502a Qual F'12

1) a) $A = LL^t$ Cholesky decomp.

is unique under the condition that L is strictly positive & real on the diagonal.

b) $A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 13 & 8 \\ 1 & 8 & 14 \end{pmatrix} = \begin{array}{ccc} l_{11} & & \\ l_{21} & l_{22} & \\ l_{31} & l_{32} & l_{33} \end{array} \quad \begin{array}{ccc} l_{11} & l_{21} & l_{31} \\ & l_{22} & l_{32} \\ & & l_{33} \end{array}$

$a_{11} = l_{11}^2 \Rightarrow l_{11} = 1$

$a_{12} = l_{11} \cdot l_{21} \Rightarrow l_{21} = 2$

$a_{13} = l_{11} l_{31} \Rightarrow l_{31} = 1$

$a_{22} = l_{21}^2 + l_{22}^2 = 4 + l_{22}^2 \Rightarrow l_{22} = 3$

$a_{23} = l_{21} \cdot l_{31} + l_{22} \, l_{32}$

$= 2 \cdot 1 + 3 \cdot l_{32} \Rightarrow l_{32} = 2$

$a_{33} = l_{31}^2 + l_{32}^2 + l_{33}^2 = 1 + 4 + l_{33}^2$

$\Rightarrow$ ~~14~~ $l_{33} = 3 \rightarrow L = \begin{pmatrix} 1 & & \\ 2 & 3 & \\ 1 & 2 & 3 \end{pmatrix}$

2) $Nx_{k+1} = Px_k + b$

a) $N, P \in \mathbb{R}^{n \times n}$

$\Rightarrow x_{k+1} = Mx_k + N^{-1}b$, $M = N^{-1}P$.

$N_\alpha = (1+\alpha)N$, $P_\alpha = P + \alpha N$

$x_{k+1} = M_\alpha x_k + N_\alpha^{-1} b$. $M_\alpha = N_\alpha^{-1} P_\alpha$.

Let v be eigenvector of M to λ.

Then $M_\alpha v = N_\alpha^{-1} P_\alpha = ((1+\alpha)N)^{-1}(P+\alpha N)$

$= \frac{1}{1+\alpha} \underbrace{N^{-1}P}_{M} + \frac{\alpha}{1+\alpha} I$

$= \frac{\lambda}{1+\alpha} + \frac{\alpha}{1+\alpha} = \frac{\lambda_k + \alpha}{1+\alpha}$

b) $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n < 1$.

we have $\tilde{\lambda}_1 = \frac{\lambda_1 + \alpha}{1+\alpha} = \frac{\lambda_1}{1+\alpha} + \frac{\alpha}{1+\alpha}$

$= \frac{\lambda_1}{1+\alpha} + \frac{1}{\frac{1}{\alpha}+1}$

$\hookrightarrow \frac{\lambda_1 + \frac{1+\lambda_1}{2}}{1 + \frac{1+\lambda_1}{2}} = \frac{1+3\lambda_1}{3+\lambda_1}$

$$\frac{\lambda_k + \alpha}{1+\alpha} < 1$$

$\lambda_k + \alpha < 1 + \alpha \iff \lambda_k < 1$ ✓

$-\lambda_k - \alpha < 1 + \alpha$

$-\lambda_k < 2\alpha + 1$

$$\frac{-1 - \lambda_k}{2} < \alpha$$

3) Let v be associated to λ. Then $Av = \lambda v$

a) So $f(A)v = \sum a_k A^k v = \sum a_k \lambda^k v$

$= f(\lambda) v$ ✓

b) $a_{1i} \neq 0$, $\sum a_{ij} = 0$, $a_{ii} = \sum_{j \neq i} |a_{ij}|$

we have by Gershgorin

$\lambda_i \in \bigcup K_i$, $K_i = \{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum |a_{ij}| \}$

now clearly all eigenvalues are non-negative.

4)

$$Ax = (U_1\ U_2)\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}V_1^T\\ V_2^T\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = (U_1\ U_2)\begin{pmatrix}\Sigma V_1^T\\ \end{pmatrix}$$

$$= U_1 \Sigma V_1 x \stackrel{!}{=} b$$

$$= (U_1\ U_2)\begin{pmatrix}\Sigma & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}$$

$$\|Ax - b\| = \|$$

$$= (U_1\ U_2)\begin{pmatrix}\Sigma y_1\\ \end{pmatrix}$$

U is eigenvector of A^TA: $\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}$

$$\begin{pmatrix}1\\1\end{pmatrix}\ \begin{pmatrix}1\\-1\end{pmatrix} \to U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$\Sigma = \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix} \qquad V_1 = Au_1$$

$\begin{pmatrix}3\\3\end{pmatrix}$ 3 3

A symm $\Rightarrow$ orth. digonalisable

$$\Rightarrow A = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$= \cancel{\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 2\\ 1 & -1\end{pmatrix}}$$

$$\Rightarrow x = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{2}}\,3$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}2 & 2\\ 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix}3/4\\ 3/4\end{pmatrix} ✓$$

$$= \frac{1}{2}\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix} ✓$$

502a Qual S'13

1) Since $\hat{A}$ is SPD, we can always divide by $\hat{A}_{11}$ (which is positive by leading principal minors.
we can always divide by that

$$LU = U^T L^T$$

$$L = U^T L^T U^{-1}$$

$$\underbrace{(U^T)^{-1} L}_{L} = \underbrace{L^T U^{-1}}_{U} = D$$

$$\Rightarrow L^T U^{-1} = D \Rightarrow L^T = DU$$

$$\Rightarrow L = U^T D$$

$$A = LU = U^T D U = U^T D^{1/2} D^{1/2} U$$

2) $$\rho L + \rho^{-1} U = \begin{pmatrix} 0 & \rho^{-1}\beta & & \\ \rho\beta & 0 & \ddots & \\ & \ddots & & \\ & & \rho\beta & \rho^{-1}\beta \end{pmatrix}$$

$$\begin{pmatrix} 0 & -\frac{\beta}{\rho\alpha} & & & \\ -\frac{\beta\rho}{\alpha} & 0 & -\frac{\beta}{\rho\alpha} & & \\ 0 & & 0 & \ddots & \\ 0 & & & \ddots & -\frac{\beta}{\rho\alpha} \\ 0 & & & -\frac{\beta\rho}{\alpha} & 0 \end{pmatrix}$$

b)

Let λ be

$$(I - \omega L)B = (1-\omega)I + \omega U$$

then $\cancel{ee}$ $(I - \omega L)\lambda v = (1-\omega)v + \omega U v$

$$\lambda v - \omega\lambda L v = (1-\omega)v + \omega U v$$

$$(\omega\lambda L + \omega U)v = \lambda v + (\omega - 1)v$$

$$\omega\sqrt{\lambda}(L + U)v = (\lambda + \omega - 1)v$$

Householder matrix

3) ~~$I - 2ww^T$~~

a) $V = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$ ✓

then $V^T B V = \begin{pmatrix} \quad \end{pmatrix}$

$BV = V\begin{pmatrix} \quad \end{pmatrix}$

$B \cdot \begin{pmatrix} V_1 & V_2 & V_3 \end{pmatrix} =$

b) $A = \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 1 \\ -2 & 0 & 3 \end{pmatrix}$

$A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow ev$

$V^T A V = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 & -2 & 2 \\ -1 & 1 & 1 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$

$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 3/\sqrt{2} & -5/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{2} & -2/\sqrt{6} \\ 1/\sqrt{3} & -2/\sqrt{2} & -8/\sqrt{6} \end{pmatrix}$

$= \begin{pmatrix} 1 & & \\ 0 & * & \\ 0 & & \end{pmatrix}$ ✓

c) $V^T A V = \begin{pmatrix} * & * & * \\ 0 & a & b \\ 0 & c & d \end{pmatrix}$ $\to$ first step is done

Then do $V_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{eigen} & \\ 0 & \text{vector} & \text{ordinary} \end{pmatrix}$

find ev of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$V_2^T V^T A V V_2 = V_2^T \begin{pmatrix} * & * & * \\ 0 & a & b \\ 0 & c & d \end{pmatrix} V_2$$

4) $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$

$\|Ax^* - b\| \le \|Ax - b\|$ for all $x \in \mathbb{R}^n$.
Show: x^* is min. norm solution $\iff$ x^* in range of A^*.

~~Let x^* be min norm solution.~~
~~Then $A^* A x^* = A^* b$~~, i.e.
we know by Fredholm alternative.
range $A^* = (\ker(A^*))^\perp$

want $(Ax^* - b) \perp$ range A

i.e. $(Ax^* - b) \in \ker(A^*)$

4) x^* is min norm sol. Then
$x^* = A^+ b$ where $A^+ = V\Sigma^{-1}U^*$, $A = U\Sigma V^*$.

$$A^* = (U\Sigma V^*)^* = V\Sigma^* U^*$$

$$\Rightarrow \quad x^* = V\Sigma^{-1}U^* b$$

$$\begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \begin{pmatrix} & u_1 & \\ & u_2 & \\ & u_3 & \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ 2u_2 \\ 3u_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

502a   Qual F'13

1) $x \in \mathbb{R}^n$, $M \in \mathbb{R}^{n \times n}$, S nonsingular, $\|x\|_S = \|Sx\|$

a) $\|x\| = 0 \Leftrightarrow x = 0$ $\quad \rightarrow \|x\|_S = 0 \Rightarrow \|Sx\| = 0 \Rightarrow Sx = 0$
$\|ax\| = |a|\,\|x\|$ $\quad \underset{\text{nons.}}{\Rightarrow} x = 0$
$\|x+y\| \leq \|x\| + \|y\|$ $\quad x = 0 \Rightarrow Sx = 0 \Rightarrow \|x\|_S = 0.$

$$\|ax\|_S = \|Sax\| = |a|\,\|Sx\| = |a|\,\|x\|_S \checkmark$$

$$\|x+y\|_S = \|Sx + Sy\| \leq \|x\|_S + \|y\|_S \checkmark$$

b) $\|x\|_S = \|Sx\| \leq \|S\|\,\|x\|$ where S is nonsingular $\checkmark$

on the other hand $\|x\| = \|SS^{-1}x\| = \|S^{-1}x\|_S$
$\leq \|S^{-1}\|_S \|x\|_S$ $\checkmark$

c) $\|M\|_S = \max_{\|x\|_S = 1} \|Mx\|_S = \max_{\|Sx\|=1} \|SMx\|$

$Sx = y \Rightarrow x = S^{-1}y$

$= \max_{\|y\|=1} \|SMS^{-1}y\|$ ✓

d) $k_S(M) \leq k(S)^2 k(M)$

$k_S(M) = \|M\|_S \|M^{-1}\|_S$

$= \|SMS^{-1}\| \|SM^{-1}S^{-1}\|_S$

$\leq \|S\| \|S^{-1}\| \|M\| \|S\| \|M^{-1}\| \|S^{-1}\|$

$= k(S)^2 k(M)$ ✓

2) a) $(0,1), (1,1), (-1,-1), (2,0)$

$y = a + bx^2$

$$\Rightarrow \begin{aligned} 1 &= a + b \cdot 0^2 \\ 1 &= a + b \cdot 1^2 \\ -1 &= a + b \cdot (-1)^2 \\ 0 &= a + b \cdot 2^2 \end{aligned} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 18 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ ✓

$$3 \times 2$$

b) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad Ax = b,$

SVD: $A^T A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$

$(2-\lambda)(6-\lambda) - 9 = 12 - 9 + \lambda^2 - 8\lambda$

$= \lambda^2 - 8\lambda + 3 \longrightarrow 4 \pm \sqrt{16-\cancel{3}3}$

$4 \pm \sqrt{13}$

$A = QR$

3×3

6

502a Qual F'13

2b) find QR: $\longrightarrow$ rank 2

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} \diagdown \\ 0 \; 0 \end{pmatrix} = \begin{pmatrix} q_1 & q_2 \end{pmatrix} \begin{pmatrix} \diagdown \end{pmatrix}$$

$$= \begin{pmatrix} r_{11} q_1 & r_{12} q_1 + r_{22} q_2 \end{pmatrix}$$

$$\Rightarrow q_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \Rightarrow r_{11} = \sqrt{2}.$$

$$\underbrace{r_{22} q_2}_{u_2} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - r_{12} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$\cancel{\begin{matrix} 1 & \\ 1 & \rightarrow \\ 1 & \end{matrix} \begin{matrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{matrix}} \qquad \begin{pmatrix} 2 - r_{12} \frac{1}{\sqrt{2}} \\ 1 \\ 1 - r_{12} \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = 0$$

$$= \frac{2}{\sqrt{2}} - \frac{r_{12}}{2} + \frac{1}{\sqrt{2}} - \frac{r_{12}}{2} = 0 \Rightarrow$$

$$\sqrt{2} - r_{12} + \frac{1}{\sqrt{2}} = 0 \Rightarrow r_{12} = \sqrt{2} + \frac{1}{\sqrt{2}}$$

$$r_{12} = \sqrt{2} + \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}.$$

$$\Rightarrow u_2 = \begin{pmatrix} 2 - \frac{3}{2} \\ 1 \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1 \\ -1/2 \end{pmatrix} \qquad r_{22} = \sqrt{\frac{3}{2}} \qquad \frac{1}{\sqrt{4}} \quad \frac{\sqrt{2}}{\sqrt{3}}$$

$$\Rightarrow \quad Q = \begin{pmatrix} 1/\sqrt{2} & \cancel{\sqrt{3/8}}\ 1/\sqrt{6} \\ 0 & \sqrt{2/3} \\ 1/\sqrt{2} & -\cancel{\sqrt{3/8}}\ -1/\sqrt{6} \end{pmatrix}, \quad R = \begin{pmatrix} \sqrt{2} & 3/\sqrt{2} \\ 0 & \sqrt{3/2} \end{pmatrix}$$

c) $b \in \text{range}(A)$, i.e. $b = a\begin{pmatrix}1\\0\\1\end{pmatrix} + b\begin{pmatrix}2\\1\\1\end{pmatrix}$.

since $\begin{pmatrix}1\\0\\1\end{pmatrix}$ $\begin{pmatrix}2\\1\\1\end{pmatrix}$ are lin. ind the solution is unique if it exists.

d) Find $A^*A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 6 \end{pmatrix}$

$(2-\lambda)(6-\lambda) - 9 = \lambda^2 - 8\lambda - \overset{+12}{9} = 0$

$4 \pm \sqrt{16-3} = 4 \pm \sqrt{13}$

$\Rightarrow$ c) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ -1 \end{pmatrix}$ $\quad x_2 = -2$, $\quad x_1 = 1$

3) $x_{k+1} = Mx_k + c$

a) Show: If $\lim x_k = x^*$ for any $x_0 \in \mathbb{C}^n$ then $r(M) < 1$.

Assume $r(M) \geq 1$. Then $\exists$ v s.t. $Mv = \lambda v$, $|\lambda| \geq 1$.

Then for $x_0 = v$ we get

$x_{k+1} = Mx_k$ ~~[illegible]~~ $+ c$

$x_1 = Mx_0 + c = \lambda v + c$

$x_2 = M(\lambda v + c) = \lambda^2 v + Mc$

does not converge.

b) $Ax = b$, lower triangular, including diag.

$A =$ ~~[illegible]~~ $X - Y \Rightarrow M = X^{-1}Y$.

$c =$ ~~[illegible]~~ $X^{-1}b$,

$(X - Y)x = b$

$Xx - Yx = b \Rightarrow x = \underbrace{X^{-1}Y}_{M^{-1}Y}x + M^{-1}b$ ✓

c) $A = L + D + U$

$$= \underbrace{L + \frac{1}{\Theta} D}_{M} + \underbrace{\left(1 - \frac{1}{\Theta}\right) D + U}_{N}$$

$$\Rightarrow \underbrace{\left(L + \frac{1}{\Theta} D\right)}_{M} - \underbrace{\left(\left(\frac{1}{\Theta} - 1\right) D - U\right)}_{N}$$

$$\Rightarrow \left(\frac{1}{\Theta} D + L\right)^{-1} \left(\left(\frac{1}{\Theta} - 1\right) D - U\right)$$

$$= -\left(\frac{1}{\Theta} D + L\right)^{-1} \left(U + \left(1 - \frac{1}{\Theta}\right) D\right)$$

$$c = \left(\frac{1}{\Theta} D + L\right)^{-1} b$$

d) $\lim x_k = x^*$ for any $x_0 \in \mathbb{C}^n \Rightarrow$

we need $\Theta \in (0,2)$ to have $r(M) < 1$, which is necessary for $\lim x_k = x^*$.

$\hookrightarrow \det M_\omega = \det($

$\det M^{-1} = \prod \frac{\Theta}{a_{ii}}$ and $\det N = \prod \left(1 - \frac{1}{\Theta}\right) a_{ii}$

$\Rightarrow \det M_\omega = (\Theta - 1)^n$

$\Rightarrow \Theta \in (0,2)$ to guarantee $r(M) < 1$

4) $x_0 = \sum \alpha_i u_i$, $\alpha_1 \neq 0$, $|\lambda_1| > |\lambda_2| \geq \ldots$

$x_{k+1} = A x_k$, $k = 0, 1, 2, \ldots$

a) $v \in \mathbb{C}^n$ fixed not orthogonal to u_1.

$q_k = \frac{v^T x_{k+1}}{v^T x_k}$ converges to λ_1 as $k \to \infty$.

we have $q_k = \frac{v^T A x_k}{v^T \cancel{A} x_k} = \frac{v^T A^{k+1} x_0}{v^T A^{k} x_0}$

$= \frac{v^T \sum \alpha_i \lambda_i^{k+1} u_i}{v^T \sum \alpha_i \lambda_i^{k} u_i} = \frac{v^T \lambda_1^{k+1} \left(\alpha_1 u_1 + \sum_{i=2}^{n} \alpha_i u_i \left(\frac{\lambda_i}{\lambda_1}\right)^{k+1}\right)}{v^T \lambda_1^{k} \left(\alpha_1 u_1 + \sum_{i=2}^{n} \alpha_i u_i \left(\frac{\lambda_i}{\lambda_1}\right)^{k}\right)}$

$\to \lambda_1 \frac{v^T \alpha_1 u_1}{v^T \alpha_1 u_1} = \lambda_1$ ✓

b) $|\lambda_2| > |\lambda_3|$ v is orth to u_1 but not to u_2

$\alpha_2 \neq 0$

$q_k = \frac{v^T \sum_{i=2} \alpha_i \lambda_i^{k+1} u_i}{v^T \sum_{i=2} \alpha_i \lambda_i^{k} u_i} = \lambda_2^{k+1} v^T \sum$

as before.

c) $\lim_{k\to\infty} (q_k - \lambda_1)(\lambda_1/\lambda_2)^k = C$ for some constant C

$$\left(\frac{v^T \sum \alpha_i \lambda_i^{k+1} u_i}{v^T \sum \alpha_i \lambda_i^k u_i} - \lambda_1 \right) \left(\left(\frac{\lambda_1}{\lambda_2} \right)^k \right)$$

$$= \frac{v^T \lambda_1^{k+1} \left(\alpha_1 u_1 + \sum \left(\frac{\lambda_i}{\lambda_1}\right)^{k+1} u_i\right)}{v^T \lambda_1^k \left(\alpha_1 u_1 + \sum \left(\frac{\lambda_i}{\lambda_1}\right)^k u_i\right)} - \lambda_1$$

$$= \lambda_1 \left(\frac{v^T \left(\alpha_1 u_1 + \sum \left(\frac{\lambda_i}{\lambda_1}\right)^{k+1} u_i\right)}{v^T \left(\alpha_1 u_1 + \underbrace{\sum \left(\frac{\lambda_i}{\lambda_k}\right)^k u_i}_{\to 0}\right)} - 1 \right) \left(\frac{\lambda_1}{\lambda_2} \right)^k$$

$$= \lambda_1 \left(\underbrace{\frac{v^T \alpha_1 u_1}{v^T \alpha_1 u_1} - 1}_{0} + \frac{v^T \sum \left(\frac{\lambda_i}{\lambda_1}\right)^{k+1} \left(\frac{\lambda_1}{\lambda_2}\right)^k}{v^T \alpha_1 u_1} \right)$$

$$\to \lambda_1 \left(\frac{v^T \frac{\lambda_2}{\lambda_1}}{v^T \alpha_1 u_1} \right) = C \quad \checkmark$$

502a Qual S'14

1) a) $\{f_k\}_{k=1}^n$ n lin. ind. functions in $L^2(a,b)$,

$Q_{ij} = \int_a^b f_i(x) f_j(x) dx$.

symmetry is obvious

pos def. let x be $(x_1, \dots, x_n)$.

Then $x^T Q x = \sum_j x_j \sum_i q_{ji} x_i$

$= \sum_j \sum_i x_j x_i q_{ji} = \sum_j \sum_i \int_a^b f_i(x) f_j(x) dx \; x_i x_j$.

$= \int_a^b \sum_j \sum_i f_i(x) f_j(x) x_i x_j \; dx$

$= \int_a^b \left(\sum_i x_i f_i\right)^2 \geq 0$ ✓.

b) g real-valued in $L_2(a,b)$, find best approx in span $\{f_k\}$

want: $\|g - f\|_{L^2}^2$ be min, i.e.

$\int (g-f)^2 dx = \int \left(g - \sum \alpha_i f_i\right)^2 dx$

$= \int g^2 - 2\int g \sum \alpha_i f_i + \int \left(\sum \alpha_i f_i\right)^2$

$= \int g^2 - 2 \sum \alpha_i \int g f_i + \sum \sum \alpha_i \alpha_j \int f_i f_j$

$$= \int g^2 - 2\sum \alpha_i \int f_i g + \sum\sum \alpha_i \alpha_j \, q_{ij}$$

$$= \int g^2 - 2\sum \alpha_i \int f_i g + \alpha^T Q \alpha = p(\vec{\alpha})$$

$$0 = -2 \int f_i g + 2Q\alpha$$

$x^T A x$

$2Ax$

$$\begin{pmatrix} \int f_1 g \\ \vdots \\ \int f_n g \end{pmatrix} = Q\alpha \to \text{solve}$$

2)

$$\begin{pmatrix} 1 & 0 & 0 \\ & \alpha_3 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & u_1 & u_2 \\ & 1 & u_3 \\ & & 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ \alpha_1 & \frac{1}{\alpha_3} & 0 \\ \alpha_2 & \alpha_4 & 1 \end{pmatrix}}_{\tilde{L}} \begin{pmatrix} A \end{pmatrix} = \begin{pmatrix} 1 & u_1 & u_2 \\ & 1 & u_3 \\ & & 1 \end{pmatrix}$$

$$A = \tilde{L}^{-1} U = LU$$

L

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha_1 & \frac{1}{\alpha_3} & 0 \\ \alpha_2 & \alpha_4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\alpha_1\alpha_3 & \alpha_3 & 0 \\ \alpha_1\alpha_3\alpha_4 - \alpha_2 & -\alpha_3\alpha_4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\alpha_2 - \alpha_1\alpha_3\alpha_4$

b) $Ax = b$

$\Leftrightarrow LUx = b \Leftrightarrow Ly = b$

$$y = L^{-1} b = \begin{pmatrix} 1 & 0 & 0 \\ \alpha_1 & \frac{1}{\alpha_3} & 0 \\ \alpha_2 & \alpha_4 & 1 \end{pmatrix} b = \begin{pmatrix} b_1 \\ \alpha_1 b_1 + \frac{1}{\alpha_3} b_2 \\ \alpha_2 b_1 + \alpha_4 b_2 + b_3 \end{pmatrix}$$

Then $y = Ux \Rightarrow$ ~~x~~

i.e. $$\begin{pmatrix} b_1 \\ \alpha_1 b_1 + \frac{1}{\alpha_3} b_2 \\ \alpha_2 b_1 + \alpha_4 b_2 + b_3 \end{pmatrix} = \begin{pmatrix} x_1 + u_1 x_2 + u_2 x_3 \\ x_2 + u_3 x_3 \\ x_3 \end{pmatrix}$$ ✓

3) $$B_J = M^{-1} N = \begin{pmatrix} 1/2 & & & \\ & 1/2 & & \\ & & \ddots & \\ & & & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & & 0 & \\ & & & 1 & 0 \end{pmatrix}$$

then $B_J x^k = \frac{1}{2} \sin\left(\frac{2\pi k}{n}\right) \quad \sin\left(\frac{\pi k}{n}\right) \cos\left(\frac{\pi k}{n}\right)$

$\frac{1}{2} \sin\left(\frac{\pi k}{n}\right) + \frac{1}{2} \sin\left(\frac{3\pi k}{n}\right)$

$\frac{1}{2} \left(\sin\left(\frac{\pi k}{n}\right) + \sin\left(\frac{3\pi k}{n}\right) \right)$

$= \frac{1}{2} \left(\sin\left(\frac{\pi k}{n}\right) + \sin\left(\frac{2\pi k}{n} + \frac{\pi k}{n}\right) \right)$

$= \frac{1}{2} \left(\sin\left(\frac{\pi k}{n}\right) + \sin\left(\frac{2\pi k}{n}\right) \cos\left(\frac{\pi k}{n}\right) + \sin\left(\frac{\pi k}{n}\right) \cos\left(\frac{2\pi k}{n}\right) \right)$

$=$

b) we have eigenvalues $\cos\left(\frac{\pi k}{n}\right)$, $k = 1, \dots, n-1$

so all eigenvalues are strictly between

-1 & 1 $\Rightarrow$ $r_\sigma(B_j) < 1$

$\Rightarrow$ convergence.

c) $$\alpha L + \alpha^{-1} U = \begin{pmatrix} 0 & \frac{1}{\alpha} & & & & \\ \alpha & 0 & \frac{1}{\alpha} & & & \\ & \alpha & 0 & \frac{1}{\alpha} & & \\ & & \alpha & 0 & \frac{1}{\alpha} & \\ & & & \alpha & 0 & \frac{1}{\alpha} \\ & & & & \alpha & 0 \end{pmatrix}$$

find eigenvalues:

$$\det(\alpha L + \alpha^{-1} U - \lambda I) = \begin{pmatrix} -\lambda & \frac{1}{\alpha} & & \\ \alpha & -\lambda & \frac{1}{\alpha} & \\ 0 & \alpha & -\lambda & \frac{1}{\alpha} \end{pmatrix}$$

$$= -\lambda \, D_{n-1} - \alpha \det\begin{pmatrix} \frac{1}{\alpha} & 0 & 0 & 0 \\ \alpha & -\lambda & \frac{1}{\alpha} & \\ 0 & \alpha & -\lambda & \frac{1}{\alpha} \end{pmatrix}$$

$$= -\lambda D_{n-1} - \alpha\left(\frac{1}{\alpha} D_{n-2}\right) = -\lambda D_{n-1} - D_{n-2}$$

$$\Rightarrow D_n + \lambda D_{n-1} + D_{n-2} = 0 \leftarrow \text{ is independent of } \alpha!$$

$$p^2 + \lambda p + 1 = 0$$

$p_{1,2} = -\frac{\lambda}{2} \pm \sqrt{\frac{\lambda^2}{4} - 1}$ satisfies rec. relation.

3) laudi: $\begin{pmatrix} 1/2 & & \\ & \ddots & \\ & & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & & 1 \\ & & & 1 & \end{pmatrix}$

$$= \begin{pmatrix} 0 & \frac{1}{2} & & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sin\left(\frac{\pi k}{n}\right) \\ \sin\left(\frac{2\pi k}{n}\right) \\ \vdots \\ \sin\left(\frac{\pi(n-1)k}{n}\right) \end{pmatrix}$$

$$\frac{1}{2}\sin\left(\frac{\pi(i-1)k}{n}\right) + \frac{1}{2}\sin\left(\frac{\pi(i+1)k}{n}\right) = c \cdot \sin\left(\frac{\pi i k}{n}\right)$$

$$\frac{1}{2}\sin\left(\frac{\pi k i}{n} - \frac{\pi k}{n}\right) + \frac{1}{2}\sin\left(\frac{\pi k i}{n} + \frac{\pi k}{n}\right)$$

$$= \frac{1}{2}\sin\left(\frac{\pi k i}{n}\right)\cos\left(\frac{\pi k}{n}\right) \mp \cancel{\frac{1}{2}\sin\left(\frac{\pi k}{n}\right)\cos\left(\frac{\pi k i}{n}\right)}$$

$$+ \frac{1}{2}\sin\left(\frac{\pi k i}{n}\right)\cos\left(\frac{\pi k}{n}\right) + \cancel{\frac{1}{2}\sin\left(\frac{\pi k}{n}\right)\cos\left(\frac{\pi k i}{n}\right)}$$

c) $\det(B_j - \lambda) = \begin{vmatrix} -\lambda & \frac{1}{2} & \\ \frac{1}{2} & -\lambda & \frac{1}{2} \\ & \frac{1}{2} & -\lambda \end{vmatrix}$

$$-\lambda D_{n-1} - \frac{1}{2} \begin{vmatrix} \frac{1}{2} & & \\ \frac{1}{2} & -\lambda & \frac{1}{2} \\ & & \end{vmatrix} = -\lambda D_{n-1} - \frac{1}{4} D_{n-2}$$

$\Rightarrow$ no α prout.

d) $\det(\lambda I - H_\omega) = 0$

$= \det($

$0 = \det(I - \omega L)\det(\lambda I - H_\omega)$

$= \det(\lambda I - \cancel{H_\omega} - \omega\lambda L - (1-\omega)I \mp \omega U)$

$= \det((\lambda - 1 + \omega)I - \omega\lambda L \mp \omega U)$

$= \det\left(\frac{\lambda - 1 + \omega}{\omega} - \lambda L \mp U\right)$

$= \det\left(\frac{\lambda - 1 + \omega}{\lambda^{1/2}\omega} - \lambda^{1/2} L - \lambda^{-1/2} U\right)$

$\Rightarrow \frac{\lambda - 1 + \omega}{\lambda^{1/2}\omega}$ is ev. of $L + U = B_J$

$\mu = \frac{\lambda - 1 + \omega}{\lambda^{1/2}\omega} \qquad \sqrt{\lambda}\,\omega\mu = \lambda - 1 + \omega$

$\lambda\omega^2\mu^2 = \lambda^2 - 2\lambda - 2\omega + 2\lambda\omega + 1 + \omega^2$

$\lambda^2 + \lambda(2\omega - \omega^2\mu^2 - 2) + (\omega - 1)^2 = 0$ ✓

$H(1) = (I - L)^{-1} U$

$=$

4) $A^TA = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 3 & 10 \end{pmatrix}$

$\begin{pmatrix} 9-\lambda & 3 \\ 3 & 10-\lambda \end{pmatrix} = (9-\lambda)(10-\lambda) - 9$

$= 90 + \lambda^2 \cancel{+} 19\lambda - 9$

$= \cancel{8}\, \lambda^2 - 19\lambda + 81$

$9.5 \pm \sqrt{\left(\frac{19}{2}\right)^2 - 81} = 9.5 \pm \sqrt{\frac{361}{4} - \frac{324}{4}}$

$= 9.5 \pm \sqrt{\frac{37}{4}} = \frac{19}{2} \pm \frac{\sqrt{37}}{2}$.

b) $\min \sigma_k \leq \min |\lambda_k|$

$\|A\|_2 = \max \sigma_k$

$\max \sigma_k = \|A\|_2 = \max_{\|x\|_2 = 1} \|Ax\|_2 \geq \|Av\|_2$

$= |\lambda_{max} v| = |\lambda_{max}| \, \|v\| = |\lambda_{max}|$

also A^{-1} has eigenvalues $\frac{1}{\lambda}$ and $\frac{1}{\sigma_k}$

Then $\max\left(\frac{1}{\sigma_k}\right) \geq \max \left|\frac{1}{\lambda}\right|$

$\Rightarrow \sigma_{min} \leq |\lambda_{min}|$

4) $A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$

$$\begin{array}{r} 190 \\ 90 \\ 81 \\ \hline \end{array}$$

$A^TA = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 3 & 10 \end{pmatrix}$

$(9-\lambda)(10-\lambda) - 9 = 90 - 9 - 19\lambda + \lambda^2$

$= \lambda^2 - 19\lambda + 81$

$9.5 \pm \sqrt{(9.5)^2 - 81} = \frac{19}{2} \pm \sqrt{\frac{361}{4} - \frac{324}{4}}$

$= \frac{19}{2} \pm \sqrt{\frac{37}{4}} \rightarrow \sigma_1 = \sqrt{\frac{19}{2} + \sqrt{\frac{37}{4}}} = \sqrt{\frac{19+\sqrt{37}}{2}}$

$\sigma_2 = \sqrt{\frac{19-\sqrt{37}}{2}}$ and

$u_1 = 9 -$

b) $\min_k \sigma_k \leq \min_k |\lambda_k|, \max_k \sigma_k \geq \max_k |\lambda_k|$

we have $\min \sigma_k = \min_k \sqrt{|\lambda(A^TA)|}$

c) 1) $A(A^*A)^{-1}A^*$

we have $A^*A = (U\Sigma V^*)^* U\Sigma V^*$

$= V\Sigma^* U^* U \Sigma V^* = V\Sigma^2 V^*$

$\Rightarrow (A^*A)^{-1} = (V\Sigma^2 V^*)^{-1} = V(\Sigma^2)^{-1}V^*$

$= U\Sigma \underbrace{V^* V}_{I} (\Sigma^2)^{-1} \underbrace{V^* V}_{I} \Sigma^* U^* = U\Sigma(\Sigma^*\Sigma)^{-1}\Sigma^* U^*$ ✓

d) $H(\omega) = (I - \omega L)^{-1}((1-\omega)I + \omega U)$

satisfies $\lambda^2 - 2(1-\omega)\lambda - \mu^2\omega^2\lambda + (1-\omega)^2 = 0$

502a Qual F'14

1) $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$

$A, D - CA^{-1}B$ nonsingular

show $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is nonsingular and find formula for inverse

a) M^{-1}: $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X & Z \\ Y & W \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Now $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$A \cdot X + BY = I$$
$$CX + DY = 0$$

$$AZ + BW = 0$$
$$CZ + DW = I$$

2) $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $m \leq n$ $\rightarrow$ ( )

rank $A = m$ (full row rank)

$\min \|x - x_0\|$ s.t. $Ax = b$ where

sol given by $x^* = A^T(AA^T)^{-1}b + (I_n - A^T(AA^T)^{-1}A)x_0$

since A has full row rank its pseudoinverse is given by $A^T(AA^T)^{-1}$, this solves $Ax = b$.

Now we need to add something from kernel

$A(I_n - A^T(AA^T)^{-1}A)x_0$

$= Ax_0 - Ax_0 = 0 \Rightarrow$ kernel, so $Ax=b$ is satisfied.

Now $\|x - x_0\|^2 =$

$x^* = A^T(AA^T)^{-1}b + \text{ker}$

$\| A^T(AA^T)^{-1}b + \text{ker} - x_0 \|$

$\lambda = (AA^T)^{-1}(2Ax_0 - 2b)$

$\Rightarrow x^* = x_0 - A^T((AA^T)^{-1})(Ax_0 - b)$

$= A^T(AA^T)^{-1}b + (I_n - A^T(AA^T)^{-1}A)x_0$ ✓

$(x - x_0)^T(x - x_0) + \lambda^T(Ax - b)$

$\nabla_x L = 2x - 2x_0 + A^T\lambda = 0 \Rightarrow x = x_0 - A^T\frac{\lambda}{2}$

$\nabla_\lambda L = Ax - b = 0 \Rightarrow A(x_0 - A^T\frac{\lambda}{2}) = b \Leftrightarrow AA^T\lambda = 2Ax_0 - 2b$

3) $A = \begin{pmatrix} -2 & \frac{1}{2} \\ -\frac{1}{2} & -2 \end{pmatrix}$

$x^{k+1} = x^k - \omega(Ax^k - b)$ converges.

$= (I - \omega A)x^k + \omega b$

$x^{k+1} - x^* = (I-\omega A)x^k + \omega b - (x^* - \omega(Ax^* - b))$

$= (I-\omega A)x^k + \cancel{\omega b} - x^* + \omega A x^* - \cancel{\omega b}$

$= (I-\omega A)x^k - (I-\omega A)x^*$

$= (I-\omega A)(x^k - x^*)$

$\Rightarrow \rho(I-\omega A) < 1$ guarantees convergence.

$I - \omega A = \begin{pmatrix} 1+2\omega & -\frac{\omega}{2} \\ \frac{\omega}{2} & 1+2\omega \end{pmatrix}$ $(2\omega+1) \pm \omega(0.5i)$

find EV:

$(1+2\omega-\lambda)^2 + \frac{\omega^2}{4} = 0$

$1 + \underline{4\omega^2} + \underline{\lambda^2} + 4\omega - \underline{4\omega\lambda} - \underline{2\lambda} + \underline{\frac{\omega^2}{4}} = 0$

$\lambda^2 - (4\omega+2)\lambda + \frac{17}{4}\omega^2 + 4\omega + 1 = 0$

$(2\omega+1) \pm \sqrt{4\omega^2 + \cancel{4\omega} + \cancel{1} - \frac{17}{4}\omega^2 - \cancel{4\omega} - \cancel{1}} = (2\omega+1) \pm \sqrt{-\frac{\omega^2}{4}}$

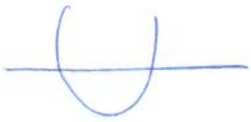
$\lambda_1 = (2w+1) + 0.5wi$

$\lambda_2 = 2w+1 - 0.5wi$

$$\|\lambda_1\|^2 = (2w+1)^2 + (0.5w)^2$$

$$4w^2 + 4w + 1 + \tfrac{1}{4}w^2 = \tfrac{17}{4}w^2 + 4w + 1 < 1$$

$$\Leftrightarrow \tfrac{17}{4}w^2 + 4w < 0$$



$$w = 0 \quad \text{or} \quad \tfrac{17}{4}w + 4 = 0$$

$$w = -\tfrac{16}{17}$$

$\Rightarrow w \in (-\tfrac{16}{17}, 0)$ gives convergence

min $\tfrac{17}{4}w^2 + 4w + 1$ $\qquad$ $\tfrac{17}{2}w + 4 = 0$

$17w = -8 \quad w = -\tfrac{8}{17}$

gives best value for w.

4) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1+\varepsilon \end{pmatrix}$ done in hw.

has ev $1, 1+\varepsilon$ with ev $\qquad$ Then $AT = TD$

$T^{-1}AT = D$ ✓

$$\begin{pmatrix} 0 & 1 \\ 0 & \varepsilon \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -\varepsilon & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$$

$$\to T = \begin{pmatrix} 1 & 1 \\ 0 & \varepsilon \end{pmatrix} \Rightarrow T^{-1} = \frac{1}{\varepsilon}\begin{pmatrix} \varepsilon & -1 \\ 0 & 1 \end{pmatrix}$$

$$T^T T = \begin{pmatrix} 1 & 0 \\ 1 & \varepsilon \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & \varepsilon \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1+\varepsilon^2 \end{pmatrix} \to .$$

<u>502 a</u> Qual F'15

1) $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon+\varepsilon^2 \end{pmatrix}$

a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is good. $\begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon \end{pmatrix} - \frac{(a_1, a_2)}{\|a_1\|} a_1$

$= \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon \end{pmatrix} - \frac{1}{1} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \end{pmatrix}$ hat norm $\sqrt{2}\,\varepsilon$

$\Rightarrow \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = q_2$

$\begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon+\varepsilon^2 \end{pmatrix} - \frac{(a_1, a_3)}{\|a_1\|} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{(a_2, a_3)}{\|a_2\|} \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon \end{pmatrix}$

$\begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon+\varepsilon^2 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{\overbrace{1+\varepsilon^2+\varepsilon^2+\varepsilon^3}^{1+2\varepsilon^2+\varepsilon^3}}{\sqrt{1+2\varepsilon^2}} \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon \end{pmatrix}$

$= \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon+\varepsilon^2 \end{pmatrix} - \begin{pmatrix} \frac{1+2\varepsilon^2+\varepsilon^3}{\sqrt{1+2\varepsilon^2}} \\ \frac{\varepsilon+2\varepsilon^3+\varepsilon^4}{\sqrt{1+2\varepsilon^2}} \\ \frac{\varepsilon+2\varepsilon^3+\varepsilon^4}{\sqrt{1+2\varepsilon^2}} \end{pmatrix} = q_3$

$q_3 = a_3 - \frac{(a_2, a_3)}{\|a_2\|} a_2 - \frac{(a_3, a_1)}{\|a_1\|} a_1$

$A = \begin{pmatrix} 1 & 0 & \\ 0 & 1/\sqrt{2} & q_3 \\ 0 & 1/\sqrt{2} & \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & \sqrt{2}\,\varepsilon & 1+\varepsilon \\ 0 & 0 & \|q_3\| \end{pmatrix}$

b) $H_w = I - 2ww^T$, $w = \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix} - \sqrt{2}\varepsilon \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} \varepsilon(1-\sqrt{2}) \\ \varepsilon \end{pmatrix}$ $\quad \|w\| = \sqrt{\varepsilon^2(1 - 2\sqrt{2} + 2) + \varepsilon^2}$

$= \sqrt{4\varepsilon^2 - 2\sqrt{2}\varepsilon^2}$

$= \varepsilon\sqrt{4 - 2\sqrt{2}}$

$H_w = I$

c) no higher powers of ε ~~...~~ using Householder
$\rightarrow$ more accurate factorization.

d) is not

2) $A \in \mathbb{R}^{n \times n}$ spd, $b \in \mathbb{R}^n$

$x_{n+\frac{1}{2}} = x_n + \omega_1(b - Ax_n)$

$x_{n+1} = x_{n+\frac{1}{2}} + \omega_2(b - Ax_{n+\frac{1}{2}})$

$\Rightarrow x_{n+1} = x_n + \omega_1(b - Ax_n) + \omega_2(b - A(x_n + \omega_1(b - Ax_n)))$

$= x_n - \omega_1 A x_n + \omega_1 b + \omega_2 b - \omega_2 A x_n$

$+ \omega_1 \omega_2 A (A x_n) - \omega_1 \omega_2 A b$

$= (I - \omega_1 A - \omega_2 A + \omega_1 \omega_2 A^2) x_n + (\omega_1 + \omega_2 - \omega_1 \omega_2 A) b$

$= (I - (\omega_1 + \omega_2) A + \omega_1 \omega_2 A^2) x_n + (\omega_1 + \omega_2 - \omega_1 \omega_2 A) b$

Then ~~x_{n+1}~~

$$x - x_{n+1} = \left(I - (\omega_1+\omega_2)A + \omega_1\omega_2 A^2\right)x$$
$$- \left(I - (\omega_1+\omega_2)A + \omega_1\omega_2 A^2\right)x_n$$

$$\Rightarrow e_{n+1} = \left(I - (\omega_1+\omega_2)A + \omega_1\omega_2 A^2\right)e_n .$$

b) Let λ be eigenvalue of A. Then

$$Kv = v - (\omega_1+\omega_2)\lambda v + \omega_1\omega_2\lambda^2 v$$

$\Rightarrow$ $1 - (\omega_1+\omega_2)\lambda_i + \omega_1\omega_2\lambda_i^2$ are eigenvalues of K.

c) Let λ_M, λ_m be largest, smallest eigenvalues of A.

min

want: $|\lambda_{k_1}, \lambda_{k_2}| < 1$.

$$1 - (\omega_1+\omega_2)\lambda_M + \omega_1\omega_2\lambda_M^2 < 1$$
$$1 - (\omega_1+\omega_2)\lambda_m + \omega_1\omega_2\lambda_m^2 < 1$$

Spd

$$(\omega_1+\omega_2)\lambda_M > \omega_1\omega_2\lambda_M^2 \Leftrightarrow (\omega_1+\omega_2) > \omega_1\omega_2\lambda_M$$
$$(\omega_1+\omega_2)\lambda_m > \omega_1\omega_2\lambda_m^2 \Leftrightarrow (\omega_1+\omega_2) > \omega_1\omega_2\lambda_m$$

only consider λ_M: now clearly $\omega_1+\omega_2$ exists s.t. $\omega_1+\omega_2 > \omega_1\omega_2\lambda_M$.

$$1 + \frac{\omega_2}{\omega_1} > \omega_2\lambda_M$$
$$\omega_2 = \frac{1}{\lambda_M}$$

rate of converge is $r_\sigma(K)$.

3) $\|Ax-b\|_2^2$, $\min \|Ax-b\|_2^2 + \alpha \|x\|_2^2$

a) x is solution if $A^TAx = A^Tb$

which always has a solution.

want Ax^*-b is orthogonal to range(A)
(best approximation)

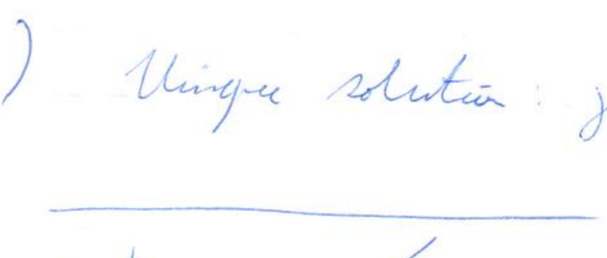
$\Rightarrow A^T(Ax^*-b) = 0 \Leftrightarrow A^TAx^* = A^Tb$ ✓

always has unique solution because $\|Ax-b\|_2^2$ is nonnegative quadratic function in n variables
OR

$\mathrm{im}(A^TA) = \mathrm{im}(A^T)$ $\quad (\mathrm{Im}(A^T) = \ker(A)^\perp$
OR $\quad = \ker(A^TA)^\perp = \mathrm{im}(A^TA))$

uniqueness of orthogonal projection.

b) Unique solution: yes



Every matrix has SVD,
so the pseudoinverse
always exists.

~~$R(A)$~~ b) $(Ax-b, Ax-b) + \alpha(x,x)$

$(A, Ax-b) +$

$2A^T(Ax-b) + 2\alpha x = 0$

$(A^TA + \alpha I)x = A^Tb$

$\Rightarrow (A^TA+\alpha I)^{-1} x = A^Tb$

c) $x^* = x_1 + x_2$, $x_1 \in R(A^T)$, $x_2 \in R(A^T)^\perp = \ker(A)$

Then $Ax^* = Ax_1$ lg

and $\|x^*\| = \|x_1\| + \|x_2\| \geq \|x_1\|$ ✓

d) as $\alpha \to 0$ a solution of (2) converges to in norm solution of (1)

$(A^TA + \alpha I) = A^Tb \qquad A = U\Sigma V^T$

$\Leftrightarrow V\Sigma^T U^T U\Sigma V^T + \alpha V V^T = V\Sigma^T U^T b$

$V(\Sigma^T\Sigma + \alpha I)\underbrace{V^T x}_{z} = V\Sigma^T U^T b$

$(\Sigma^T\Sigma + \alpha I) z = \Sigma^T U^T b$

$$z_i = \begin{cases} \dfrac{\sigma_i u_i^T b}{\sigma_i^2 + \lambda} \\ 0 \end{cases}$$

$z \xrightarrow[\alpha \to 0]{} x^+$

4) $A^H = -A$ matrix

a) $I+A$ invertible

we have $A_{ii} = 0 \quad \forall i$ and hence

$(iA)^* = -i(A^H) = -i \cdot -A = iA \Rightarrow iA$ is hermitian and has purely real e.v. $\Rightarrow A$ has purely imaginary eigenvalues

$\Rightarrow I+A$ cannot have eigenvalue 0 $\Rightarrow I+A$ is invertible.

b) $U^*U = \left((I+A)^{-1}(I-A)\right)^*\left((I+A)^{-1}(I-A)\right)$

$= (I-A)^*(I+A)^{-*}(I+A)^{-1}(I-A)$

$I+A = \begin{pmatrix} 1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & 1 \end{pmatrix} \quad I-A = \begin{pmatrix} 1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & 2 & 1 \end{pmatrix} = (I+A)^*$

$= (I+A)(I+A)^{-1*}(I+A)^{-1}(I+A)^* = I$

c) $U = (I+A)^{-1}(I-A)$

$\Leftrightarrow (I+A)U = I-A$

$U + AU = I - A$

$AU + A = I - U$

$A(U+I) = I - U$

$A = (I-U)\underbrace{(U+I)^{-1}}_{\text{exists since } -1 \notin \sigma(U)}$ is a

and $A^* = (U+I)^{-*}(I-U)^*$ is skew hermitian

d) we want $D = \cancel{B}U^T B U$

$$U D = B U$$

e) $|\lambda| = |\lambda v| = \|Av\| = (v^T A^T A v)^{1/2}$ $\leq \|A\|_2 \overbrace{\|v\|}^{1} = \sigma_1$ ✓

$\leq (A^T A)^{1/2} (v^T v)^{1/2} = \|A^T A\|^{1/2}$

↓

largest singular value.

or $\|A\|_2 =$

$\det(C) = \det(U \Sigma V^T) = \det(U)\det(V^T)\det(\Sigma)$

$= \prod \sigma_i.$

502a Qual S'15

1) A $n\times n$, spd

a) $\underbrace{l_{11}}_{1} u_{11} = a_{11}$ ✓

~~[illegible]~~

we have $\det(A) = \det(LU) = \underbrace{\det(L)}_{1} \det(U)$

$= \prod_{j=1}^{k} u_{jj}$ and

$\det(A_{k-1}) = \det(L_{k-1}) \det(U_{k-1}) = \prod_{j=1}^{k-1} u_{jj}$

$\Rightarrow u_{kk} = \dfrac{\det(A_k)}{\det(A_{k-1})}$

b) $A = R^T R$

$A = LU = LDD^{-1}U = A^T$

$\rightarrow LDD^{-1}U = (LDD^{-1}U)^T = U^T D^{-1} D L^T$

~~$DD^{-1}U = L^{-1}U^T$~~

$\boxed{L}DD^{-1}\boxed{UL^{-T}} = U^T D^{-1} \boxed{D}$

$\underset{\nabla}{UL^{-T}} = \underset{\Delta}{\underbrace{U^T L^{-1}}} = D$

$\Rightarrow U = DL^T$

$\Rightarrow A = LU = LDL^T = LD^{1/2} D^{1/2} L^T = R^T R$

c) just get LDL^T. Then no square roots, cause diags are in D.

2) $A \in \mathbb{R}^{m \times n}$, $m \geq n$

a) $\begin{pmatrix} I_m & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$

$r + Ax = b \longrightarrow A^T r + A^T A x = A^T b$

$A^T r = 0 \longrightarrow 0$

$\Rightarrow$

a) ✓

b) (let $\min \|Ax - b\|$ have a unique solution)

$\Leftarrow$ let system have a unique solution.

$\Rightarrow A^T r = 0 \Rightarrow r = 0$ (and x is unique)

$\Rightarrow Ax = b \Rightarrow \min \|b - Ax\|$ is unique.

Let $\|b - Ax\|$ have a unique solution. Then

$A^T A x = A^T b$ is unique $\rightarrow$ A has full column rank $\rightarrow$ A

$(r \in \ker(A^T) = \operatorname{range}(A)^\perp$

3) $X_{k+1} = X_k + X_k(I - AX_k)$

$= X_k + X_k - X_k A X_k$

$= 2X_k - X_k A X_k$

(let converge then $AX = X + X - XAX$ (X)

Let $AX_{k+1} = AX_k + AX_k(I - AX_k)$

$$X_{k+1} - X^* = X_k + X_k(I - AX_k) - X^* - X^*(I - AX_k)$$

$$= X_k - X^* + (X_k - X^*)(I - AX_k)$$

$$= (X_k - X^*)(I + I - AX_k)$$

$$= (X_k - X^*)(2I - AX_k)$$

$$\Rightarrow E_{k+1} = E_k(2I - AX_k)$$

$$\Rightarrow E_{k+1} = E_{k-1}(2I - AX_{k-1})(2I - AX_k)$$

$$\Rightarrow \|E_{k+1}\| \leq \|E_0\| \, \|2I - AX_0\| \, \|2I - AX_1\| \dots \|2I - AX_k\|$$

4) $A = \begin{pmatrix} 3 & \alpha & \beta \\ -1 & 7 & -1 \\ 0 & 0 & 5 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$

a) $|\alpha|, |\beta| \le 1$

$$\begin{pmatrix} 1 & & \\ & 2 & \\ & & 4 \end{pmatrix} \begin{pmatrix} 3 & \alpha & \beta \\ -1 & 7 & -1 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1/2 & \\ & & 1/4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & & \\ & 2 & \\ & & 4 \end{pmatrix} \begin{pmatrix} 3 & \alpha/2 & \beta/4 \\ -1 & 3.5 & -1/4 \\ 0 & 0 & 5/4 \end{pmatrix} = \begin{pmatrix} 3 & \alpha/2 & \beta/4 \\ -2 & 7 & -1/2 \\ 0 & 0 & 5 \end{pmatrix}$$

$\{z : |z-3| \le \underbrace{|\frac{\alpha}{2}| + |\frac{\beta}{4}|}_{\le \frac{3}{4}}\}$ no overlap.

$\{z : |z-7| \le \frac{5}{2}\}$

and 5. possible overlap.

b) $A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}$ ✓

$U^T A U$ = Schur:

$$\begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix}$$ ✓

then $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = P$

$P' U' A U P$ ✓

502a Qual (no year)

1) This means in every column we have

a) $|f_i| + |e_i| < |d_i|$ → by Gershgorin 0 is not an eigenvalue.

→ invertible.

b) go for $D_3 = L_3 F_2 + U_3$

---

2) $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$

a) $\min_{x \in S^*, \|x\|_2 = 1} \|Ax\| = \max_{\dim S = 2} \min_{x \in S, \|x\|_2 = 1} \|Ax\|_2$.

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is in kernel

so $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a suitable space.

(since they're not sent to 0)

b)

b) SVD: Q_1

$$A = U \Sigma V^T = u_1 \sigma_1 v_1^T + \dots$$

so $A^*A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 1 & 5 \\ 1 & 2 & 1 \\ 5 & 1 & 5 \end{pmatrix}$

want

$\begin{pmatrix} & \cancel{1} & 1 \\ 1 & & \end{pmatrix}$

3) a) $P = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & & \ddots \\ 1 & & & & \end{pmatrix}$

$P^0 = Id$, $P^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ & & & & \ddots \\ 1 & & & & \\ & 1 & & & \end{pmatrix}$

P^{n-1}, $P^n = P^0 = I$,

$\shortparallel$

$\begin{pmatrix} 1 & & & & 1 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \end{pmatrix}$ ✓

b) Show $F_{k+1} = \overline{\omega}(k-1) F_k$

~~F~~ clear by definition.

unitary: $F^*F = I$

$(F^*F)_{i,j} = \sum (f)^*_{ik} f_{kj} =$

$\sum \overline{f_{ki}} f_{kj} = \frac{1}{n} \overline{\omega}^{(k-1)(i-1)} \omega^{(k-1)(j-1)}$

$= 1$ for $i=j$, 0 rest

d) we have $A_\alpha F$

follows from a – c ✓

4) a) $z^{m+1} = \begin{pmatrix} 1/2 & 1/2 & & \\ 0 & 1/2 & 1/2 & \\ & & & \\ 1/2 & \dots & \dots & 1/2 \end{pmatrix} z^m$

b) $\lim_{m\to\infty} z_k^m = \hat{z}$, $\hat{z} = \frac{1}{4}\sum_{j=1}^{4} z_j^0$.

502a Qual S'17

1) a) $\det B = \det(AD^{-1}) = \frac{\det(A)}{\det(D)}$

$= \frac{\det(A)}{\prod \|A_j\|_1}$

$\Rightarrow |\det B| \left(\prod \|A_j\|_1\right) = |\det A|$

and $\det B < 1$

because all colums have length less than 1

so resulting parallelepiped has volume < 1 $= |\det B|$.

2) A nonsingular.

a) $A = QR$, Q orthogonal and R is upper triangular.

$u_j = a_j - (a_{j-1} \cdot e_{j-1}) e_1 - \cdots (a_{j-1} \cdot e_1) e_1$

$e_k = \frac{u_k}{\|u_k\|}$ $\quad a_2 = e_2 \|u_2\| + (a_2 e_1) e_1$

clear

b) $QRx = b \iff \overset{a_2 e_2}{R}x = Q^t b$

$Rx = \underbrace{Q^T b}_{n^2}$ mult.

and then $1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \frac{n^2+n}{2}$

$\Rightarrow \frac{3}{2}n^2 + \frac{n}{2}$ $\quad O(n^2)$ ✓

3) $\min \|Ax-b\|_2$

s.t. $Cx=d$.

a) $C=0,\ d=0$

b) $A=I,\ b=0$

c) $\min \|A(x_0+Nz)-b\|_2$

s.t. $C(x_0+Nz)=d$

x_0 s.t. $Cx_0=d$, and $Nz\in \ker(C)$.

So N contains colums of kernel vectors of C.

Clear what x_0 and N are!

Then

$\min \|Ax_0+ANz-b\|_2$

$\|ANz-(b-Ax_0)\|_2$

$\tilde{A}=AN,\ \hat{b}=b-Ax_0$, N, x_0 given

d) $Cy=d$

$$\begin{pmatrix} 4 & 16 & 12 \\ \cancel{-2} & \cancel{-8} & \cancel{-6} \\ 1 & 4 & 3 \\ \cancel{-1} & \cancel{-4} & \cancel{-3} \end{pmatrix} \begin{pmatrix} 12 \\ \cancel{-6} \\ 3 \\ \cancel{-3} \end{pmatrix}$$

x_4 belibig

x_2 belibig

$$\begin{pmatrix} 4 & 16 & 12 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$4x_1 + \cancel{4x} \; 16x_2 + 12x_3 = 12$$

x_1, x_2 free $\Rightarrow x_3 = 1 - \frac{1}{3}x_1 - \frac{4}{3}x_2$

$$\Rightarrow \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{x_0} + a \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1/3 \end{pmatrix}}_{v_1} + b \underbrace{\begin{pmatrix} 0 \\ 1 \\ -4/3 \end{pmatrix}}_{v_2}$$

$\Rightarrow$

$$A \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1/3 & -4/3 \end{pmatrix} = \quad -2/3$$

$$b - Ax_0 = b - \begin{pmatrix} 2 \\ 3 \\ 2 \\ 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \\ 0 \end{pmatrix}$$

$$T^T T x = T^T t$$

the normal equations.

4) $y^k = x^k + w_0(b - Ax^k), \quad x^{k+1} = y^k + w_1(b - Ay^k)$

$$x^{k+1} = x^k + w_0(b - Ax^k) + w_1(b - A(x^k + w_0(b - Ax^k)))$$

$$= \underline{x^k} - \underline{w_0 A x^k} + w_0 b + w_1 b - \underline{w_1 A x^k}$$

$$- w_1 w_0 A b + \underline{w_1 w_0 A^2 x^k}$$

$$= (I - w_0 A - w_1 A + w_1 w_0 A^2) x_k$$

$$+ (w_0 I + w_1 I - w_1 w_0 A) b$$

$$= (I - (w_0 + w_1) A + w_0 w_1 A^2) x_k + ((w_0 + w_1) I - w_0 w_1 A) b$$

a) $B = \underbrace{w_0 w_1}_{\mu} \left(A^2 - \frac{w_0 + w_1}{w_0 w_1} A + \frac{1}{w_0 w_1} I \right)$ ✓

b) $T_2(\lambda) = \lambda^2 - \frac{1}{2}$

Let $x^2 + ax + b$ be $q(\lambda)$.

Then max is either $1 + a + b$ or $1 - a + b$

$1 + a + b \leq 0.5$ also $2a \leq 0$

$1 - a + b \leq 0.5$ $\quad a \leq 0.$

$2 + 2b \leq 1$

$2b \leq -1$

$b \leq -\frac{1}{2}$

$1 + a + b \geq -0.5$

$1 - a + b \geq -0.5$

$\rightarrow 2 + 2b \geq -1$

$2b \geq -3$

$b \geq -\frac{3}{2}$

$2a \geq 0$

$\Rightarrow a = 0$

$1+b$ and $1+b$

and b must be small $\Rightarrow b = -\frac{1}{2}$ ✓

c) $(I - (w_0 + w_1) A + w_0 w_1 A^2)$

$\lambda_{max} \rightarrow \quad 1 - (w_0 + w_1)\lambda_{max} + w_0 w_1 \lambda_{max}^2 < 1$

$1 - (w_0 + w_1)\lambda_{min} + w_0 w_1 \lambda_{min}^2 < 1$

$\Rightarrow \quad 1 - (w_0 + w_1) + w_0 w_1 < 1 \quad \leftarrow \lambda_{max} = 1$

$1 + (w_0 + w_1) + w_0 w_1 < 1 \quad \leftarrow \lambda_{min} = -1$

$w_0 w_1 - w_0 - w_1 < 0$

$w_0 w_1 + w_0 + w_1 < 0$

$w_0 - \frac{w_0}{w_1} < 1$

$w_0 + \frac{w_0}{w_1} < -1$

$2 w_0 < 0 \quad \Rightarrow \quad w_0 < 0$

$w_1 - 1 - \frac{w_1}{w_0} < 0$

$\Rightarrow \quad w_1 \left(1 - \frac{1}{w_0}\right) < 1$

$w_1 < \frac{1}{1 - \frac{1}{w_0}} = \frac{w_0}{w_0 - 1} \quad \frac{-\frac{1}{2}}{-\frac{1}{2} - 1} = \frac{-\frac{1}{2}}{-\frac{3}{2}} = \frac{1}{3}$

i.e. $w_0 = -\frac{1}{2}, \quad w_1 = \frac{1}{3}$

$-\frac{1}{6} + \frac{1}{2} - \frac{1}{3} \qquad -\frac{1}{6} + \frac{3}{6} - \frac{2}{6} = 0$

$-\frac{1}{6} - \frac{1}{2} + \frac{1}{3} \qquad -\frac{1}{6} - \frac{3}{6} + \frac{2}{6} = -\frac{1}{3}$ ✓

502a Qual F'17

1) a) $C = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$

det C =

$\det(A)\det(A - BA^{-1}B) = \det(A^2 - ABA^{-1}B)$

$\begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} I & 0 \\ -A^{-1}B & I \end{pmatrix} = \begin{pmatrix} A-BA^{-1}B & B \\ 0 & A \end{pmatrix}$

$\Rightarrow \det \quad = \det C = \det(A)\det(A - BA^{-1}B)$

b) $A+B$ and $A-B$ are non-singular.

assume $A+B$ is singular

$\Rightarrow \quad (A+B)x = 0 \qquad Ax = -Bx$

then $\quad C\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{matrix} Ax + Bx = 0 \\ Bx + Ax = 0 \end{matrix} = 0$

$\Rightarrow$ C singular $\lightning$

$(A-B)x = 0 \Rightarrow$

$C\begin{pmatrix} x \\ -x \end{pmatrix} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}\begin{matrix} x \\ -x \end{matrix} = \begin{matrix} Ax - Bx = 0 \\ Bx - Ax = 0 \end{matrix}$ $\lightning$

c) $Cx = b$, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$(A+B)\,y_1 = b_1 + b_2$, $(A-B)\,y_2 = b_1 - b_2$

$x_1 = \frac{1}{2}(y_1 + y_2)$, $x_2 = \frac{1}{2}(y_1 - y_2)$

We have $Cx = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{matrix} \frac{1}{2}(y_1 + y_2) \\ \frac{1}{2}(y_1 - y_2) \end{matrix}$

$$= \begin{matrix} \cancel{A}\ \frac{1}{2}\big((A+B)y_1 + (A-B)y_2\big) \\ \frac{1}{2}\big((A+B)y_1 \cancel{+} - (A-B)y_2\big) \end{matrix} = \begin{matrix} \frac{1}{2}(b_1 + \cancel{b_2} + b_1 - \cancel{b_2}) \\ \frac{1}{2}(\cancel{b_1} + b_2 - \cancel{b_1} + b_2) \end{matrix}$$

$$= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \quad \checkmark$$

d) Smaller system to solve.

$O(n^3)$ and so...

4) a) $x^{k+1} = x^k + \tilde{A}^{-1}(b - Ax^k)$

$$= (I - \tilde{A}^{-1}A)x + \tilde{A}^{-1}b$$

$\|A\| < 1$, then $(I - A$

$\|A\| \, \|\tilde{A}^{-1} - A^{-1}\| \geq \|\tilde{A}^{-1}A - I\|$

<u>S 18</u>

1) $LU = A$

$$\begin{pmatrix} l_1 & 0 \\ l_2^T & 1 \end{pmatrix} \begin{pmatrix} u_1 & u_2 \\ 0 & u_n \end{pmatrix} = \begin{pmatrix} R & v \\ u^T & 0 \end{pmatrix}$$

$l_1 \cdot u_1 = R$, $l_1 \cdot u_2 = v$

$l_2^T u_1 = u^T$, $l_2 u_2 + u_n = 0$.

$u_1 = R$, $l_1 = Id$, $u_2 = v$,

$l_2^T R = u^T \Rightarrow R^T l_2 = u \Rightarrow l_2 = (R^{-T}u)^T$

$u_n = -(R^{-T}u)^T v$

4) a)

$$\| A - \sum_{i=1}^{k} x_i y_i^T \|_F^2$$

$$= \| U \Sigma V^T - \sum_{i=1}^{k} \sigma_i u_i v_i^T \|_F^2$$

$$= \| \Sigma - \sum_{i=1}^{k} U^T \sigma_i u_i v_i^T V \|_F^2$$

$$= \| \Sigma - \sum_{i=1}^{k} \sigma_i e_i e_i^T \|_F^2$$

$$= \sum_{j=k+1}^{n} \sigma_i \sigma$$

3) $A^T A$

$$2 \frac{3}{4} = \frac{11}{4} \cdot 8$$

$$22 s = 8 \cdot \frac{14}{17} - 4$$

$$s = \frac{8}{22} \frac{14}{17} - \frac{4}{22}$$

$$= \frac{4}{11} \cdot \frac{14}{17} - \frac{2}{11}$$

$$s = \frac{56}{187} - \frac{28}{187} = \frac{28}{187}$$

2) a) $|\omega - 1| < 1 \Rightarrow M^* + N$ pos. def.

$$M^* + N = \left(\frac{1}{\omega} D - E\right)^* + \left(\frac{1}{\omega} - 1\right) D + F$$

$$= \frac{1}{\omega} D^* - \cancel{F} + \left(\frac{1}{\omega} - 1\right) D + \cancel{F}$$

$$= \left(\frac{2}{\omega} - 1\right) D$$

$$\frac{2}{\omega} - 1 > 0$$

BWL ist kacke

$$\Rightarrow 2 > \omega$$

$$\Rightarrow$$

b) $$\|M^{-1} N\|_A = \max_{\|x\|_A = 1} \|M^{-1} N x\|_A$$

$$\max_{x^* A x = 1} \sqrt{x^T N^* M^{-*} A M^{-1} N x}$$

$$= \sqrt{x^T (M^{-1} N)^* A (M^{-1} N) x}$$

d) $$\det(I_\omega) = \frac{1}{\prod d_{ii}}.$$

G is orthogonal

$$\|A_1\|^2 = \|A_0\|^2$$

$$\Rightarrow \|D_0 + E_0\|_F^2 = \|D_1 + E_1\|_F^2$$

$$\|D_0\|^2 + \|E_0\|^2 = \|D_1\|^2 + \|E_1\|^2$$

$$\Rightarrow \|E_1\|^2 = \|E_0\|^2 + \|D_0\|^2 - \|D_1\|^2.$$

1) a) Show $\frac{\beta_k}{\lambda_1} - 1 = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k\right)$

i.e. $\frac{\frac{\beta_k}{\lambda_1} - 1}{\left(\frac{\lambda_2}{\lambda_1}\right)^k} = \left(\frac{\beta_k}{\lambda_1} - 1\right) \cdot \left(\frac{\lambda_1}{\lambda_2}\right)^k$

$= \frac{\beta_k \, \lambda_1^{(k-1)}}{\lambda_2^{k}} - \frac{\lambda_1^{k}}{\lambda_2^{k}}$ is bounded.

$$\beta_k = \frac{x_{k-1}^T x_k}{x_{k-1}^T x_{k-1}} = \frac{(\Sigma \alpha_i u_i \lambda_i^{k-1})^T (\Sigma \alpha_i u_i \lambda_i^{k})}{(\Sigma \alpha_i u_i \lambda_i^{k-1})^T (\Sigma \alpha_i u_i \lambda_i^{k-1})}$$

2) a) $A = V^{-1} \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_m \end{pmatrix}$ ✓

$$J_k(\lambda)^m = \sum_{j=0}^{k-1} \binom{m}{m-j} \lambda^{m-j} J_k(0)^j$$

$$\left(\begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \end{pmatrix} \right)$$

$\boxed{\text{Fall 18-19}}$

502a S'19

1) a) $x^T \Phi^T \Phi x = (\Phi x)^T (\Phi x) \geq 0$

and since $x \neq 0$ we know $\Phi x \neq 0$

(Φ are linearly independent vectors)

> 0

b) $\|\cdot\|$ $P_\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $P_\Phi x = \varphi$ where

$\varphi = \arg\min_{\psi \in \mathrm{span}\{\varphi_i\}_1^m} \|x - \psi\|$ show P_Φ is linear.

$P_\Phi(ax+y) = \arg\min_{\varphi \in \mathrm{span}\{\varphi_i\}_1^m} \|(ax+y) - \psi\|$

---

we know $\psi = \cancel{\Phi} (\Phi^T \Phi)^{-1} \Phi^T \cancel{\Phi} (ax+y)$
$= a(\Phi^T \Phi)^{-1} \Phi^T x + (\Phi^T \Phi)^{-1} \Phi^T y \to$ linear, since least squares are linear.

c) $P^* = P$, $P^2 = P$

$P = (\Phi^T \Phi)^{-1} \Phi^T$, then

$\cancel{P^* = ((\Phi^T \Phi)^{-1} \Phi^T)^T = \Phi (\Phi^T \Phi)^{-T} =}$

$\cancel{P^2 = (\Phi^T \Phi)^{-1} \Phi^T (\Phi^T \Phi)^{-1} \Phi^T =}$

b) min $\|\phi c - x\|$

$\Rightarrow c = (\phi^T\phi)^{-1}\phi^T x$

and so ϕc is the element in span ...

$\Rightarrow \varphi = \phi(\phi^T\phi)^{-1}\phi^T x$

but the linearity holds

c) $P^T = \phi\underbrace{(\phi^T\phi)^{-T}}_{(\phi^T\phi)^{-1}}\phi^T$ ✓

clearly

$P^2 = \phi(\phi^T\phi)^{-1}\underbrace{\phi^T\phi(\phi^T\phi)^{-1}}_{I}\phi^T$ ✓

d) span is equal to orthogonal complement

of subspace spanned by $\{\varphi_i\}$

$(\text{range}(\phi))^\perp = \ker(\phi^T)$.

~~null~~ ~~span~~ ϕ^T

2) a) Let (x,y) be in $\mathbb{R}^2$ s.t. $\left\|\begin{pmatrix}x\\y\end{pmatrix}\right\|_2 = 1$ i.e.

$y = \sqrt{1-x^2}$

Then $\left\|A\begin{pmatrix}x\\y\end{pmatrix}\right\|_2 = \left\|\begin{pmatrix}x+\sqrt{1-x^2}\\ x\varepsilon + \varepsilon\sqrt{1-x^2}\end{pmatrix}\right\|_2$

$= \sqrt{x^2 + 2x\sqrt{1-x^2} + (1-x^2) + x^2\varepsilon^2 - 2x\varepsilon^2\sqrt{1-x^2} + \varepsilon^2(1-x^2)}$

$= \dots$

Do $\frac{\partial}{\partial x}$ and set $=0$ to maximize

$\Rightarrow x = y = 1/\sqrt{2}$ so

$\|A\|_2 = \left\|A\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}\right\|_2 = \sqrt{2}$

b) Find $A^{-1} = \begin{pmatrix}1/2 & -1/2\varepsilon\\ 1/2 & 1/2\varepsilon\end{pmatrix}$

$x = \begin{pmatrix}0\\1\end{pmatrix}$ achieves max of $\|A^{-1}x\|_2$ when

ε is small $\Rightarrow \|A^{-1}\|_2 = \left\|A^{-1}\begin{pmatrix}0\\1\end{pmatrix}\right\|_2 = \frac{1}{\sqrt{2}\varepsilon}$

S'19

1 a) $x^T \phi^T \phi x = (\phi x)^T (\phi x) > 0$

for $x \neq 0$ since ϕ has linearly indep. columns

b) $\varphi = \underset{\varphi \in \operatorname{span}\{\varphi_i\}}{\arg\min} \|x - \varphi\|$

i.e. $\min \|\phi e - x\|$

$\rightsquigarrow$ Normal eq. $e = (\phi^T \phi)^{-1} \phi^T x$

so $\varphi = \phi e = \phi (\phi^T \phi)^{-1} \phi^T x$ (*)

and so clearly P_ϕ is linear

c) Compute $\phi^* = \phi$ and $\phi^2 = \phi$ using (*)

d) $(\text{range } \phi)^\perp = \ker(\phi^T)$.

2) $A = \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon \end{pmatrix}$ $\qquad x^2+y^2=1$

$y = \sqrt{1-x^2}$

a) $\| A x \|_2 = \max \left\| A \begin{pmatrix} x \\ \sqrt{1-x^2} \end{pmatrix} \right\|_2$

$= \left\| \begin{matrix} x + \sqrt{1-x^2} \\ -x\varepsilon + \varepsilon\sqrt{1-x^2} \end{matrix} \right\|_2 = \sqrt{x^2 + 2x\sqrt{1-x^2} + (1-x^2) + \cancel{x^2\varepsilon^2} - 2x\varepsilon^2\sqrt{1-x^2} + \varepsilon^2(1-\cancel{x^2})}$

$= 1 + 2x\sqrt{1-x^2} + \varepsilon^2 - 2x\varepsilon^2\sqrt{1-x^2}$

$\frac{\partial}{\partial x} = 2\left(\sqrt{1-x^2} - x \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\right) - 2\varepsilon^2\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right) = 0$

$(1-\varepsilon^2)\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}}\right) = 0$

$1 - x^2 - x^2 = 0 \Rightarrow 1 - 2x^2 = 0$

$\frac{1}{2} = x^2 \Rightarrow x = \frac{1}{\sqrt{2}} = y$

$\rightarrow A \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \left\| \begin{matrix} 2/\sqrt{2} \\ 0 \end{matrix} \right\| = \frac{2}{\sqrt{2}} = \sqrt{2}$ ✓

b) $A^{-1} = \frac{1}{\det} \begin{pmatrix} \varepsilon & -1 \\ \varepsilon & 1 \end{pmatrix} = \frac{1}{2\varepsilon} \begin{pmatrix} \varepsilon & -1 \\ \varepsilon & 1 \end{pmatrix}$

$= \begin{pmatrix} 1/2 & -1/2\varepsilon \\ 1/2 & 1/2\varepsilon \end{pmatrix}$ and is large when ✓

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \left\| \begin{matrix} -1/2\varepsilon \\ 1/2\varepsilon \end{matrix} \right\| = \sqrt{\frac{1}{4\varepsilon^2} + \frac{1}{4\varepsilon^2}} = \sqrt{\frac{1}{2\varepsilon^2}} = \frac{1}{\varepsilon}\frac{1}{\sqrt{2}}.$

c) want $(A+\delta A)x = 0$

$\Rightarrow \delta A x = -Ax$

$\|A^{-1}\delta A x\| = \|-x\| = \|x\|$

$\Rightarrow \dfrac{\|A^{-1}\delta A x\|}{\|x\|} = 1$

$\Rightarrow \dfrac{\|A^{-1}\|\,\|\delta A\|\,\cancel{\|x\|}}{\cancel{\|x\|}} \geq 1$

$\Rightarrow \|\delta A\| \geq \dfrac{1}{\|A^{-1}\|}$ ✓

d) want $\|\delta A\|_2 = \dfrac{1}{\|A^{-1}\|} = \varepsilon\sqrt{2}$.

take $\delta A = \begin{pmatrix} 0 & 0 \\ \varepsilon & -\varepsilon \end{pmatrix}$. then $A+\delta A$ is singular

and $\|\delta A\|_2 =$ $\delta A \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\varepsilon}{\sqrt{2}} + \frac{\varepsilon}{\sqrt{2}} \end{pmatrix}$

$= \begin{pmatrix} 0 \\ \frac{2\varepsilon}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2}\,\varepsilon \end{pmatrix}$ ✓

3 a)

Don't know how to show

$|\omega^{-1}(1-\omega-\lambda)| \geq |\lambda|$.

But then

strictly diag.

$$|A_{\lambda,\omega\, i,i}| \geq |\lambda|\,|a_{ii}| > |\lambda| \sum_{i\neq j} |a_{ij}|$$

$$\geq \sum_{j<i} |\lambda|\,|a_{ij}| + \sum_{j>i} |a_{ij}| \longrightarrow A_{\lambda,\omega}$$

is strictly diag dom.

c) want $x \neq 0$ s.t.

$$(A + \delta A)x = 0$$

$$\Rightarrow \delta A \, x = -Ax$$

$$\| A^{-1} \delta A \, x \| = \| x \|$$

So $\| x \| = \| A^{-1} \delta A \, x \| \leq \| A^{-1} \| \, \| \delta A \| \, \| x \|$

$$\Rightarrow \| \delta A \| \geq \frac{1}{\| A^{-1} \|}$$

d) want $\| \delta A \|_2 = \dfrac{1}{\| A^{-1} \|_2} = \varepsilon \sqrt{2}$

take $\delta A = \begin{pmatrix} 0 & 0 \\ \varepsilon & -\varepsilon \end{pmatrix}$. Then $A + \delta A$

is singular & $\| \cancel{A} \, \delta A \|_2 = \sqrt{2} \, \varepsilon$.

3) A strictly diag. dom.

$A = D - E - F$

a) $0 < \omega < 1$, $|\lambda| \geq 1$

show that $A_{\lambda,\omega} = \omega^{-1}(1-\omega-\lambda)D + F + \lambda E$

is strictly diagonally dominant.

(we have $|1-\omega-\lambda| \geq |\lambda| - (1-\omega)$ and

hence $|\omega^{-1}(1-\omega-\lambda)| \geq \left|\frac{\lambda}{\omega}\right| - \frac{1}{\omega} + 1$

$\geq \left|\frac{\lambda}{\omega}\right| - \frac{1}{\omega} \geq$)

~~[illegible]~~

$1-\omega + |1-\omega-\lambda| \geq |\lambda|$ . . .

~~[illegible]~~ since $|\omega^{-1}(1-\omega-\lambda)| \geq |\lambda| \geq 1$

we have $|A_{\lambda,\omega_{ii}}| \geq |\lambda|\,|a_{ii}| > |\lambda| \sum_{i\neq j} |a_{ij}|$

$\geq \sum_{j<i} |\lambda||a_{ij}| + \sum_{j>i} |a_{ij}| \rightarrow A_{\lambda,\omega}$ strictly

diag dom.

b) $B_{SOR} = (\omega^{-1}D - E)^{-1}((\omega^{-1}-1)D + F)$

Show: $B_{SOR} - \lambda I$ is nonsingular for all $|\lambda| \geq 1$ and $0<\omega<1$.

$$B_{SOR} - \lambda I = (\omega^{-1}D - E)^{-1}((\omega^{-1}-1)D + F)$$

$$- \lambda (\omega^{-1}D - E)^{-1}(\omega^{-1}D - E)$$

$$= (\omega^{-1}D - E)^{-1}((\omega^{-1}-1)D + F - \lambda\omega^{-1}D + \lambda E)$$

$$= (\omega^{-1}D - E)^{-1}((\omega^{-1}-1-\lambda\omega^{-1})D + F + \lambda E)$$

$$= (\omega^{-1}D - E)^{-1}\underbrace{(\omega^{-1}(1-\omega-\lambda)D + F + \lambda E)}$$

strictly d.d. hence invertible by Gershgorin.

$\rightarrow$ nonsingular.

c) we want all $|\lambda|$ eigenvalues to be < 1 for convergence. Since $B_{SOR} - \lambda I$ is nonsingular for all $|\lambda| \geq 1$ we know $|\lambda| < 1$ for all eigenvalues and hence convergence!

4) $A =$

a) A is close to upper triangular.

$\rightarrow$ eigenvalues close to $-2, 5, 15, 9$.

b) $Z_1 \{z \in \mathbb{C} : |z+2| \leq 310\}$

$Z_2 \{z \in \mathbb{C} : |z-5| \leq 1100.01\}$

$Z_3 \{z \in \mathbb{C} : |z-15| \leq 10.021\}$

$Z_4 \{z \in \mathbb{C} : |z-9| \leq 0.01\}$

all eigenvalues are in $Z_1 \cup Z_2 \cup Z_3 \cup Z_4$.

c) $\det(A - \lambda I) = \cancel{\det(I}$

$\det(T)\det(T^{-1})\det(A-\lambda I)$

$= \det(T^{-1}AT - \lambda T^{-1} I T)$

$= \det(T^{-1}AT - \lambda I)$ ✓.

d) $\cancel{I = 2}$?

502a F'19

1) a) Let A, B be lower triag, n-dimensional.

Then $C = A \cdot B$ and for $j > i$ we have

$$C_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

$$= \underbrace{\sum_{k=1}^{i} a_{ik} b_{kj}}_{i<j\text{, so } b_{kj}=0} + \underbrace{\sum_{k=i+1}^{n} a_{ik} b_{kj}}_{a_{ik}=0 \text{ in this sum}} = 0$$

$\Rightarrow$ C lower triag

b) Let A be non-sing. lower triag. Denote $A^{-1} = (\tilde{a})_{ij}$

Then $I = A A^{-1}$, i.e.

$$I_{ij} = \delta_{ij} = \sum_{k=1}^{n} a_{ik} \tilde{a}_{kj} = \sum_{k=1}^{i} a_{ik} \tilde{a}_{kj} \quad (*)$$

For $i=1$ we have
$$1 = a_{11} \tilde{a}_{11}$$
$$0 = a_{11} \tilde{a}_{12} \Rightarrow \tilde{a}_{1j} = 0 \text{ for } j > 1.$$
$$\vdots$$
$$0 = a_{11} \tilde{a}_{1n}$$

~~analogously for~~

$(*)$ shows $\delta_{ij} = 0$ for $j > i$ and hence $\sum_{k=1}^{i} a_{ik} \tilde{a}_{kj} = 0$

for $j>i$. Hence $\tilde{a}_{ij}=0$ for $j>i$ and so A^{-1} is lower triag.

c) Let R be the row echolon form of A. Divide every row s.t. we get ones on the diagonal. This is the matrix U.

Matrix L is obtained by "book-keeping", i.e. if we ~~subtract~~ add α times row 1 ~~from~~ to row 2, we get $L_{21} = -\alpha$.

d) By induction:

$n=1$ $A=(a)\neq 0$ has LU decomp.

Suppose $A_k = (a_{ij})_{\substack{i=1\\j=1}}^{k}$ (submatrix of A consisting of first k rows and columns) has LU decomp. $L_k U_k$

Then $A_{k+1} = \begin{pmatrix} A_k & v \\ w^T & a_{k+1,k+1} \end{pmatrix} = \begin{pmatrix} L_k & 0 \\ d^T & b \end{pmatrix} \begin{pmatrix} U_k & e \\ 0 & 1 \end{pmatrix}$

and so $A_k = L_k U_k$

$$v = L_k e$$

$$w^T = d^T U_k$$

$$a_{k+1,k+1} = d^T e + b$$

Since A is pos. def. all A_k (submatrices) are non-singular. Hence L_k, U_k are non singular and we can solve for e, d, b.

$\Rightarrow A_{k+1}$ has LU decomp.

By induction, A has LU decomp.

Induction:

$$\begin{pmatrix} A_k & v \\ w^T & a_{k+1,k+1} \end{pmatrix} = \begin{pmatrix} L_k & 0 \\ d^T & b \end{pmatrix} \begin{pmatrix} U_k & e \\ 0 & c \end{pmatrix}$$

$A_k = L_k U_k$ ✓

$v = L_k e$

$w^T = d^T U_k$

$d^T e + b = a_{k+1,k+1}$

2) a) look at

$$I - AX_{n+1} = I - AX_n - cA(AX_n - I)$$

$$= I - AX_n + cA(I - AX_n)$$

$$= (I + cA)(I - AX_n)$$

$$\Rightarrow I - AX_{n+1} = (I + cA)^n (I - AX_0)$$

Hence we need $\|(I+cA)^n\| \xrightarrow[n\to\infty]{} 0$ in order to have a convergent scheme.

b) Want eigenvalues of $I+cA$ to be as small as possible in magnitude:

$I + cA$ has eigenvalues $\mu_i = 1 + c\lambda_i$, i.e.

$\mu_r = 1 + c, \ldots, \mu_1 = 1 + 5c$.

$|\mu_i|$ is minimal if $|\mu_r| = |\mu_1|$,

i.e. $1 + c = -1 - 5c \Rightarrow c = -\frac{1}{3}$.

Then $r_\sigma(I + cA) = \frac{2}{3}$

c) Now

$\mu_r = 1 - c$ and $\mu_1 = 1 + 5c$

In order to have $r_\sigma(I + cA) < 1$ (convergence)

we need $c > 0$ (see μ_r).

But then $\mu_1 > 1 \Rightarrow$ no value of c

will make scheme convergent.

Modified scheme:

$$I - AX_{n+1} = I - AX_n - AC(AX_n - I)$$

$$= I - AX_n + AC(I - AX_n)$$

$$= (I + AC)(I - AX_n)$$

Let ~~CA~~ $C = dA$. ~~then~~ for $d \in \mathbb{R}$. Then

$I + AC = I + dA^2$ has eigenvalues

~~$\mu_r = 1 + (-1)^2 = 2, \ldots, \mu_1 = 1 + 5^2 = 26$~~

and ~~then~~ $\mu_r = 1 + (-1)^2 d = 1 + d, \ldots$

$\mu_1 = 1 + 5^2 d = 1 + 26d$ and we can

use the strategy from b) to find an optimal d.

3) see Spring '19 exam

4) $A \in \mathbb{C}^{5\times 5}$, $p(\lambda) = (\lambda-1)^3(\lambda-2)^2$

a : algebraic mult. g : geometric mult.

a) & b)

① $\begin{pmatrix} 2 & 0 & & & \\ & 2 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$

$a(2)=2,\ a(1)=3,\ g(2)=2$
$g(1)=3$

② $\begin{pmatrix} 2 & 1 & & & \\ & 2 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$

$a(2)=2,\ a(1)=3$
$g(2)=1,\ g(1)=3$

③ $\begin{pmatrix} 2 & 1 & & & \\ & 2 & 0 & & \\ & & 1 & 1 & \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$

$a(2)=2,\ a(1)=3,\ g(2)=1,$
$g(1)=2$

④ $\begin{pmatrix} 2 & 1 & & & \\ & 2 & 0 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$

$a(2)=2,\ a(1)=3,\ g(2)=1,\ g(1)=1$

⑤ $\begin{pmatrix} 2 & 1 & & & \\ & 2 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$

$a(2)=2,\ a(1)=3,\ g(2)=1,\ g(1)=2$

⑥ $\begin{pmatrix} 2 & 0 & & & \\ & 2 & 0 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$

$a(2)=2,$
$a(1)=3,\ g(2)=2,\ g(1)=1$

⑦ $\begin{pmatrix} 2 & 0 & & & \\ & 2 & 0 & & \\ & & 1 & 1 & \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}$

$a(2)=2,\ a(1)=3,\ g(2)=2,\ g(1)=2$

⑧ $\begin{pmatrix} 2 & 0 & & & \\ & 2 & 0 & & \\ & & 1 & 0 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$

$a(2)=2,\ a(1)=3,\ g(2)=2,\ g(1)=2$

~~b)~~ c)

1 $p_0(\lambda) = (\lambda-1)(\lambda-2)$

2 $p_0(\lambda) = (\lambda-1)(\lambda-2)^2$

3 $p_0(\lambda) = (\lambda-1)^2(\lambda-2)^2$

4 $p_0(\lambda) = (\lambda-1)^3(\lambda-2)^2$

5 $p_0(\lambda) = (\lambda-1)^2(\lambda-2)^2$

6 $p_0(\lambda) = (\lambda-1)^3(\lambda-2)$

7 $p_0(\lambda) = (\lambda-1)^2(\lambda-2)$

8 $p_0(\lambda) = (\lambda-1)^2(\lambda-2)$

S'20

1) $\|A\|_F = \sqrt{\sum_{k=1}^{\min\{n,m\}} \sigma_k^2}$

Let $U \Sigma V^T$ be the svd of A.

Then $\|A\|_F = \|U \Sigma V^T\|_F = \|\Sigma\|_F$

$= \sqrt{\sum \sigma_k^2}$.

Now why is $\|\cdot\|_F$ invariant under orthogonal transformations?

$\|QA\|_F = \sqrt{\sum_i \sum_j |(QA)_{ij}|^2}$

$= \sqrt{\sum_i \sum_j \left(\sum_k q_{ik} a_{kj}\right)^2}$

$= \sqrt{\sum_i \sum_j \left(\sum_k (q_{ik} a_{kj})^2 + \sum_{k \neq p} q_{ik} a_{kj} q_{ip} a_{pj}\right)}$

$= \sqrt{\sum_k \sum_j \underbrace{\left(\sum_i q_{ik}^2\right)}_{1} a_{kj}^2 + \sum_{k \neq p} \sum_j \underbrace{\sum_i q_{ik} q_{ip}}_{0} (a_{kj} a_{pj})}$

$= \|A\|_F$.

2) A symm, pos def. , $b \in \mathbb{R}$

$r_0 = b - Ax_0$

a) CG finds x^* in 1 step $\Leftrightarrow$ r_0 is 0 or an eigenvector of A

„$\Leftarrow$" $r_0 = 0 \to$ clear.

Let r_0 be an ev.

then $\alpha_{(1)} = \dfrac{r_0^T r_0}{r_0^T A r_0} = \dfrac{1}{\lambda}$

& $A x_{(1)} = A(x_{(0)} + \frac{1}{\lambda} r_0) = \cancel{Ax_0 + \lambda(A(b - Ax_{(0)}))}$

$= \cancel{x_0 + Ar_0 = x_0 + A(b - Ax_0)}$

$\cancel{Ax_0 + \lambda^2 r_0 = Ax_0 + \lambda^2(b - Ax_0)}$

$= Ax_0 + r_0 = Ax_0 + b - Ax_0 = b$ ✓.

b) $\cancel{\dfrac{10^2}{10^{2/3}}} \Rightarrow A \quad \dfrac{10^{6/3}}{10^{4/3}} \quad , \quad 10^{4/3} \cdot 10^{-6/3}$

$10^{8/3} \cdot 10^{-\frac{6}{3}} + 10^{4/3}\, 10^{-6/3}$

$10^{2/3} + 10^{-2/3}$

$\underbrace{\dfrac{10^2}{10^{4/3}} + \dfrac{10^{6/3}}{10^{8/3}}}$ $\quad 10^{2/3} + 10^{-2/3}$ ✓

3) $AXA = A$

$XAX = X$

$(AX)^* = AX$

$(XA)^* = XA$

a) Let X & Y be MP.

Then ~~$A = AXA$, $A = AYA$~~

$\Downarrow$

~~$A = AXAXAYA$~~

~~$A = AXA \rightarrow AX = AY$~~

~~$A = AYA$~~

$X = XAX$

$X = XAYAX$

So $X = XAY$
$X = YAX$ $\rightarrow X = Y$.

b) $A = V\Sigma W^*$

i) clear

ii) nachrechnen: clear

iii) $V\Sigma W^* W \Sigma^+ V^* V \Sigma^* W^* = V\Sigma W^*$

& so on.

c) $\|Ax-b\|_2 = \|V\Sigma W^* - b\|_2$

$= \|\Sigma W^* x - V^* b\|_2$

$= \|\Sigma y - V^* b\|$

Σ is diagonal $\Rightarrow$ $y^+ = \Sigma^+ V^* b$

$x^+ = W^* \Sigma^+ V^* b$

f) Gershgorin: $|\lambda - 2| < 200$

$|\lambda - 1| < 100 + \frac{1}{100}$

$|\lambda - 2| < \frac{2}{100}$

$T = \begin{matrix} 1 & & \\ & \alpha & \\ & & \alpha^2 \end{matrix}$

$\Rightarrow TAT^{-1} = \begin{pmatrix} 2 & 10^2 & 10^2 \\ \alpha 10^{-2} & \alpha & \alpha 10^2 \\ \alpha^2 10^{-2} & \alpha^2 10^{-2} & 2\alpha^2 \end{pmatrix} T^{-1}$ $\alpha =$

$= \begin{pmatrix} 2 & \frac{10^2}{\alpha} & \frac{10^2}{\alpha^2} \\ \alpha 10^{-2} & 1 & \frac{10^2}{\alpha} \\ \alpha^2 10^{-2} & \alpha 10^{-2} & 2 \end{pmatrix}$

$|\lambda - 2| < \frac{10^2}{\alpha} + \frac{10^2}{\alpha^2}$

$|\lambda - 2| < \alpha^2 10^{-2} + \alpha 10^{-2}$

$|\lambda - 1| < \frac{10^2}{\alpha} + \alpha 10^{-2}$