|}{\|b\|}$

$\Rightarrow \dfrac{\|\delta x\|}{\|x\|} \leq \|A\|.\|A^{-1}\|.\dfrac{\|\delta b\|}{\|b\|}$.

c) $\|A\|_1 = \max\{2, 2+\varepsilon\} = 2+\varepsilon$

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1+\varepsilon & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & \varepsilon & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \varepsilon+1/\varepsilon & -1/\varepsilon \\ 0 & 1 & -1/\varepsilon & 1/\varepsilon \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \varepsilon+1/\varepsilon & -1/\varepsilon \\ 1/\varepsilon & 1/\varepsilon \end{bmatrix}, \quad \|A^{-1}\|_1 = \max\left\{\frac{\varepsilon+2}{\varepsilon}, \frac{2}{\varepsilon}\right\} = \frac{\varepsilon+2}{\varepsilon}$$

$$\Rightarrow K(A) = \frac{(\varepsilon+2)^2}{\varepsilon}$$

So, if $\varepsilon = 10^{k}$ then $\log_{10} K(A) \approx \log 4 - k \approx k$ (for large k)

So, we will lose k digits of accuracy. This means,

$\rightarrow$ If $\varepsilon = 10$ we will appr. lose 1 digit of accuracy

$\rightarrow$ If $\varepsilon = 10$ " " " " 10 " " " "

So, calling a matrix ill-conditioned depends on the required accuracy

2. Least Squares

a) We know that x is the LS-solt to $Ax=b$ iff x is the solt of NE which are $A^TAx = A^Tb$.

Now observe that

$$A^TAx = A^TA(R^{-1}Q^Tb) = (QR)^T(QR)R^{-1}Q^Tb = R^T\underbrace{Q^TQ}_{=I}\underbrace{RR^{-1}}_{=I}Q^Tb = R^TQ^Tb$$

$$= (QR)^Tb = A^Tb$$

This shows x is solution of NE, $A^TAx = A^Tb$. So, x is the LS-solt to $Ax=b$

(3)

b) Let us denote A by $[a_1\ a_2\ a_3]$

$$[q_1\ q_2\ q_3]\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33}\end{bmatrix}$$

$a_1 = r_{11} q_1$

$a_2 = r_{12} q_1 + r_{22} q_2$, $r_{12} = q_1^T a_2$

$a_3 = r_{13} q_1 + r_{23} q_2 + r_{33} q_3$ $r_{13} = q_1^T a_3$, $r_{23} = q_2^T a_3$

$r_{11} = \|a_{11}\| = 1$, $q_1 = [0\ 0\ 1]^T$ $\boxed{r_{11} = 1}$

$r_{12} = [0\ 0\ 1]\begin{bmatrix}0\\4\\1\end{bmatrix} = 1$ $\boxed{r_{12} = 1}$

$r_{22} q_2 = a_2 - r_{12} q_1 = \begin{bmatrix}0\\4\\1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\4\\0\end{bmatrix}$, $r_{22} = \|a_2 - r_{12} q_1\| = 4$, $q_2 = [0\ 1\ 0]^T$

$\boxed{r_{22} = 4}$

$r_{13} = [0\ 0\ 1]\begin{bmatrix}5\\1\\1\end{bmatrix} = 1$ $r_{23} = [0\ 1\ 0]\begin{bmatrix}5\\1\\1\end{bmatrix} = 1$ $\boxed{r_{13} = 1}$ $\boxed{r_{23} = 1}$

$r_{33} q_3 = a_3 - r_{13} q_1 - r_{23} q_2 = \begin{bmatrix}5\\1\\1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix} - \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}5\\0\\0\end{bmatrix}$, $r_{33} = \|a_3 - r_{13} q_1 - r_{23} q_2\| = 5$

$\boxed{r_{33} = 5}$

$q_3 = [1\ 0\ 0]^T$

So,
$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

3 Eigenvalue Problems

a) Suppose λ is a nonzero eigenvalue of BC and v is the corresp. eigenvector. So, $BCv = \lambda v$ and we must have $Cv \neq 0$. Then observe that

$$\lambda(Cv) = C(\lambda v) = C(BCv) = (CB)(Cv)$$

Since $Cv \neq 0$, this shows that λ is an eigenvalue of CB.

Converse (if λ is an eigenvalue of CB then it is also an eigenvalue of BC) can be shown in the same way. So, the nonzero eigenvalues of BC and CB are the same.

imensional vector. Consider the equation .

$$\text{(LS)} \qquad Ax = b.$$

State a necessary and sufficient condition on the rank of the augmented matix (A, b) for I
e a solution. Construct a simple case for m = 3, n= 2 where rank(A) = 2 and there is no
ition.

etting $f(x) = \|Ax - b\|_2^2$, define a *least squares solution*, x^*, of (LS). Write the normal
ations that correspond to equation LS. Show that $f(x) = \langle A^TAx, x\rangle - 2\langle A^T b, x\rangle + \langle$
re $\langle\, ,\rangle$ denotes inner-product. Compute the gradient $\nabla f(x)$ and prove that x^* satisfies the
nal equations. When is x^* unique ?

Define the pseudo-inverse, $A^\dagger$, of A . Define the singular values and singular vectors of A
e the result on the singular values decomposition (SVD) of A and its relation to $A^\dagger$. Outl
steps of an algorithm to compute the SVD.

4

b) Now observe that we have

$$T_1 - \lambda I = \begin{bmatrix} a_1-\lambda & b_2 & & & \\ b_2 & a_2-\lambda & \ddots & & 0 \\ & \ddots & \ddots & \ddots & \\ & & & & b_n \\ 0 & & & b_n & a_n-\lambda \end{bmatrix}$$

Let us remove the first row and last column of $T_1 - \lambda I$, then the resulting $n-1 \times n-1$ matrix is upper triangular and has nonzero diagonal. So, it is nonsingular and has full rank. So, we must have

$$\operatorname{rank}(T_1 - \lambda I) \geq n-1.$$

On the other hand, since λ represents an arbitrary eigenvalue, we have $\operatorname{null}(T_1 - \lambda I) \geq 1$. But since $\operatorname{rank}(T_1-\lambda I) + \operatorname{null}(T_1 - \lambda I) = n$, we see that $\operatorname{rank}(T_1 - \lambda I) = n-1$ and $\operatorname{null}(T_1 - \lambda I) = 1$. Since geometric multiplicity of λ is 1, its algebraic multiplicity has to be 1. So, T_1 has n distinct eigenvalu
Similarly T_2 has n+1 distinct eigenvalues.

Finally, since

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$$

eigenvalues of T are eigenvalues of T_1 and T_2. So, T has at least one eigenvalue having multiplicity 1.

c)
$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix}$$

$q_1 \; q_2$ $\begin{matrix} r_{11} & r_{12} \\ 0 & r_{22} \end{matrix}$

$a_2 = r_{12} q_1 + r_{22} q_2$

Compute QR factorization of A:

$\|a_1\| = 1 \Rightarrow r_{11} = 1$ and $q_1 = [\cos\alpha \;\; \sin\alpha]^T$.

$$r_{12} = q_1^T a_2 = [\cos\alpha \;\; \sin\alpha]\begin{bmatrix} \sin\alpha \\ 0 \end{bmatrix} = \cos\alpha \sin\alpha$$

$$r_{22} q_2 = \begin{bmatrix} \sin\alpha \\ 0 \end{bmatrix} - \begin{bmatrix} \cos^2\alpha \sin\alpha \\ \cos\alpha \sin^2\alpha \end{bmatrix} = \begin{bmatrix} \sin^3\alpha \\ -\cos\alpha\sin^2\alpha \end{bmatrix}$$

$$r_{22} = \sqrt{\sin^6\alpha + \cos^2\alpha \sin^4\alpha} = \sin^2\alpha \quad \text{and} \quad q_2 = [\sin\alpha \;\; -\cos\alpha]^T$$

Hence,

$$Q = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} \qquad R = \begin{bmatrix} 1 & \cos\alpha\sin\alpha \\ 0 & \sin^2\alpha \end{bmatrix}$$

$\cos\alpha + \cos\alpha\sin^2\alpha$

$\sin\alpha - \cos^2\alpha\sin\alpha$

$$A^{(1)} = RQ = \begin{bmatrix} 1 & \cos\alpha\sin\alpha \\ 0 & \sin^2\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\alpha(1+\sin^2\alpha) & \sin^3\alpha \\ \sin^3\alpha & -\cos\alpha\sin^2\alpha \end{bmatrix}$$

4. Iterative Methods

a) Let λ denote an arbitrary eigenvalue of A, and v denote the corresp eigenvector. So, $Av = \lambda v$. Then, we obtain,

$$|\lambda| \cdot \|v\| = \|\lambda v\| = \|Av\| \underset{\substack{\downarrow \\ \text{consistent} \\ \text{norm}}}{\leq} \|A\| \cdot \|v\| \xRightarrow[v \neq 0]{v \text{ is eigenvector}} |\lambda| \leq \frac{\|A\| \cdot \|v\|}{\|v\|} = \|A\|.$$

Since λ is arbitrary, we get $\max_{1 \leq i \leq n} |\lambda_i| \leq \|A\|$, i.e. $\rho(A) \leq \|A\|$.

b) The iteration can be written as

$$x^{(k+1)} = x^{(k)} + \omega(b - Ax^{(k)}) = x^{(k)} + \omega b - \omega A x^{(k)} = (I - \omega A)x^{(k)} + \omega b$$

For convergence, we must have $\rho(I - \omega A) < 1$ (necessary & suff. cond. for conv

The eigenvalues of $I - \omega A$ are (since $\omega > 0$)

$$1 - \omega\lambda_1 \geq 1 - \omega\lambda_2 \geq \cdots \geq 1 - \omega\lambda_n$$

So, we must have $1 - \omega\lambda_n > -1$ and $1 - \omega\lambda_1 < 1$

$\Rightarrow 1 - \omega\lambda_1 < 1 \Rightarrow -\omega\lambda_1 < 0 \xRightarrow{\omega > 0} -\lambda_1 < 0 \Rightarrow \lambda_1 > 0$

We also see that λ_1 must be positive. Note that in this case $\lambda_n > 0$

Define the eigenvalues and eigenvectors of A. How many eigenvalues are there? Prove yc
wer. Are all eigenvalues necessarily real? Explain your answer. Now suppose A is symme
hat $A = A^T$. What can be said about the eigenvalues and eigenvectors? Prove your answe
Let $A = A^T$. Using 2D rotations, describe the Jacobi method to compute the eigenvalues
eigenvectors of A. Give 2 possible algorithms for applying the rotations. What is the rate
vergence to the eigenvalues? λ_1, λ_n $\lambda_1 + \lambda_n > \lambda_n$ $\frac{1}{\lambda_n} < \left(\frac{1}{\lambda_1+\lambda_n}\right)$
Give the main equations for the QR algorithm to compute the eigenvalues of A by compu
quence of matrices A_k. Prove that the A_k have the same eigenvalues as A. Now assume t
ience A_k converges to U. What is the structure of U if A has all real eigenvalues? What is
cture of U if A has some complex eigenvalues? Describe the shifted QR algorithm.

6

$\Rightarrow 1-\omega\lambda_n > -1 \iff \omega\lambda_n < 2 \quad (1)$

Now, since $0<\omega<2/\rho$ and $\lambda_n>0$, we have

$$0<\omega\lambda_n < \frac{2\lambda_n}{\rho} = 2\underbrace{\frac{\lambda_n}{\rho}}_{<1} < 2$$

So, $\omega\lambda_n<2$ and from (1) we get $1-\omega\lambda_n>-1$ and hence

$\rho(1-\omega A)<1$ and iteration converges with any initial point

c) We know the eigenvalues of $I-\omega A$ are

$$1-\omega\lambda_1 \geq 1-\omega\lambda_2 \geq \dots \geq 1-\omega\lambda_n.$$

Now, since we showed all eigenvalues of $I-\omega A$ are positive, we have

$\rho=\lambda_n$ So, $\omega_1 = \frac{1}{\lambda_n}$ and $\omega_2 = \frac{1}{\lambda_1+\lambda_n}$. So, $\omega_2 < \omega_1$

Eigenvalues of $I-\omega_1 A$

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \to -\lambda_1 \geq -\lambda_2 \geq \dots \geq -\lambda_n \to -\frac{\lambda_1}{\lambda_n} \geq -\frac{\lambda_2}{\lambda_n} \geq \dots \geq -\frac{\lambda_n}{\lambda_n}$$

$$\to 1-\frac{\lambda_1}{\lambda_n} \geq 1-\frac{\lambda_2}{\lambda_n} \geq \dots \geq 1-\frac{\lambda_n}{\lambda_n} = 0 \qquad \rho(I-\omega_1 A) = 1-\frac{\lambda_1}{\lambda_n}$$

Eigenvalues of $I-\omega_2 A$

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \to -\lambda_1 \geq -\lambda_2 \geq \dots \geq -\lambda_n \to -\frac{\lambda_1}{\lambda_1+\lambda_n} \geq -\frac{\lambda_2}{\lambda_1+\lambda_n} \geq \dots \geq -\frac{\lambda_n}{\lambda_1+\lambda_n}$$

$$\to 1-\frac{\lambda_1}{\lambda_1+\lambda_n} \geq 1-\frac{\lambda_2}{\lambda_1+\lambda_n} \geq \dots \geq 1-\frac{\lambda_n}{\lambda_1+\lambda_n} \qquad \rho(I-\omega_2 A) = 1-\frac{\lambda_1}{\lambda_1+\lambda_n}$$

$\Rightarrow \rho(I-\omega_2 A) \geq \rho(I-\omega_1 A)$

$\Rightarrow$ The method with ω_1 converges faster than the one with ω_2

3b
4b- eigenvectors of a nonsquare matrix?
4c → also of A we have here B.
4d

Screening Exam in Numerical Analysis – Fall 2007

Name ______________________

Solve any three out of the following four problems (do not attempt more than three).

1. **linear equations**

a) Define the condition number of a matrix A and explain briefly how it is related to the numerical solving of the system $Ax = b$. Prove that $\|A\| \cdot \|A^{-1}\| \geq 1$ for any operator norm $\|\cdot\|$, provided that A^{-1} exists.

b) Let $A \in R^{n\times n}$ be a nonsingular matrix. Show that the condition number $\kappa(A)$ satisfies

$$\kappa(A) \geq \frac{\|A\|}{\|B - A\|}$$

for any singular matrix $B \in R^{n\times n}$.

c) Suppose $0 < |\varepsilon| < 1$. Given matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon \end{bmatrix}.$$

Show that

$$\kappa_\infty(A) \geq \frac{3}{2\varepsilon}.$$

where κ_∞ is the condition number with respect to infinity norm.

2. **iterative methods**

a) Let $B \in R^{n\times n}$ and $c \in R^n$. Denote the spectral radius by $\rho(B)$. Suppose $\rho(B) < 1$, then show that

$$\lim_{n\to\infty} B^n = 0.$$

b) Prove that A stationary iterative method

$$x^{(n+1)} = Bx^{(n)} + c$$

is convergent to the unique solution of $(I - B)x = c$ for all initial vector $x^{(0)}$ provided that $\rho(B) < 1$

provided that $\rho(B) < 1$.

c) Consider the iteration of (b). Let $\|B\| \leq \beta < 1$, and $\|x^{(k)} - x^{(k-1)}\| \leq \varepsilon$ for some k. Prove that

$$\|x - x^{(k)}\| \leq \frac{\beta\varepsilon}{1-\beta}$$

1

3. eigenproblems

a. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 10 & 0 \\ 1 & 1 & 9.8 \end{pmatrix}$.

i. Explain the slow rate of convergence of the power method with A.

ii. Choose a suitable shift σ so that power method converges to the largest eigenvalue of A. Explain the improvement of the rate of convergence with $A - \sigma I$.

directly or eigenproblems

b. Show that a product (both HR and RH) of a nonsingular right triangular matrix (R) with a Hessenberg matrix (H) is a Hessenberg matrix.

4. least squares

Consider the following least squares minimization problem: find a minimum norm solution of the functional

$$J(x) = \|Bx - c\|^2,$$

where B is a $n \times m$ real matrix and $c \in \mathbb{R}^m$ is given.

(a) Write down the minimum norm solution of the least square minimization problem as a solution of system of linear equation of the form $Ax = y$ and determine whether or not a unique solution of $Ax = y$ exists.

(b) Express the minimum norm solution of the least squares minimization problem as a function of the eigenvectors of the matrix A. Justify your answer.

A penalty method approach for solving this problem is to consider an alternative minimization problem of finding x_λ that minimizes

$$J_\lambda(x) = \|Bx - c\|^2 + \lambda\|x\|^2,$$

for $\lambda > 0$.

→ min-norm solt of $\|Bx - c\|^2$?? see (d)

(c) Write down the solution of penalty method as a function of the eigenvectors of the matrix A.

of the matrix A.

(d) Show that x_λ tends toward the solution x of the original least square minimization problem as λ tends toward infinity.

Fa 2007:

1 Linear Equations

a) We denote the condition number of a matrix by $K(A)$ and it is defined as (for any subordinate/compatible matrix norms)

$$K(A) = \|A\| \, \|A^{-1}\|.$$

Suppose we have $Ax = b$ and we are perturbing b as δb. That is $b + \delta b = \tilde{b}$. Let δx represent the corresponding change in x. So,

$$A(x+\delta x) = b + \delta b \quad \Rightarrow \quad A\delta x = \delta b \qquad (*)$$

Then, assuming A is invertible, we have $\delta x = A^{-1}\delta b$. So,

$$\|\delta x\| \leq \|A^{-1}\| \cdot \|\delta b\| = \|A^{-1}\| \cdot \|\delta b\| \cdot \frac{\|Ax\|}{\|b\|} \leq \|A\| \cdot \|A^{-1}\| \cdot \frac{\|\delta b\|}{\|b\|} \|x\|$$

$$\Rightarrow \frac{\|\delta x\|}{\|x\|} \leq \|A\| \cdot \|A^{-1}\| \cdot \frac{\|\delta b\|}{\|b\|} = K(A) \frac{\|\delta b\|}{\|b\|}$$

So, if $K(A)$ is small (ie. close to 1) then we say that a small change in b causes a small change in x. But if $K(A)$ is big, then a small change in b may cause a big change in x which means that the problem is sensitive to perturbation and this makes the problem hard to solve.

Above we obtained $\|\delta x\| \leq \|A^{-1}\| \cdot \|\delta b\|$. Since $\delta b = A\delta x$, we get

$\|\delta x\| \leq \|A^{-1}\| \cdot \|A\delta x\| \leq \|A\| \cdot \|A^{-1}\| \cdot \|\delta x\|$ Dividing by $\|\delta x\|$, we get $\|A\| \cdot \|A^{-1}\| \geq 1$.

b) Firstly since B is singular, $A^{-1}B$ is also singular since $\det(A^{-1}B) = \det(A^{-1})\det(B)$. Since $\det(C) = \prod_{i=1}^{n} \lambda_i(C)$ for any $n \times n$ matrix C, this means that $A^{-1}B$ has zero eigenvalue. Then we know $\lambda_i(A^{-1}B - I) = \lambda_i(A^{-1}B) - 1$ for all i. So, $A^{-1}B - I$ has -1 has an eigenvalue. Since we know $|\lambda_i(C)| \leq \|C\|$ $\forall i$ and for any matrix C, we deduce $1 \leq \|A^{-1}B - I\|$. Then,

$$1 \leq \|A^{-1}B - A^{-1}A\| = \|A^{-1}(B-A)\| \leq \|A^{-1}\| \cdot \|B - A\| \quad \Rightarrow \quad \frac{1}{\|B-A\|} \leq \|A^{-1}\|$$ Multiplying both sides by $\|A\|$, we get

$$K(A) = \|A\| \cdot \|A^{-1}\| \geq \frac{\|A\|}{\|B-A\|}$$

c) $K_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ -1 & \varepsilon & \varepsilon & 0 & 1 & 0 \\ 1 & \varepsilon & \varepsilon & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & \varepsilon-1 & \varepsilon+1 & 1 & 1 & 0 \\ 0 & \varepsilon+1 & \varepsilon-1 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & \frac{\varepsilon+1}{\varepsilon-1} & \frac{1}{\varepsilon-1} & \frac{1}{\varepsilon-1} & 0 \\ 0 & \varepsilon+1 & \varepsilon-1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2\varepsilon}{\varepsilon-1} & \frac{\varepsilon}{\varepsilon-1} & \frac{1}{\varepsilon-1} & 0 \\ 0 & 1 & \frac{\varepsilon+1}{\varepsilon-1} & \frac{1}{\varepsilon-1} & \frac{1}{\varepsilon-1} & 0 \\ 0 & 0 & \frac{-4\varepsilon}{\varepsilon-1} & \frac{-2\varepsilon}{\varepsilon-1} & \frac{-\varepsilon-1}{\varepsilon-1} & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{2\varepsilon}{\varepsilon-1} & \frac{\varepsilon}{\varepsilon-1} & \frac{1}{\varepsilon-1} & 0 \\ 0 & 1 & \frac{\varepsilon+1}{\varepsilon-1} & \frac{1}{\varepsilon-1} & \frac{1}{\varepsilon-1} & 0 \\ 0 & 0 & 1 & 1/2 & \frac{\varepsilon+1}{4\varepsilon} & \frac{1-\varepsilon}{4\varepsilon} \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -1/2 & \frac{1-\varepsilon}{4\varepsilon} & \frac{1+\varepsilon}{4\varepsilon} \\ 0 & 0 & 1 & 1/2 & \frac{1+\varepsilon}{4\varepsilon} & \frac{1-\varepsilon}{4\varepsilon} \end{array}\right]$$

$\|A\|_\infty = \max\{3, 1+2|\varepsilon|\} = 3$ since $0 < |\varepsilon| < 1$

$\|A^{-1}\|_\infty = \max\left\{1, \frac{|\varepsilon|+1}{2|\varepsilon|}\right\} = \frac{1+|\varepsilon|}{2|\varepsilon|}$

So, $K_\infty(A) = \frac{3+3|\varepsilon|}{2|\varepsilon|} \geq \frac{3}{2|\varepsilon|} \geq \frac{3}{2\varepsilon}$

2. Iterative Methods

a) Suppose $\rho(B) < 1$. That is $\max\limits_{i \in \{1,\dots,n\}} |\lambda_i| < 1$. We know that for any $n \times n$ matrix B, there exists a nonsingular matrix X such that $B = XJX^{-1}$ where

$$J = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix}$$

and each J_i has form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda_i \end{bmatrix} \qquad i = 1, \ldots k.$$

Then, we know that

$$J^n = \begin{bmatrix} J_1^n & & & 0 \\ & J_2^n & & \\ & & \ddots & \\ 0 & & & J_k^n \end{bmatrix}$$

Suppose each J_i is $P_i \times P_i$ where $P_i \geq 1$ with $P_1 + P_2 + \cdots + P_k = n$. We know also that there is $m_i \in \mathbb{N}$ such that

$$J_i^{m_i} = \begin{bmatrix} \lambda_i^m & & & \\ & \lambda_i^m & \star & \\ & & \ddots & \\ 0 & & & \lambda_i^m \end{bmatrix}$$

and the entries of $(\star)$ part of J_i^m consists of some powers of $\lambda_i^{r_{ij}}$ where $1 \leq r_{ij} \leq m_i$ and as m_i increases r_{ij} also increases. But since $|\lambda_i| < 1$ for all i, we say that all entries of $J_i^{m_i}$ approaches 0 as $m_i \to \infty$. Thus all blocks on diagonal of J^n approaches 0 matrices. That is, $J^n \to 0$ as $n \to \infty$. Since we have

$$B^n = X J^n X^{-1}$$

for any n, taking limit of both sides as $n \to \infty$, we get $\lim_{n\to\infty} B^n = 0$.

b) Suppose that x represents the solution of $(I-B)x = c$. That is, $x = Bx + c$. Then, for any $n \in \mathbb{N}$,

$$x - x^{(n+1)} = Bx + c - (Bx^{(n)} + c)$$

$$\Rightarrow \quad x - x^{(n+1)} = B(x - x^{(n)})$$

Then, we can write $x - x^{(n+1)} = B(x - x^{(n)}) = B^2(x - x^{(n-1)}) = \cdots = B^{n+1}(x - x^{(0)})$

$$\Rightarrow x - x^{(n+1)} = B^{n+1}(x - x^{(0)}).$$

Taking limit as $n \to \infty$, we get $x - x^{(n+1)} \to 0$ as $n \to \infty$ because of $\rho(B) < 1$ and by part (a). So, $x^{(n)} \to x$ as $n \to \infty$, where x is a solution of $(I-B)x = c$.

Since $\rho(B)<1$, we know $-1<\lambda_i(B)<1$ for $i=1,2,\dots,n$. We know that eigenvalues of $-B$ are $-\lambda_i(B)$ for $i=1,2,\dots,n$ and eigenvalues of $I-B$ are $1-\lambda_i(B)$ for $i=1,2,\dots,n$. Here, $-1<\lambda_i(B)<1$ implies $0<1-\lambda_i(B)<1 \iff 0<\lambda_i(I-B)<1$ for $i=1,2,\dots,n$. Thus $I-B$ has no zero eigenvalues. Since $\det(A)=\prod_{i=1}^{n}\lambda_i(A)$ for any matrix A, we get $\det(I-B)\neq 0$. So, $I-B$ is invertible and the system $(I-B)x=c$ has a unique solution. Thus the iteration $x^{(n+1)}=Bx^{(n)}+c$ converges to the unique solution of $(I-B)x=c$ when $\rho(B)<1$

c) Firstly, let us fix k for which $\|x^{(k)}-x^{(k-1)}\|\leq\varepsilon$ holds. Then observe that for some $N>k$,

$$\|x-x^{(k)}\| = \|x-x^{(k)}+x^{(k+1)}-x^{(k+1)}+x^{(k+2)}-x^{(k+2)}+\dots+x^{(N)}-x^{(N)}\| \qquad (\star)$$

$$\leq \|x^{(k+1)}-x^{(k)}\|+\|x^{(k+2)}-x^{(k+1)}\|+\dots+\|x^{(N)}-x^{(N-1)}\|+\|x-x^{(N)}\|$$

At this point observe that for any m,

$$x^{(m+1)}-x^{(m)} = B(x^{(m)}-x^{(m-1)}) = \dots = B^{m-k+1}(x^{(k)}-x^{(k-1)})$$

Substitution into $(\star)$ yields that

$$\|x-x^{(k)}\| \leq \|B(x^{(k)}-x^{(k-1)})\|+\|B^2(x^{(k)}-x^{(k-1)})\|+\dots+\|B^{N-k}(x^{(k)}-x^{(k-1)})\|+\|x-x^{(N)}\|$$

norm properties

$$\leq \|B\|\,\|x^{(k)}-x^{(k-1)}\|+\|B\|^2\|x^{(k)}-x^{(k-1)}\|+\dots+\|B\|^{N-k}\|x^{(k)}-x^{(k-1)}\|+\|x-x^{(N)}\|$$

$$= (\|B\|+\|B\|^2+\dots+\|B\|^{N-k})\underbrace{\|x^{(k)}-x^{(k-1)}\|}_{\leq\varepsilon}+\|x-x^{(N)}\|$$

$\|B\|\leq\beta$

$$\leq \underbrace{(\beta+\beta^2+\dots+\beta^{N-k})}_{\leq\frac{\beta}{1-\beta}\text{ since }\beta<1}\varepsilon+\|x-x^{(N)}\|$$

$$\leq \frac{\beta\varepsilon}{1-\beta}+\|x-x^{(N)}\|$$

Taking limit of both sides as $N\to\infty$

$$\lim_{N\to\infty}\|x-x^{(k)}\| \leq \frac{\beta\varepsilon}{1-\beta}+\underbrace{\lim_{N\to\infty}\|x-x^{(N)}\|}_{=0\text{ by (b)}}$$

Thus, we obtain

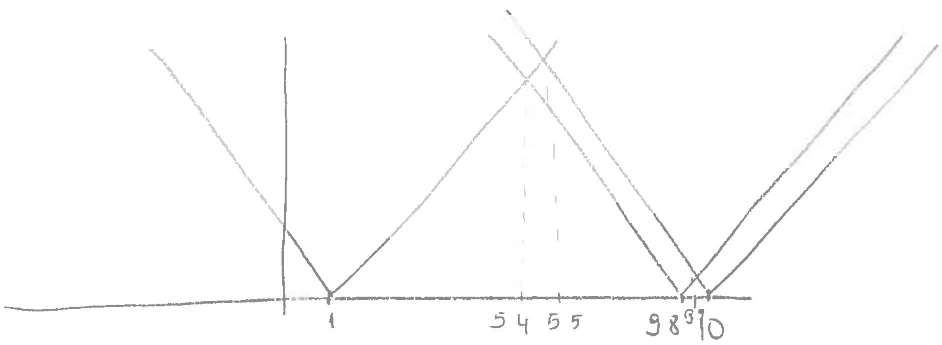
$$\|x-x^{(k)}\|\leq\frac{\beta\varepsilon}{1-\beta}$$

3. Eigenproblems

a)
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 10 & 0 \\ 1 & 1 & 9.8 \end{bmatrix}$$

i) Since A is uppertriongulor, its eigenvalues are $\lambda_1 = 10$, $\lambda_2 = 9.8$, $\lambda_3 = 1$. Rate of convergence of power method is determined for this motrix by $\frac{9.8}{10} = 0.98$ which is very close to 1. So the rate of convergence of power method will be slow

ii) Eigenvalues of $A - \sigma I$ will be $10-\sigma$, $9.8-\sigma$ and $1-\sigma$. We want $|10-\sigma| > |9.8-\sigma|$ and $|10-\sigma| > |1-\sigma|$ and we want to make $\frac{\lambda_2}{|10-\sigma|}$ as small as possible where λ_2 is the grater one of $|9.8-\sigma|$ and $|1-\sigma|$, according to the choice of σ. Then observe



Because of the reasons above, we cannot choose $\sigma \geqslant 5.4$. Also if $\sigma = 5.4$ $|10-\sigma| = 4.6$, $|9.8-\sigma| = 4.4$, $|1-\sigma| = 4.4$ and the rate of convergence is $\frac{4.4}{4.6}$ which is smaller than 0.98. Also as σ gets smaller than 5.4 the rate of convergence getting bigger again. So, choose $\sigma = 5.4$. Becouse of the explanations above rate of convergence is faster with shift $\sigma = 5.4$

b)

6) If the system is overdetermined,

→ If A is full rank, then the unique solution exists and it is also min-norm solution and $x = (A^T A)^{-1} A^T y$.

→ If A is rank defizient then min-norm solution can be uniquely determined and if $A = U \Sigma V^T$ is the SVD factorizati and $r = \text{rank}(A) < n$, then

$$x = \sum_{i=1}^{r} \frac{u_i^T y}{\sigma_i} v_i$$

is the min-norm solution.

If the system is underdetermined, then either there are infinitely many solutions or there is no solution. In the case A is full rank, the solution can be written as

$$x = A^T (A A^T)^{-1} y + (I - A^T (A A^T)^{-1} A) z$$

for some arbitrary z. To get min-norm solution, we set $z = 0$ and get

$$x = A^T (A A^T)^{-1} y$$

as the min-norm solution.

Screening Exam in Numerical Analysis, Spring 2007

There are questions on 4 chapters covered in M502a.

I. *Linear systems*

1. Give a definition of a Symmetric Positive Definite (SPD) matrix $A \in R^{n\times n}$.

2. Show that no pivoting is necessary during Gaussian Elimination of a SPD matrix.

3. Show that matrix $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$ is SPD .

II. *Iterative solutions.*

Consider solving a system $Ax = b$, where $x, b \in R^n$, $A \in R^{n\times n}$ using semi direct iterations, i.e. splitting the system into $Mx = (M - A)x + b$ and iterating as $Mx_{k+1} = (M - A)x_k + b$, or $x_{k+1} = Qx_k + c$, where the iteration matrix $Q = I - M^{-1}A$.

1. Define M for the Jacobi, Gauss Seidel and SOR iterations.

2. Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.

a. Find Q for the Jacobi, Gauss Seidel and SOR iterations.

b. Find the optimal overrelaxation parameter ω for this Q_{SOR}.

c. Compute spectral radius of all three Q matrices and compare their rate of convergence.

III. *Eigenvalue problems*

1. Prove the *Gerschgorin's theorem*: All the eigenvalues of the matrix $A \in C^{n\times n}$ lie in the union of the Gerschgorin disks in the complex plane

$$D_i = \{z : |z - a_{ii}| \leq r_i\}, \quad r_i = \sum_{j\neq i, j=1}^{n} |a_{ij}|, i = 1, 2, \ldots, n.$$

Moreover, if the union $\mathcal{M}$ of k Gershgorin disks D_i is disjoint from the remaining disks, then $\mathcal{M}$ contains precisely k eigenvalues of A.

2. Consider the matrix

$$B = \begin{pmatrix} -5 & 1 & 0 & 0 \\ a & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

with $1 \leq a \leq 3$. Show that the dominant eigenvalue of B is real.

IV. *Least squares*

1. Given the matrix $A \in \mathbb{R}^{m\times n}$ with rank n. Show that $A^T A$ is symmetric positive definite.
2. Suppose $b \in \mathbb{R}^m$. Show that $x = (A^T A)^{-1} A^T b$ minimizes $\|b - Ax\|_2$.
3. If the matrix is given by

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Find pseudoinverse $A^\dagger$.

1. Linear Systems

1. $A \in \mathbb{R}^{n\times n}$ matrix is called symmetric positive definite if $A = A^T$ and $x^T A x > 0$ for every nonzero $x \in \mathbb{R}^n$.

2. We want to show no pivoting is necessary during GE of an SPD matrix. That is, all of $a_{11}, a_{22}^{(1)}, a_{33}^{(2)}, \ldots, a_{n-1,n-1}^{(n-2)}$ are nonzero.

Firstly, we know that $a_{11} \neq 0$ since A is SPD. Then let A_k represents $k\times k$ matrix formed by the first k rows and first k columns of A. Again since A is SPD, we know $\det(A_k) \neq 0$ for $k = 1, 2, \ldots, n-1$.

Now, observe A_2:

$$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

During GE, we multiply first row by $-\frac{a_{21}}{a_{11}}$ and add it to the second row. But this does not change determinant and the matrix will be

$$A^{(1)}_2 = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}^{(1)} \end{bmatrix}$$

So, $0 \neq \det(A^{(1)}_2) = a_{11} \cdot a_{22}^{(1)} \Rightarrow a_{22}^{(1)} \neq 0$.

Similarly observe

$$A_k^{(k-2)} \begin{bmatrix} a_{11} & a_{12} & & - & - & a_{1k} \\ & a_{22}^{(1)} & & - & - & a_{2k}^{(1)} \\ & & a_{33}^{(2)} & - & - & a_{3k}^{(2)} \\ & & & \ddots & & \vdots \\ & 0 & & & a_{k-1,k-1}^{(k-2)} & \\ & & & & a_{k,k-1}^{(k-2)} & a_{kk}^{(k-2)} \end{bmatrix}$$

$\det(A_k^{(k-2)}) = \det(A_k^{(k-1)})$. Since $0 \neq \det(A_k^{(k-2)})$ by above reasoning, we see $\det(A_k^{(k-1)}) = a_{11} a_{22}^{(1)} a_{33}^{(2)} \cdots a_{kk}^{(k-1)} \neq 0$ since $a_{ii}^{(i-1)} \neq 0$ for $i = 1, 2, \ldots, k-1$, we deduce $a_{kk}^{(k-1)} \neq 0$. Thus at no stage of GE, we need pivoting.

3. We see that A is symmetric. We know a symmetric matrix is positive definite iff all its principal minors are positive

$\rightarrow a_{11} > 0$

$$\rightarrow \quad A^{(1)} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \qquad a_{22}^{(1)} = 1 > 0 \quad \Rightarrow \det(A_2) = 1 > 0$$

$$\rightarrow \quad A^{(2)} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \qquad a_{33}^{(2)} = 1 > 0 \quad \Rightarrow \det(A_3) = 1 > 0$$

$$\rightarrow \quad A^{(3)} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad a_{44}^{(3)} = 1 > 0 \quad \Rightarrow \det(A_4) = \det(A) = 1 > 0$$

Thus A is SPD.

2. Iterative Solutions

1. Jacobi: $A = L + D + U$

$$Ax = b \Rightarrow Dx = -(L+U)x + b \Rightarrow x = -D^{-1}(L+U)x + D^{-1}b$$

$$M_j = D \qquad Q_j = -D^{-1}(L+U)$$

Gauss-Seidel: $A = L + D + U$

$$Ax = b \Rightarrow (L+D)x = -Ux + b \Rightarrow x = -(L+D)^{-1}Ux + (L+D)^{-1}b$$

$$M_{G\text{-}S} = L + D, \qquad Q_{G\text{-}S} = -(L+D)^{-1}U$$

SOR: $A = L + D + U$

$$Ax = b \Rightarrow (\omega L + \omega D + \omega U)x = \omega b \Rightarrow (D + \omega L - D + \omega D + \omega U)x = \omega b$$

$$\Rightarrow (D+\omega L)x = [(1-\omega)D - \omega U]x + \omega b$$

$$\Rightarrow x = (D+\omega L)^{-1}[(1-\omega)D - \omega U]x + \omega (D+\omega L)^{-1} b$$

$$M_{SOR} = D + \omega L, \quad Q_{SOR} = (D+\omega L)^{-1}[(1-\omega)D - \omega U]$$

2. $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

a) $L = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$ $\quad D = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad U = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$

$$\rightarrow Q_j = \begin{bmatrix} -1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$\rightarrow L + D = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} \qquad (L+D)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 1/4 & 1/2 \end{bmatrix}$$

$$Q_{G-S} = \begin{bmatrix} -1/2 & 0 \\ -1/4 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 \\ 0 & 1/4 \end{bmatrix}$$

$$\rightarrow D + \omega L = \begin{bmatrix} 2 & 0 \\ -\omega & 2 \end{bmatrix} \qquad (D+\omega L)^{-1} = \begin{bmatrix} 1/2 & 0 \\ \omega/4 & 1/2 \end{bmatrix}$$

$$(1-\omega)D - \omega U = \begin{bmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{bmatrix}$$

$$Q_{SOR} = \begin{bmatrix} 1/2 & 0 \\ \omega/4 & 1/2 \end{bmatrix} \begin{bmatrix} 2(1-\omega) & \omega \\ 0 & 2(1-\omega) \end{bmatrix} = \begin{bmatrix} 1-\omega & \omega/2 \\ \omega(1-\omega)/2 & \frac{\omega^2}{4} + 1 - \omega \end{bmatrix}$$

b) $$\omega_{opt} = \frac{2}{1 + \sqrt{1 + \rho(Q_j)^2}}$$

Eigenvalues of Q_j are: $\lambda^2 - \frac{1}{4} = 0 \Rightarrow \lambda = \frac{1}{2}, \lambda = -\frac{1}{2}$

So, $\rho(Q_j) = \frac{1}{2}$. Thus,

$$\omega_{opt} = \frac{2}{\cdots} = \frac{2}{\cdots} = \frac{4}{\cdots}$$

c) We found $\rho(Q_j) = \frac{1}{2}$. We know $\rho(Q_{SOR}) = w_{opt} - 1$. So,

$\rho(Q_{SOR}) = \frac{2-\sqrt{3}}{2+\sqrt{3}}$.

Eigenvalues of Q_{G-S} are 0 and $1/4$. So, $\rho(Q_{G-S}) = \frac{1}{4}$

$$\rho(Q_{SOR}) = \frac{4(2-\sqrt{3})}{4(2+\sqrt{3})}, \quad \rho(Q_j) = \frac{2(2+\sqrt{3})}{4(2+\sqrt{3})}, \quad \rho(Q_{G-S}) = \frac{2+\sqrt{3}}{4(2+\sqrt{3})}$$

$$\Rightarrow \rho(Q_{SOR}) \leq \rho(Q_{G-S}) \leq \rho(Q_j)$$

So, SOR method converges faster. Gauss-Seidel converges slower than SOR but faster than Jacobi and thus Jacobi converges slower than both.

3. Eigenvalue Problems

1. Let λ be an arbitrary eigenvalue of A, and v be the correspondi eigenvector. So, $Av = \lambda v$, which means

$$\sum_{j=1}^{n} a_{ij} v_j = \lambda v_i \quad \text{for } i = 1, 2, \dots, n$$

$$\Rightarrow \lambda v_i - a_{ii} v_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} v_j, \quad \forall i$$

$$\Rightarrow |\lambda v_i - a_{ii} v_i| \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| |v_j|, \quad \forall i$$

Let v_k be the largest component in magnitude of v. So, in particular,

$$|\lambda - a_{kk}| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{ij}| \cdot \frac{|v_j|}{|v_k|} \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{ij}| = r_k$$

So, $\lambda \in D_k$ where λ is arbitrary. Thus all the eigenvalues lie in $\bigcup_{k=1}^{n} D_k$.

2. On the last page

4. Least Squares

1. Let $A \in \mathbb{R}^{m \times n}$ be rank n.

→ We know that for any matrices $B \in \mathbb{R}^{a \times b}$ and $C \in \mathbb{R}^{b \times c}$, $(BC)^T = C^T B^T$ and $(B^T)^T = B$.

So, $(A^TA)^T = A^T(A^T)^T = A^TA$, showing A^TA is symmetric.

Let $x \neq 0$ be arbitrary.

→ Observe $Ax \neq 0$. Because if $Ax = 0$ then $A^TAx = 0$ and we know A^TA is invertible since A is full rank. So, we get $x = 0$, a contradiction. Thus $Ax \neq 0$

Now, observe

$$x^TA^TAx = (Ax)^TAx = \|Ax\| > 0 \quad \text{since } Ax \neq 0$$

Thus A^TA is positive definite.

2. Suppose $b \in \mathbb{R}^m$.

Firstly observe that x satisfies the normal equations

$$A^TAx = A^TA(A^TA)^{-1}A^Tb = A^Tb$$

Let $y \in \mathbb{R}^n$ be arbitrary. Then, observe

$$\begin{aligned}\|b - Ay\|_2^2 &= \|b - Ay + Ax - Ax\|_2^2 \\ &= \|b - Ax\|_2^2 + (b - Ax)^T(Ax - Ay) + (Ax - Ay)^T(b - Ax) + \|Ax - Ay\|_2^2 \\ &= \|b - Ax\|_2^2 + \underbrace{2(x-y)^TA^T(b - Ax)}_{\substack{= 0 \text{ since } x \text{ satisfies} \\ A^TAx = A^Tb}} + \underbrace{\|Ax - Ay\|_2^2}_{\geq 0} \\ &\geq \|b - Ax\|_2^2\end{aligned}$$

So, $\|b - Ax\|_2 \leq \|b - Ay\|_2$ for any $y \in \mathbb{R}^n$

3. $$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix}$$

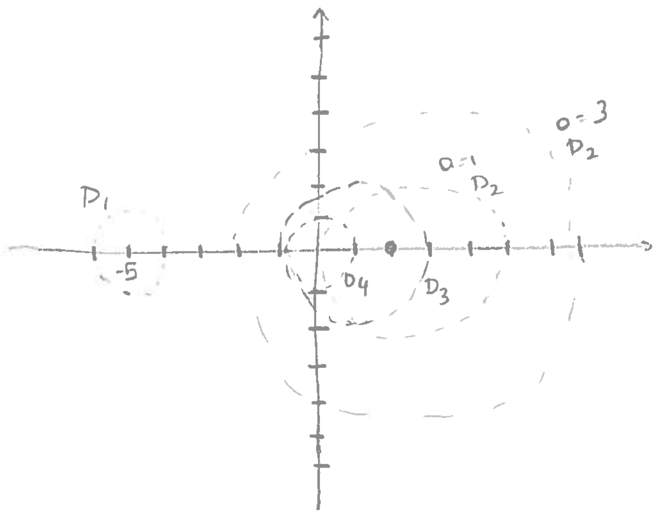
$$(A^TA)^{-1} = \frac{1}{6}\begin{bmatrix} 5 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 5/6 & -3/6 \\ -3/6 & 3/6 \end{bmatrix}$$

$$A^{+} = (A^{T}A)^{-1}A^{T} = \begin{bmatrix} 5/6 & -3/6 \\ -3/6 & 3/6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/6 & 5/6 \\ 0 & 3/6 & -3/6 \end{bmatrix}$$

III-2) From theorem, the disks are as follows:

$D_1: |z+5| < 1$ $\quad D_2: |z-2| < a+1$ $(1 \le a \le 3)$, $D_3: |z-1| < 2$, $D_4: |z| < 1$

So,



For any value of a between 1 and 3, D_1 is disjoint from the other disks. So, it contains only one eigenvalue

Now, suppose it is complex. Let $\lambda = x+iy$ with $y \neq 0$. Then $x-iy$ is also an eigenvalue of A. And observe

$$|x+iy-5| = |x-iy-5|$$

So, $x-iy$ lies in the disk D_1 which is a contradiction. So, λ must be real.

Screening Exam on Numerical Analysis – Fall 2006

1. **Gaussian-Seidel Method**

   Consider the $n \times n$ matrix A_n defined by

$$A_n = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

   (a) Show that the vectors $\vec{v}_k^n$ defined by

$$\vec{v}_k^n = \begin{pmatrix} \sin \frac{k\pi}{n+1} \\ \sin \frac{2k\pi}{n+1} \\ \vdots \\ \sin \frac{nk\pi}{n+1} \end{pmatrix}$$

   are eigenvectors of matrix A_n. Find the associated eigenvalues.

   (b) Show that Gauss Seidel method converges for solving equation $A_n x = b$.

   (c) Find the limit of the condition number of A_n as n tends toward infinity.

2. **Least square**

   Suppose you are given the singular value decomposition (SVD) of a matrix A.

   (a) Explain how to use this SVD of A to obtain a simple solution to the least square problem

$$\min_x \|Ax - b\|_2$$

   (b) Using your work in part a), give an algorithm to solve the least square problem. Show also, that your algorithm can be used to obtain the minimum norm solution.

3. **Numerical Integral**

(a) A Legendre polynomial $L(x)$ of degree n satisfies

- $\int_{-1}^{1} L(x)p(x)dx = 0$ for any polynomial $p(x)$ with degree less than n.
- $L(1) = 1$

Find $L(x)$ of degree 3.

(b) Show that if f and g are polynomials of degree less than n, if $x_i, i = 1, 2, \ldots, n$ are the roots of Legendre polynomial with degree n, and if

$$\gamma_i = \int_{-1}^{1} l_i(x)dx$$

with

$$l_i(x) = \prod_{k=1, k\neq i}^{n} \frac{x - x_k}{x_i - x_k}, \; i = 1, 2, \ldots, n$$

then

$$\int_{-1}^{1} f(x)g(x)dx = \sum_{i=1}^{n} \gamma_i f(x_i)g(x_i).$$

(Hint: one can write $f(x)g(x) = L(x)q(x) + r(x)$, where $L(x)$ is Legendre polynomial with degree n.)

4. **Linear Equation**

We say a vector $X = (x_1, \ldots, x_n)'$ is an *oscillation* vector, if $x_i x_{i+1} < 0$ for all $1 \leq i \leq n-1$. Let $n \geq 2$ be an integer and t be a real number with $0 < t < 1$. Let A_n be $n \times n$ tridiagonal matrix with

$$A_n = \begin{pmatrix} t & 1 & & & \\ -1-t & t & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1-t & t & 1 \\ & & & -1-t & t \end{pmatrix}.$$

$X = (x_1, \ldots, x_n)'$ is the solution of linear system $A_n X = e_n$, where $e_n = (0, \ldots, 0, 1)'$ is a n-dimensional unit column vector.

(a) Prove that X is an oscillation vector, when $n = 2$.

(b) Prove that X is an oscillation vector for all $n \geq 2$.

(Remark: This partially explains the typical oscillation behavior of Galerkin finite element solution for BVP, since error vector satisfies similar linear equation.)

1. Gauss-Seidel Method

a) Observe that

$$A_n v_k^n = \begin{bmatrix} 2\sin\frac{k\pi}{n+1} - \sin\frac{2k\pi}{n+1} \\ -\sin\frac{k\pi}{n+1} + 2\sin\frac{2k\pi}{n+1} - \sin\frac{3k\pi}{n+1} \\ -\sin\frac{2k\pi}{n+1} + 2\sin\frac{3k\pi}{n+1} - \sin\frac{4k\pi}{n+1} \\ \vdots \\ -\sin\frac{(n-2)k\pi}{n+1} + 2\sin\frac{(n-1)k\pi}{n+1} + \sin\frac{nk\pi}{n+1} \\ -\sin\frac{(n-1)k\pi}{n+1} + 2\sin\frac{nk\pi}{n+1} \end{bmatrix}$$

$*\ \sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2}$

$*\ \sin(a+b) = \sin a\cos b + \cos a \sin b$

For $m = 1, 2, 3, \cdots, n-2$,

$$-\sin\frac{mk\pi}{n+1} + 2\sin\frac{(m+1)k\pi}{n+1} - \sin\frac{(m+2)k\pi}{n+1}$$

$$= -\left(2\sin\frac{(m+1)k\pi}{n+1}\cdot\cos\frac{k\pi}{n+1}\right) + 2\sin\frac{(m+1)k\pi}{n+1}$$

$$= \left(2 - 2\cos\frac{k\pi}{n+1}\right)\sin\frac{(m+1)k\pi}{n+1}$$

Then consider

$$2\sin\frac{k\pi}{n+1} - \sin\frac{2k\pi}{n+1} = 2\sin\frac{k\pi}{n+1} - 2\sin\frac{k\pi}{n+1}\cos\frac{k\pi}{n+1} = \left(2-2\cos\frac{k\pi}{n+1}\right)\sin\frac{k\pi}{n+1}$$

$$-\sin\frac{(n-1)k\pi}{n+1} + 2\sin\frac{nk\pi}{n+1} = -\left(\sin\frac{nk\pi}{n+1}\cos\frac{k\pi}{n+1} \underbrace{- \cos\frac{nk\pi}{n+1}\sin\frac{k\pi}{n+1}}_{= +\cos\frac{k\pi}{n+1}\sin\frac{nk\pi}{n+1}}\right) + 2\sin\frac{nk\pi}{n+1}$$

$$\frac{k\pi}{n+1} + \frac{nk\pi}{n+1} = \frac{(n+1)k\pi}{n+1} = k\pi$$

$$= \left(2 - 2\cos\frac{k\pi}{n+1}\right)\sin\frac{nk\pi}{n+1}$$

So, we see that

$$A_n v_k^n = \left(2 - 2\cos\frac{k\pi}{n+1}\right) v_k^n$$

Thus v_k^n's are the eigenvectors and $\left(2-2\cos\frac{k\pi}{n+1}\right)$ are the corresponding eigenvalues for each $k = 1, 2, \cdots, n$.

b) Since $-1 < \cos\frac{k\pi}{n+1} < 1$, $\forall k$, we have $0 < 2 - 2\cos\frac{k\pi}{2} < 4$, $\forall k$.

So, all eigenvalues are positive. Thus, A_n is a positive definite matrix Obviously, it is symmetric. Hence A_n is an SPD matrix. We know that Gauss-Seidel method converges for SPD matrices.

c) We know that $\|A_n\|_2 = \sqrt{\text{max eigenvalue of } A_n^TA_n}$ (max singular value)

Since A_n is SPD, it is invertible and $\|A_n^{-1}\|_2 = \dfrac{1}{\sqrt{\text{min eigenvalue of } A_n^TA_n}}$ (1/(min singular value)). Since $A_n = A_n^T$, $A_n^TA_n = A_n^2$ and we know that the eigenvalues of A_n^2 are the squares of eigenvalues of A_n. So,

$\|A_n\|_2 = $ max eigenvalue of A_n, $\|A_n^{-1}\|_2 = 1/(\text{min. eigenvalue of } A_n)$

$$\Rightarrow \|A_n\|_2 = 2 - 2\cos\frac{n\pi}{n+1} = 2 + 2\cos\frac{\pi}{n+1}, \quad \|A_n^{-1}\|_2 = \frac{1}{2 - 2\cos\frac{\pi}{n+1}}$$

$$\text{So,}\quad K(A_n) = \|A_n\|\cdot\|A_n^{-1}\| = \frac{2 + 2\cos\frac{\pi}{n+1}}{2 - 2\cos\frac{\pi}{n+1}} = \frac{1 + \cos\frac{\pi}{n+1}}{1 - \cos\frac{\pi}{n+1}}$$

$$\lim_{n\to\infty} K(A_n) = \frac{\lim_{n\to\infty}\left(1 + \cos\frac{\pi}{n+1}\right)}{\lim_{n\to\infty}\left(1 - \cos\frac{\pi}{n+1}\right)} = \frac{2}{0} = \infty$$

2. Least-squares

a) Suppose $A = U\Sigma V^T$ is the SVD of A. We know from normal equations, $(A^TA)^{-1}A^Tb$ is the unique solution when A is full rank. Consider firstly the case A is full-rank, and see

$$
\begin{aligned}
(A^TA)^{-1}A^T &= (V\Sigma^TU^TU\Sigma V^T)^{-1}(V\Sigma^TU^T) \\
&= (V\Sigma^T\Sigma V^T)^{-1}(V\Sigma^TU^T) \\
&= V(\Sigma^T\Sigma)^{-1}V^TV\Sigma^TU^T \\
&= V\Sigma^{-1}(\Sigma^T)^{-1}\Sigma^TU^T \\
&= V\Sigma^{-1}U^T
\end{aligned}
$$

So, we can immediately compute pseudoinverse of A and find the solution which is A^+b.

If A is not full rank, say r is the rank of A. Let u_i and v_i denote the first columns of U and V. Then, we know

$$x = \sum_{i=1}^{r} \frac{u_i^T b}{\sigma_i} v_i$$

is the least squares solution when A is rank deficient

b) Another way to use SVD to compute the solution of $Ax=b$, compute full SVD of A. That is

$$\underset{m\times n}{A} = \underset{m\times m}{U}\,\underset{m\times n}{\Sigma}\,\underset{n\times n}{V^T}$$

where $U^TU = UU^T = I_m$, $V^TV = VV^T = I_n$

→ Solve $Uy=b$ by multiplying U^T from left: $U^TUy = U^Tb \Rightarrow y = U^Tb$

→ Solve the diagonal system $\Sigma z = y$

→ Solve $V^Tx = y$ by multiplying V from left: $VV^Tx = Vy \Rightarrow x = Vy$

Now, observe (for $m \geq n$)

$\|Ax-b\|_2 = \|U\Sigma V^Tx - b\|_2$ by invariance of norm under unitary matrix multip.

$= \|\Sigma V^Tx - \hat{b}\|_2$, $\hat{b} = U^Tb$

$= \|\Sigma y - \hat{b}\|_2$, $y = V^Tx$

$= \sum_{i=1}^{k} |\sigma_{ii} y_i - \hat{b}_i|^2 + \sum_{i=k+1}^{m} |\hat{b}_i|^2$, $k \leq m$ is the rank of A.

So, either A is full rank or it is rank deficient, by choosing $(y_i = 0,\ i \geq k+1)$

$$y_i = \frac{\hat{b}_i}{\sigma_{ii}} \quad \text{for } i = 1, \ldots, k$$

$$y_i = \text{arbitrary} \quad \text{for } i = k+1, \ldots, n$$

we get the minimum-norm solution. Observe that if A is full rank $(k=n)$ we directly find the minimum-norm solution which is also unique in this case.

3. Numerical Integral —

4. Linear Equation

a) For $n=2$,

$$A_2 = \begin{bmatrix} t & 1 \\ -1-t & t \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the system will be

$$\left.\begin{array}{l} t x_1 + x_2 = 0 \\ -x_1 - t x_1 + t x_2 = 1 \end{array}\right\} \quad \left.\begin{array}{l} \\ -x_1 + (1+t) x_2 = 1 \Rightarrow -t x_1 + t(1+t) x_2 = t \end{array}\right\} \quad (t + t^2 + 1) x_2 = t$$

$$\Rightarrow x_2 = \frac{t}{t^2+t+1} \quad \text{and} \quad x_1 = \frac{-1}{t^2+t+1}$$

Then, $x_1 x_2 = \dfrac{-t}{(t^2+t+1)^2} < 0$ since $0<t<1$.

Thus $\begin{bmatrix} -1/(t^2+t+1) \\ t/(t^2+t+1) \end{bmatrix}$ is an oscillation vector

b) Adding each row to the following row, the matrix will be

$$\left[\begin{array}{ccccccc|c}
t & 1 & & & & & & 0 \\
-1 & t+1 & 1 & & & & & 0 \\
 & 0 & t+1 & 1 & & & & 0 \\
 & & 0 & t+1 & 1 & & & 0 \\
 & & & \ddots & \ddots & \ddots & & \vdots \\
 & & & & 0 & t+1 & 1 & 0 \\
 & & & & & 0 & t+1 & 1
\end{array}\right]$$

Then adding $\frac{1}{t} \times$ (1st row) to 2nd row, we get

$$\left[\begin{array}{ccccccc|c}
t & 1 & & & & & & 0 \\
0 & t+1+\frac{1}{t} & 1 & & & & & 0 \\
 & 0 & t+1 & 1 & & & & 0 \\
 & & 0 & t+1 & 1 & & & 0 \\
 & & & \ddots & \ddots & \ddots & & \vdots \\
 & & & & 0 & t+1 & 1 & 0 \\
 & & & & & 0 & t+1 & 1
\end{array}\right]$$

By backward substition,

$$x_n = \frac{1}{t+1}$$

$$(t+1)x_{n-1} + x_n = 0 \Rightarrow x_{n-1} = \frac{-1}{(t+1)^2}$$

$$(t+1)x_{n-2} + x_{n-1} = 0 \Rightarrow x_{n-2} = \frac{1}{(t+1)^3}$$

$$\vdots$$

$$(t+1)x_3 + x_4 = 0 \Rightarrow x_3 = x_{n-(n-3)} = \frac{(-1)^{n-3}}{(t+1)^{n-2}}$$

$$(t+1+\tfrac{1}{t})x_2 + x_3 = 0 \Rightarrow x_2 = \frac{(-1)^{n-2}}{(t+1)^{n-2}(t+1+\frac{1}{t})}$$

$$tx_1 + x_2 = 0 \Rightarrow x_1 = \frac{(-1)^{n-1}}{t(t+1)^{n-2}(t+1+\frac{1}{t})}$$

Now, observe that

$$x_1 = \frac{(-1)^{n-1}}{t(t+1)^{n-2}(t+1+\frac{1}{t})}, \quad x_2 = \frac{(-1)^{n-2}}{(t+1)^{n-2}(t+1+\frac{1}{t})},$$

$$x_m = \frac{(-1)^{n-m}}{(t+1)^{n-m+1}} \quad \text{for} \quad m = 3, 4, \ldots, n.$$

Since the denominators of all x_n's are positive, we can multiply only numerators to decide if the multiplication is positive or negative. Note that neither of x_n is 0.

$$x_1 x_2 < 0 \quad \text{since} \quad (-1)^{n-1}(-1)^{n-2} < 0$$

$$x_2 x_3 < 0 \quad \text{since} \quad (-1)^{n-2}(-1)^{n-3} < 0$$

$$x_m x_{m+1} < 0 \quad \text{since} \quad (-1)^{n-m}(-1)^{n-m-1} < 0 \quad \text{for all} \quad m = 3, 4, \ldots, n-1.$$

Thus $X = (x_1, \ldots, x_n)$ which is a solution of $A_n X = e_n$ with $n \geq 3$ is an oscillation vector.

1 2.b

1 2. c) what value of κ would suggest ill-conditioning?

Screening Exam in Numerical Analysis, Spring 2006

1.Iterative methods for solving (*) $Au = f$

1. Consider a square matrix $A \in R^{n \times n}$ such that $A_{ij} = \begin{cases} 2 & \text{for } i = j \\ -1 & \text{for } |i - j| = 1 \\ 0 & \text{otherwise} \end{cases}$

The eigenvalues of A are known to be $\lambda_i(A) = 2 + 2cos(\frac{\pi i}{n+1})$, i=1,...,n.
Compute the following estimates as a function of n (you may use the big-Oh notation, i.e. $O(n)$, $O(n^2)$, etc. in the final answer):

a. The condition number of A, $\kappa(A)$,
b. Spectral radius, ρ_J of the Jacobi iteration matrix for (*),
c. Number of Jacobi iterations required.

2. State the assumptions and present the conclusions of the theorem for finding the optimal SOR parameter ω_{opt} (and the corresponding spectral radius, $\rho_{\omega_{opt}}$ of the SOR iteration matrix).

Compute the following estimates for the matrix in (1):

a. Optimal SOR parameter, ω_{opt} and the spectral radius, $\rho_{\omega_{opt}}$,
b. Number of optimal SOR iterations required.

Numerical Analysis Screening Exam Feb. 2006

Problem 2. Solution of Linear Equations by Direct Methods

Given a non-singular n x n real matrix $A = (a_{ij})$ and an n-dimensional vector b.

(a) Write the equations for an algorithm to compute the LU decomposition of A without pivoting. (Assume that the LU factors exist.) How many arithmetic operations are required ?

(b) Write the equations for forward and back substitution to solve $Ax = b$, given that $A = LU$. How many operations are required?

(c) Define the condition number, κ , of A. Use κ to specify a formula for an estimate of rounding error in solving the system $Ax = b$. Define what is meant by an ill-conditioned system. What value of κ would suggest ill-conditioning?

Numerical Analysis Exam Feb 2006

3.(**Least squares**) Let A be a linear operator mapping a real n-dimensional vector space, V^n, into a real m-dimensional vector space V^m. Both spaces have an inner-product $\langle u, v\rangle$. The *adjoint* of A is a linear operator A^* mapping V^m into V^n and satisfying the relation $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x \in V^n$ and $y \in V^m$.

(a) Let bases be chosen and A represented by an m×n matrix relative to these bases. With the usual definition of $<u, v>$, show that A^* is represented by the transpose of the matrix.

(b) The *pseudoinverse* of A is a linear operator $A^\dagger$ on V^m to V^n satisfying the Moore-Penrose conditions :

(Gen) $A A^\dagger A = A$ and $A^\dagger A A^\dagger = A^\dagger$; (Sym) $(A A^\dagger)^* = A A^\dagger$ and $(A^\dagger A)^* = A^\dagger A$.

Assume $A^\dagger$ exists. It can be proved that $A A^\dagger = P_{R(A)}$ (projection onto the range)

Use this to prove the following Theorem.

Theorem . $A^\dagger b$ is a solution of the least squares problem, that is,

$$\|A A^\dagger b - b\| \le \|Ax - b\| \text{ for all } x \in V^n.$$

(Hint: $AA^\dagger b = b_{R(A)}$, where $b = b_{R(A)} + b_{R(A)^\perp}$. For any x ,

$\|Ax - b\|^2 = \|Ax - b_{R(A)}\|^2 + \|b_{R(A)^\perp}\|^2 \ge \|b_{R(A)^\perp}\|^2$. Justify each step.)

(c) State the Singular Values Decomposition (SVD) Theorem for $A^\dagger$.

NumericAL Analysis Screening Feb 06

4. Eigenvalue problems

Suppose that A is a matrix of order $m \times m$.

1. Give an argument to support the idea that an algorithm for finding eigenvalues needs to be an iterative one.
2. State the QR algorithm with shifts. Indicate its order of convergence.
3. Explain how Householder reflectors and Hessenberg matrices are used in eigenvalue computations (QR algorithm).

1 Iterative Methods

1. $A_{ij} = \begin{cases} 2 & , i=j \\ -1 & , |i-j|=1 \\ 0 & , \text{otherwise} \end{cases}$

We know $\lambda_i(A) = 2+2\cos\left(\frac{\pi i}{n+1}\right)$, $i=1,2,\dots,n$.

a) We see that $0<\lambda_i(A)<4$ $\forall i$ since $-1<\cos\left(\frac{\pi i}{n+1}\right)<1$ $\forall i$. So, the matrix is invertible.

Since also $A^T = A$, we have $A^TA = A^2$ and also note that the eigenvalues of A^2 are $\lambda_i(A)^2$ for $i=1,2,\dots,n$. Then,

$$\|A\|_2 = \sqrt{\text{max eigenvalue of } A^TA} = 2+2\cos\frac{\pi}{n+1}$$

$$\|A^{-1}\|_2 = \frac{1}{\sqrt{\text{min eigenvalue of } A^TA}} = \frac{1}{2+2\cos\frac{n\pi}{n+1}} \qquad * \cos\frac{n\pi}{n+1} = -\cos\frac{\pi}{n+1}$$

So, $K(A) = \|A\|_2\,\|A^{-1}\|_2 = \dfrac{1+\cos\frac{\pi}{n+1}}{1-\cos\frac{\pi}{n+1}}$

b) We know $B_J = -D^{-1}(L+U)$. Since $L+U = A-D$, we get $B_J = -D^{-1}(A-D) = I - D^{-1}A$.

Let (λ, v) be an eigenpair of A. Then $Av=\lambda v$ and $-D^{-1}Av = -\lambda D^{-1}v = -\frac{\lambda}{2}v$ since $D^{-1} = \text{diag}(1/2,\dots,1/2)$. So, $-D^{-1}A$ has eigenvalues $-1-\cos\frac{\pi i}{n+1}$, $i=1,2,\dots,n$. Hence $I-D^{-1}A$ has eigenvalues $-\cos\frac{\pi i}{n+1}$ for $i=1,2,\dots,n$. Then $\|B_J\|_2 = \cos\frac{\pi}{n+1}$ and since $\rho(B_i) \le \|B\|$ we deduce $\rho(B_J) \le \cos\frac{\pi}{n+1}$

c) We have $\|x^{(k+1)}-x^{(k)}\| \le \|B\|\,\|x^{(k)}-x^{(k-1)}\| \le \dots \le \|B\|^k\,\|x^{(1)}-x^{(0)}\|$.

So, $\|x^{(k+1)}-x^{(k)}\| \le \left(\cos\frac{\pi}{n+1}\right)^k \|x^{(1)}-x^{(0)}\|$. Now, let $k=2^m$ for some m. Since $\cos a\cos b = \frac{1}{2}\left(\cos(a+b)+\cos(a-b)\right)$, we get

$$\left(\cos\frac{\pi}{n+1}\right)^k = \left(\cos\frac{\pi}{n+1}\cos\frac{\pi}{n+1}\right)\cdots\left(\cos\frac{\pi}{n+1}\cdot\cos\frac{\pi}{n+1}\right) \quad \to k/2 = 2^{m-1} \text{ groups}$$

$$= \left(\frac{1}{2}\right)^{2^{m-1}}\left(\cos\frac{2\pi}{\dots}\right)^{2^{m-1}}$$

Continuing on this way, we get

$$\left(\cos\frac{\pi}{n+1}\right)^{2^m} = \left(\frac{1}{2}\right)^{2^{m-1}}\left(\frac{1}{2}\right)^{2^{m-2}}\cdots\left(\frac{1}{2}\right)^{2^{m-m}}\cos\left(\frac{2^m\pi}{n+1}\right)$$

$$= \left(\frac{1}{2}\right)^{2^m\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^m}\right)}\cdot\cos\left(\frac{2^m\pi}{n+1}\right)$$

$$= \left(\frac{1}{2}\right)^{\cancel{2^m}\cdot\frac{2^m-1}{\cancel{2^m}}}\cos\left(\frac{2^m\pi}{n+1}\right)$$

$$= \frac{1}{2^{2^m-1}}\left|\cos\left(\frac{2^m\pi}{n+1}\right)\right| \quad \text{, since we know } \cos\frac{\pi}{n+1} > 0.$$

Replacing k back,

$$\left(\cos\frac{\pi}{n+1}\right)^k = \frac{1}{2^{k-1}}\left|\cos\left(\frac{k\pi}{n+1}\right)\right|$$

So, $\|x^{(k+1)}-x^{(k)}\| \leq \frac{\left|\cos\frac{k\pi}{n+1}\right|}{2^{k-1}}\|x^{(1)}-x^{(0)}\| \leq \frac{1}{2^{k-1}}\|x^{(1)}-x^{(0)}\|$

Thus, the smallest natural number k_0 with $\frac{1}{2^{k_0-1}} \approx \varepsilon_{mach}$ is the number of required iterations for Jacobi iteration to converge.

(If B_j is consistently ordered and weakly cyclic of index 2)

2. We know that SOR method converges when $0<\omega<2$. Let B_j be the associated Jacobi matrix. If B_j^2 has only nonnegative eigenvalues and $\rho(B_j)<1$ then the optimal parameter ω_{opt} for SOR method is given by

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\rho(B_j)^2}}$$

and in this case $\rho(B_{SOR}) = \omega_{opt}-1 = \dfrac{1-\sqrt{1-\rho(B_j)^2}}{1+\sqrt{1-\rho(B_j)^2}}$

a) We see that $B_j = -D(L+U)$ is as follows

$$B_j = \begin{bmatrix} 0 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ & 1/2 & 0 & \ddots & \\ & & \ddots & \ddots & 1/2 \\ & & & 1/2 & 0 \end{bmatrix} = -\frac{1}{2}(A-2I)$$

So, eigenvalues of B_j are $-\cos\left(\frac{\pi i}{n+1}\right)$, $i=1,\ldots,n$, and so,

$\rho(B_j) = \cos\left(\frac{\pi}{n+1}\right)$ and so,

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\cos^2\left(\frac{\pi}{n+1}\right)}} = \frac{2}{1+\sin\left(\frac{\pi}{n+1}\right)}$$

$$\rho_{SOR} = w_{opt} - 1 = \frac{2}{1+\sin\left(\frac{\pi}{n+1}\right)} = \frac{1-\sin\left(\frac{\pi}{n+1}\right)}{1+\sin\left(\frac{\pi}{n+1}\right)}$$

2. Solution of Linear Equations by Direct Methods

a) for $k = 1, 2, \dots, n-1$

for $j = k+1, \dots, n$

compute $m_{j,k} = \dfrac{a_{j,k}^{(k-1)}}{a_{k,k}^{(k-1)}}$ — 1

(nonzero parts) multiply k^{th} row of $A^{(k-1)}$ by $m_{j,k}$ — $n-k+1$

add it to the j^{th} row of $A^{(k-1)}$ — $n-k+1$

end

end

Number of required arithmetic operations:

$$\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}(2n-2k+3) = \sum_{k=1}^{n-1}(n-k)(2n-2k+3)$$

$$= \sum_{k=1}^{n-1}(2n^2+3n) + \sum_{k=1}^{n-1}2k^2 + \sum_{k=1}^{n-1}(-4n-3)k$$

$$= (n-1)(2n^2+3n) + 2\,\frac{(n-1)\cdot n\,(2n-1)}{6} - (4n+3)\frac{(n-1)n}{2}$$

$$= 2n^3+n^2-3n + \frac{(n^2-n)(2n-1)}{3} - \frac{(4n+3)(n^2-n)}{2}$$

$$= 2n^3+n^2-3n + \frac{2n^3-3n^2+n}{3} - \frac{4n^3-n^2-3n}{2}$$

$$= \frac{2}{3}n^3 + \frac{1}{2}n^2 - 2n$$

b) $Ax=b \Leftrightarrow LUx=b$. Calling $Ux=y$, Firstly solve $Ly=b$ for y and then solve $Ux=y$ for x.

$\rightarrow Ly=b$ (forward substitution)

$y_1 = b_1/\ell_{11}$ $\longrightarrow$ 1 multiplication

for $k = 2, \dots, n$

$y_k = \left(b_k - \sum_{j=1}^{k-1}\ell_{kj}y_j\right)/\ell_{kk}$ $\longrightarrow$ k multiplication, 1 addition

end

# of equations: $1 + \sum^{n}(k+1) = \cancel{1} + (n-1) + \frac{n(n+1)}{2} - \cancel{1} = \frac{1}{2}n^2 + \frac{3}{2}n - 1$

$\to Ux = y$ (backward substitution)

$x_n = y_n / u_{nn}$ — 1 multiplication

for $k = 1, 2, \dots, n-1$

$$x_{n-k} = \left(b_{n-k} - \sum_{j=1}^{k+1} u_{n-k,n-k-1+j} \cdot y_{n-k-1+j} \right) / u_{n-k,n-k}$$ — k+2 multiplication, 1 addition

end

# of equations: $1 + \sum_{k=1}^{n-1} (k+2) = 1 + \frac{(n-1)n}{2} + 2n - 2 = \frac{1}{2}n^2 + \frac{3}{2}n - 1$

So, in total $n^2 + 3n - 2$ operations required.

c) $K(A) = \|A\| \cdot \|A^{-1}\|$, provided A^{-1} exists.

Suppose the vector b is perturbed by δb and $b + \delta b = \tilde{b}$. Let $\tilde{x}$ be the corresponding solution such that $A\tilde{x} = \tilde{b}$. Then observe

$$\|x - \tilde{x}\| = \|A^{-1}b - A^{-1}\tilde{b}\| \le \|A^{-1}\| \, \|b - \tilde{b}\| = \|A^{-1}\| \cdot \|b - \tilde{b}\| \cdot \frac{\|Ax\|}{\|b\|} \le \|A\| \cdot \|A^{-1}\| \cdot \frac{\|b - \tilde{b}\|}{\|b\|} \cdot \|x\|$$

$$\Rightarrow \frac{\|x - \tilde{x}\|}{\|x\|} \le K(A) \cdot \frac{\|b - \hat{b}\|}{\|b\|}.$$

A system $Ax = b$ is called ill-conditioned, if $K(A)$ is a large number. In this case, even if the relative error of perturbation is small, the relative error in solution may be high.

3. Least-squares

a) Let $\{u_1, \dots, u_n\}$ be a basis for V^n and $\{w_1, \dots, w_m\}$ be a basis for V^m. Then, for any $1 \le i \le n$, and $1 \le j \le m$

$$Au_i = \sum_{k=1}^{m} a_{ki} w_k \qquad A^* w_j = \sum_{k=1}^{n} a^*_{kj} u_k$$

By definition, observe that

$$\langle Au_i, w_j \rangle = \left(\sum_{k=1}^{m} a_{ki} w_k \right)^T w_j = a_{ji},$$

$$\langle u_i, A^* w_j \rangle = u_i^T \left(\sum_{k=1}^{n} a^*_{kj} u_k \right) = a^*_{ij}$$

By assumption $\langle Au_i, w_j \rangle = \langle u_i, A^* w_j \rangle$ and so, $a^*_{ij} = a_{ji}$ $\forall i, j$.

Thus A^* is represented by A^T

b) We can write $b = b_{R(A)} + b_{R(A)^\perp}$ for any vector b and subspace. Since AA^+ is the projection onto $R(A)$, we know $b_{R(A)} = AA^+b$. So, $b = AA^+b + b_{R(A)^\perp}$ which implies that $b_{R(A)^\perp} = b - AA^+b$

On the other hand, we know $Ax \in R(A)$ for any x. Then $Ax - b_{R(A)} \in R(A)$. Now, for any x, observe

$$
\begin{aligned}
\|Ax - b\|^2 &= \|Ax - b_{R(A)} - b_{R(A)^\perp}\|^2 \\
&= \|Ax - b_{R(A)}\|^2 + \|b_{R(A)^\perp}\|^2 \quad \text{by Pythagorean Thm} \\
&\geq \|b_{R(A)^\perp}\|^2 \quad \text{since } \|Ax - b_{R(A)}\|^2 \geq 0 \\
&= \|b - AA^+b\|^2 \\
&= \|A(A^+b) - b\|^2
\end{aligned}
$$

So, for any x, $\|A(A^+b) - b\| \leq \|Ax - b\|$. Thus A^+b is a least-squares solution.

c) We know by theorem, for any matrix $A \in \mathbb{R}^{m\times n}$, there exists orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ and diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ such that

$$A = U\Sigma V^T$$

In this case, we know that A^+ is given by

$$A^+ = V\Sigma^{-1}U^T$$

4 Eigenvalue Problems:

1. We know that eigenvalues of an $m\times m$ matrix A are the roots of its characteristic polynomial which is of order m. Since there is no general formula for finding roots of a polynomial of order m when $m \geq 5$ (Abel's Theorem), we deduce that any algorithm for finding eigenvalues needs to be iterative.

b) Let σ be a suitable shift to find eigenvalues of A by QR algorithm Then,

$A^{(0)} = A$

for $k = 0, 1, 2, \ldots$

compute QR factorization: $A^{(k)} - \sigma I = Q_k R_k$

form $A^{(k+1)}$: $A^{(k+1)} = R_k Q_k + \sigma I$.

end

If $\lambda_1, \ldots, \lambda_m$ are eigenvalues of A then $\lambda_1 - \sigma, \ldots \lambda_n - \sigma$ are the eigenvalues of $A - \sigma I$ Its order of convergence is given by

$$\left| \frac{\lambda_i - \sigma}{\lambda_{i-1} - \sigma} \right|$$

where $|\lambda_1 - \sigma| > |\lambda_2 - \sigma| > \cdots |\lambda_n - I|$, and this ratio is the biggest among $i = 2, 3, \ldots, n$.

c) Householder matrices are orthogonal, because of that the eigenvalues of A and $H^T A H$ are the same for any matrix A and any Householder matrix H.

QR factorization of a full and dense matrix A requires $O(n^3)$ flops. Thus, n QR iterations will require $O(n^4)$ flops. So, instead of applying this method first, we can reduce full matrix A to an Hessenberg matrix H by Householder matrices so that A and H have the same eigenvalues. QR iteration of an Hessenberg matrix requires $O(n^2)$ flops. So, QR iteration method with the initial reduction of A to a Hessenberg matrix will be an $O(n^3)$ method.

1b 4c (last part)

2c (last part) 4d

)

Screening Exam in Numerical Analysis, Sept. 12,2005
Do all 4 problems

1. Linear equations

Consider a linear system

$$Ax = b \qquad (1)$$

where A is a square matrix of order n and vectors x and b have dimension n.

a. Give conditions under which a solution to (1) exists.

b. Give formulas (using matrix notation) that define the most commonly used algorithms to compute a solution to (1). Compare their computational cost and accuracy.

(c). Define the row and column rank of a matrix. How are they related to part (a)? Determine the rank of the following 3x3 matrix A

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$$

Find an integer-valued vector b for this matrix

(i) for which Ax = b has no solutions,

(ii) for which Ax = b has infinitely many solutions, and give the explicit form of these solutions (depending on some parameter c).

a) If A is full rank (equivalently nonsingular), the system $Ax=b$ has a unique solution.

If A is not full rank (singular) then either there is no solution or there are infinitely many solutions.

b) LU factorization:

→ Obtain LU factorization of A: $A=LU$

→ Solve $Ly=b$ for y

→ Solve $Ux=y$ for x.

c) Column rank of a matrix is the number of linearly independe columns of A Similarly, the row rank is the number of linearly independent rows.

Let a_1, a_2, a_3 be the columns of A. Then, let $c_1a_1+c_2a_2+c_3a_3=0$ and solve this for c_1, c_2, c_3

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad (*)$$

So, we see that there are 2 linearly independent rows. Since the row rank = column rank for any matrix, we deduce the rank of A is 2.

i) If we choose $b=[1\ 1\ 2]^T$, at the last stage of (*) we get

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Because of the last row there's no solution.

ii) Observing last stage in (*), we can choose c_3 as free So, let $c_3=c$. Then $-3c_2=6c$ implying $c_2=-2c$ and then $c_1-4c-6c=0$ implying $c_1=10c$. Thus, if we choose $b=[0,0,0]^T$, we get the solutions as

$$x=c\cdot\begin{bmatrix} 10 \\ -2 \\ 1 \end{bmatrix}$$

for any $c\in\mathbb{R}$

Numerical Analysis Screening Exam. Sept. 12, 2005

Problem 2. Iterative Solution of Linear Equations

Let A be a real non-singular n x n matrix and b a real n-dimensional vector. Consider the equation Ax = b. Let A = M -N be a splitting of A such that M is non-singular.

(a)Use the splitting to write the equation for an iteration that computes a sequence of vectors, $x^{(m)}$. If the sequence converges, prove that the limit is the solution of Ax = b.

(b)Define $Q = M^{-1}N$. Give a sufficient condition on a norm of Q for the convergence of $x^{(m)}$. Prove convergence using this condition. Define the rate of convergence.

(c)Give the splitting for the Jacobi iteration. State a property of A in terms of its diagonal elements that implies the sufficient condition in (b) holds for the Jacobi iteration. Discuss how to parallelize the Jacobi iteration on a cluster of computers.

(d) Give the splitting for the Gauss-Seidel iteration and state a property of A in terms of its elements that is sufficient for convergence. Prove it. Define the SOR family of relaxation iterations with parameter ω which yields the Gauss -Seidel iteration when ω = 1. Discuss how to choose ω to minimize the spectral radius of Q_ω.

a) $Ax = b \Leftrightarrow (M-N)x = b \Leftrightarrow Mx = Nx + b \Leftrightarrow x = M^{-1}Nx + M^{-1}b$ (1)

So, $x^{(m+1)} = M^{-1}Nx^{(m)} + M^{-1}b$, $m = 0, 1, 2, \ldots$ (2)

If $\{x^{(m)}\}$ converges then, say $\lim_{n\to\infty} x^{(m)} = x^*$ and, taking limit as $m \to \infty$ of both sides of (2), we get

$$x^* = M^{-1}Nx^* + M^{-1}b$$

which is the same as the last equation in (1). Since all statements are "$\Leftrightarrow$" in (1), we deduce $Ax^* = b$, i.e x^* is a solution of $Ax = b$

b) A sufficient condition for convergence of (let $M^{-1}b = c$)

$$x^{(m+1)} = Qx^{(m)} + c$$

for any initial vector $x^{(0)}$, is $\|Q\| < 1$.

For any m, (since $Ax = b \Leftrightarrow x = Qx + c$)

$x^{(m)} - x = Qx^{(m-1)} + \cancel{c} - Qx - \cancel{c} = Q(x^{(m-1)} - x) = \dots = Q^m(x^{(0)} - x)$

So, $\|x^{(m)} - x\| \le \|Q\|^m \|x^{(0)} - x\|$ Since $\|Q\| < 1$, if we let $m \to \infty$, we get $\|x^{(m)} - x\| \to 0$ as $m \to \infty$ So, $x^{(m)} \to x$ as $m \to \infty$ where x is the solution of $Ax = b$.

Then, observe

$$\frac{\|x^{(m)} - x\|}{\|x^{(m-1)} - x\|} = \|Q\| \implies \lim_{m \to \infty} \frac{\|x^{(m)} - x\|}{\|x^{(m-1)} - x\|} = \|Q\|$$

and $\|Q\|$ is the rate of convergence

c) $A = L + U + D$

$Ax = b \Leftrightarrow (L+U+D)x = b \Leftrightarrow Dx = -(L+U)x + b \Leftrightarrow x = -D^{-1}(L+U)x + D$

So,

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$$

and $-D^{-1}(L+U)$ is the Jacobi iteration matrix

If A is row diagonally dominant then,

$$-D^{-1}(L+U) = \begin{bmatrix} -1/a_{11} & & \\ & \ddots & \\ & & -1/a_{nn} \end{bmatrix} \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & & \vdots \\ a_{31} & a_{32} & \ddots & a_{n-1,n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & a_{n,n-1} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\frac{a_{12}}{a_{11}} & \cdots & -\frac{a_{1n}}{a_{11}} \\ +\frac{a_{21}}{a_{22}} & 0 & \cdots & -\frac{a_{2n}}{a_{22}} \\ \vdots & & \ddots & \\ -\frac{a_{n1}}{a_{nn}} & -\frac{a_{n2}}{a_{nn}} & \cdots & 0 \end{bmatrix}$$

Then, for any i, summation of absolute values of a row of $-D^{-1}(L+U)$

$$\sum_{j=1}^{n} \frac{|a_{ij}|}{|a_{ii}|} = \frac{\sum_{j=1}^{n} |a_{ij}|}{|a_{ii}|} < 1 \quad \text{by assumption}$$

So, $\|-D^{-1}(L+U)\|_\infty < 1$ and the iteration converges.

2. d) $A = L + D + U$

$$Ax = b \iff (L+D+U)x = b \iff -(L+D)x = +Ux + b$$

$$\iff x = -(L+D)^{-1}Ux - (L+D)^{-1}b$$

$$x^{(k+1)} = -(L+D)^{-1}Ux^{(k)} - (L+D)^{-1}b.$$

Here $-(L+D)^{-1}U$ is the Gauss-Seidel iteration matrix.

Again if A is row diagonally dominant, then Gauss-Seidel iteration converges

Let (λ, v) be an eigenpair for $-(L+D)^{-1}U$ Then

$$-(L+D)^{-1}Uv = \lambda v \iff -Uv = \lambda(L+D)v.$$

$$\Rightarrow \quad -\sum_{j=k+1}^{n} a_{kj}v_j = \lambda \sum_{j=1}^{k} a_{kj}v_j \quad, \text{for} \quad k = 1, 2, \ldots, n.$$

$$\Rightarrow \quad \lambda a_{kk}v_k = -\sum_{j=k+1}^{n} a_{kj}v_j - \lambda \sum_{j=1}^{k-1} a_{kj}v_j \quad, \quad k = 1, 2, \ldots, n.$$

?

Now, let $|v_i|$ be the largest component (having magnitude 1) of v_k

Then,

$$|\lambda| . |a_{ii}| \leq \sum_{j=k+1}^{n} |a_{ij}| + |\lambda| \sum_{j=1}^{k-1} |a_{ij}|$$

$$\Rightarrow \quad |\lambda| \leq \frac{\sum_{j=k+1}^{n} |a_{ij}|}{|a_{ii}| - \sum_{j=1}^{k-1} |a_{ij}|}$$

Since A is row diagonally dominant $|a_{ii}| > \sum_{j=1}^{k-1} |a_{ij}| + \sum_{j=k+1}^{n} |a_{ij}|$. So, we say $|\lambda| < 1$ which is the necessary and sufficient condition for the convergence.

→ $A = L + D + U$

$$Ax = b \iff (L+D+U)x = b \iff (wD + L + D - wD + U)x = b$$

$$\iff (wD+L)x = [(1-w)D + U]x + b \iff x = (wD+L)^{-1}[(1-w)D+U]x + b.$$

So, $(wD+L)^{-1}[(1-w)D+U]$ is the SOR iteration matrix.

We need to choose ω so that $\rho(B_{SOR})$ is minimized for convergence rate. We know that when A is consistently ordered and $\rho(B_J) < 1$ and eigenvalues of B_J are real, (B_J is Jacobi iteration matrix) then optimal parameter is given as

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(B_J)^2}}$$

In this case $\rho(B_{SOR}) = \omega_{opt} - 1$ is minimized.

Numerical Analysis Screening Exam Sept 12, 2005

3. Eigenvalue problems

Suppose that A is a matrix of size mxm.
(a) Prove Hadamard's circle theorem on the location of the eigenvalues of A:
Theorem. Let Ps be the sum of the moduli of the elements along the s-th row excluding
the diagonal element a(s,s). Then each eigenvalue of A lies inside or on the boundary of
at least one of the circles |z - a(s,s)| = Ps.

(b) Give estimates based on Hadamard's circle theorem for the eigenvalues z(1), z(2), Z(3)
of the matrix

$$\begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & e \\ 0 & e & 1 \end{pmatrix}$$

where |e| < 1 .

(c) Establish the tighter bound |z(3) - 1| <= e^2 on the smallest eigenvalue z(3).
(Hint: Find a suitable diagonal matrix similarity transformation.)

(d) Explain the QR algorithm with shifts. Indicate the order of convergence of this algorithm and under what conditions the order of convergence is achieved.

a) Let (λv) be an eigenpair for the matrix A. Then

$$Av = \lambda v$$

and

$$\sum_{j=1}^{n} a_{ij} v_j = \lambda v_i \qquad \text{for} \quad i = 1, 2, \dots, n$$

Then

$$\sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} v_j = (\lambda - a_{ii}) v_i \qquad \text{for} \quad i = 1, 2, \dots, n$$

Take i so that $|v_i|$ is the max in magnitude among other entries. Then,

$$|\lambda - a_{ii}| \le \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \frac{|v_j|}{|v_i|} < \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| .$$

b) $|z(1)-8|<1$, $|z(2)-4|<1+|e|<2$, $|z(3)-1|<|e|<1$ (1)

c) Let $X=\begin{bmatrix}1&&\\&1&\\&&e\end{bmatrix}$ Then $X^{-1}=\begin{bmatrix}1&&\\&1&\\&&1/e\end{bmatrix}$, and

$$\begin{bmatrix}1&&\\&1&\\&&e\end{bmatrix}\underbrace{\begin{bmatrix}8&1&0\\1&4&e\\0&e&1\end{bmatrix}}_{A_1}\begin{bmatrix}1&&\\&1&\\&&1/e\end{bmatrix}=\underbrace{\begin{bmatrix}8&1&0\\1&4&1\\0&e^2&1\end{bmatrix}}_{A_2}$$

A_1 and A_2 has the same eigenvalues, comparing

$|z(1)-8|<1$, $|z(2)-4|<2$, $|z(3)-1|<e^2$

by (1) we deduce we found a better estimate for $z(3)$ which is $|z(3)-1|<e^2$.

d) We shift eigenvalues of A^k by σ : $A^k-\sigma I$

Then find QR factorization : $A^k-\sigma I$ Q^kR^k

Then form the matrix again $A^{k+1}=R^kQ^k+\sigma I$

If A has eigenvalues $\lambda_1,\dots,\lambda_n$ then $A\cdot\sigma I$ has eigenvalues $\lambda_1-\sigma,\dots,\lambda_n-\sigma$. We take the absolute values of $\lambda_1-\sigma,\dots,\lambda_n-\sigma$ and let order them and μ_1 represents the largest and μ_n the smallest

$$\mu_1\geq\dots\geq\mu_n$$

(For any μ_i, $\exists k\in\{1,\dots,n\}$ s.t $\mu_i=\lambda_k-\sigma$).

Without shift if $|\lambda_1|>|\lambda_2|>\dots>|\lambda_n|$ then the convergence rate is determined by

$$\left|\frac{\lambda_2}{\lambda_1}\right|^k$$

If we choose σ suitable then we can make this rate smaller

$$\frac{\mu_1}{\mu_2}^k$$

and improve the rate of convergence.

Numerical Analysis Screening Exam Sept. 12, 2005

Problem 4 .Least Squares.

Let A be an mxn real matrix.

(a) Show that A is a linear operator mapping the real n-dimensional vector space, V^n, into the real m-dimensional vector space V^m. Both spaces have an inner-product $\langle u, v\rangle$. Show that the transpose of A is a linear operator A^* mapping V^m into V^n and satisfying the relation

$$\langle Ax, y\rangle = \langle x, A^*y\rangle \text{ for all } x \in V^n \text{ and } y \in V^m.$$

(b) Define the l_2 norms . Define the null space N_A of A and the range R_A. The orthogonal complement of N_A is the subspace $N_A^\perp = \{x : \langle x, u\rangle = 0 \text{ for all } u \in N_A\}$. Prove that A maps $N_A^\perp$ onto R_A in a 1:1 way, so that $\dim(N_A^\perp) = \text{rank}(A)$ and $V^n = N_A \oplus N_A^\perp$.

(c) Let $f(x) = |Ax - b|^2$, where $x \in V^n$ and $b \in V^m$.
Prove that f has a global minimum. (Hint: To minimize f(x) first find the vector in R(A) (= R_A) which is closest to b, where $b = b_{R(A)} + b_{R(A)}{}^\perp$. Then there exists x^- such that $Ax^- = b_{R(A)}$.) Is the minimum unique? Explain your answer.

(d) The *pseudoinverse* of A is a linear operator $A^\dagger$ on V^m to V^n satisfying the Moore-Penrose conditions :

(Gen) $A A^\dagger A = A$ and $A^\dagger A A^\dagger = A^\dagger$;
(Sym) $(A A^\dagger)^* = A A^\dagger$ and $(A^\dagger A)^* = A^\dagger A$.

Assume $A^\dagger$ exists. Show that $A^{\dagger\dagger} = A$.

It can be proved that $R(A^\dagger) = N(A)^\perp$ and $N(A^\dagger) = R(A)^\perp$ .Use this to show that $A A^\dagger = P_{R(A)}$, where P denotes the orthogonal projection. ?

Then prove the following Theorem:

Theorem . $A^\dagger b$ is a solution of the least squares problem, that is,

$$|A A^\dagger b - b| \le |Ax - b| \text{ for all } x \in V^n.$$

(Hint: $AA^\dagger b = b_{R(A)}$.)

(e) Describe a method for computing $A^\dagger$. (Hint: State the Singular Values Decomposition (SVD) Theorem or describe the QR method.)

a) If $A \cdot V^n \to V^m$ then A is $m \times n$ and A^* is $n \times m$. So, A^* is a map from V^m to V^n

Let us firstly show the equation is satisfied Let $x \in V^n$, $y \in V^m$ be orbitrary. Then,

$$\langle Ax, y\rangle = y^T A x = \sum_{i=1}^{m} y_i \cdot \Big(\sum_{j=1}^{n} a_{ij} x_j \Big) = \sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} x_j$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{m} y_i a_{ij} x_j = \sum_{j=1}^{n} x_j \underbrace{\sum_{i=1}^{m} a_{ij} y_i}_{(A^* y)_j} = x^T A^* y = \langle x, A^* y\rangle$$

Now, let $y_1, y_2 \in V^m$ be arbitrary and α be a scalar Then for any $x \in V^n$ by using inner product properties,

$$\langle x, A^*(\alpha y_1 + y_2)\rangle = \langle Ax, \alpha y_1 + y_2\rangle$$

$$= \alpha\langle Ax, y_1\rangle + \langle Ax, y_2\rangle$$

$$= \alpha\langle x, A^* y_1\rangle + \langle x, A^* y_2\rangle$$

So, A^* is a linear map

b) l_2 norm vector-2, matrix-2 norms? If so ✓

Let A be $m \times n$

$N(A) = \{x \in V^n : Ax = 0\}$,

$R(A) = \{y \in V^m : y = Ax \text{ for some } x \in V^n\}$

c) Let $b = b_{R(A)} + b_{R(A)^\perp}$ Then since $b_{R(A)} \in R(A)$, $Ax = b_{R(A)}$ has a solution, say $\tilde{x}$. Then for any other solution x,

$$\|Ax - b\|^2 = \|\underbrace{Ax - b_{R(A)}}_{\in R(A)} - b_{R(A)^\perp}\|^2$$

$$\geq \|Ax - b_{R(A)}\|^2 + \|b_{R(A)^\perp}\|^2$$

$$\geq \|b_{R(A)^\perp}\|^2$$

$$= \|A\tilde{x} - b_{R(A)} - b_{R(A)^\perp}\|^2 \quad \text{since} \quad A\tilde{x} - b_{R(A)} = 0$$

$$= \|A\tilde{x} - b\|^2$$

So, $\|A\tilde{x} - b\| \leq \|Ax - b\|$ where x is an arbitrary solution. Thus, $f(x)$ has a global minimum.

d) Suppose A^+ exists then, by definition, if we find a matrix X such that

$$A^+XA^+ = A^+ \quad \text{and} \quad XA^+X = X$$
$$(A^+X)^* = A^+X \quad \text{and} \quad (XA^+)^* = XA^+$$

then $A^{++} = X$. But since A^+ satisfies

$$AA^+A = A \quad \text{and} \quad A^+AA^+ = A^+$$
$$(AA^+)^* = AA^+ \quad \text{and} \quad (A^+A)^* = A^+A$$

we see that such an X exists. We also know Moore-Penrose inverse is unique. So, $X = A$, i.e. $A^{++} = A$.

$$\|\underbrace{AA^+b}_{= b_{R(A)}} - b_{R(A)} - b_{R(A)^\perp}\|^2 = \|b_{R(A)^\perp}\|^2 \quad \text{for any } x \in V^n$$
$$\leq \|\underbrace{Ax - b_{R(A)}}_{\in R(A)}\|^2 + \|b_{R(A)^\perp}\|^2$$
$$= \|Ax - b_{R(A)} - b_{R(A)^\perp}\|^2$$

$\|AA^+b - b\| \leq \|Ax - b\|$, for any $x \in V^n$.

e) Find SVD of A: $A = U\Sigma V^T$

Then we know $A^+ = V\Sigma^+U^T$, where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ 0 & & & 0 \end{bmatrix}_{m\times n} \Rightarrow \Sigma^+ = \begin{bmatrix} 1/\sigma_1 & & & \\ & \ddots & & 0 \\ & & 1/\sigma_r & \\ 0 & & & 0 \end{bmatrix}_{n\times m}$$