USC Geometry and Topology Graduate Exams

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1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable. Compute $f^*(dx_1 \wedge dx_2)$

   Proof. We can distribute the pullback and commute it with the differential operator to obtain:

   $$f^*(dx_1 \wedge dx_2) = df^*(x_1) \wedge df^*(x_2) \tag{0.1}$$ {GT.1993.1.1}

   By definition $f^*(x_1) = x_1(f)$ and $f^*(x_2) = x_2(f)$. Thus, (0.1) becomes:

   $$f^*(dx_1 \wedge dx_2) = d(x_1(f)) \wedge d(x_2(f)) \tag{0.2}$$

   If we now let $f(x_1, x_2) = (u(x_1, x_2), v(x_1, x_2))$, then we obtain $d(x_1(f)) = u_{x_1} dx_1 + u_{x_2} dx_2$ while $d(x_2(f)) = v_{x_1} dx_1 + v_{x_2} dx_2$, so that:

   $$\begin{aligned} f^*(dx_1 \wedge dx_2) &= (u_{x_1} dx_1 + u_{x_2} dx_2) \wedge (v_{x_1} dx_1 + v_{x_2} dx_2) & (0.3)\\ &= u_{x_1} v_{x_1} dx_1 \wedge dx_1 + u_{x_1} v_{x_2} dx_1 \wedge dx_2 + u_{x_2} v_{x_1} dx_2 \wedge dx_1 + u_{x_2} v_{x_2} dx_2 \wedge dx_2 & (0.4)\\ &= u_{x_1} v_{x_2} dx_1 \wedge dx_2 + u_{x_2} v_{x_1} dx_2 \wedge dx_1 \quad \text{because } dx_1 \wedge dx_1 = dx_2 \wedge dx_2 = 0 & (0.5)\\ &= (u_{x_1} v_{x_2} - u_{x_2} v_{x_1}) dx_1 \wedge dx_2 \quad \text{because } dx_1 \wedge dx_2 = -dx_2 \wedge dx_1 & (0.6) \end{aligned}$$

   □

2. Let $f : M \to N$ be a differentiable map between two manifolds, such that f is bijective and such that its tangent map $T_x f : T_x M \to T_{f(x)} N$ is an isomorphism for every $x \in M$. Show that f s a diffeomorphism.

   Proof. Let (U_i, ϕ) and (V_j, ψ) be charts on M and N respectively. Then we can consider the map $\psi f \phi^{-1}$. The tangent map is the Jacobian matrix of this map and since we know this is an isomorphism, it is both defined and its inverse is well defined as well by the inverse function theorem. This is precisely the same as f and f^{-1} being differentiable and thus f is a diffeomorhpism. □

3. Let $B^2 = \{x \in \mathbb{R}^2; ||x|| \leq 1\}$ be the unit disk inthe plane. Let $f : B^2 \to B^2$ be a continuous map such that $f(x) = x$ for every $x \in S^1 = \{x \in \mathbb{R}^2; ||x|| = 1\}$. Show that f is surjective.

   Proof. If f isn't surjective, then there exists a point $y \in B^2$ such that $f(x) - y$ is never 0 for all $x \in B^2$. Thus $g(x) = \frac{f(x)-y}{|f(x)-y|}$ is well defined and exists on S^1. We can consider the homotopy $H(x,t) = \frac{x-ty}{|x-ty|}$ which is a homotopy between 1_{S^1} and $g \circ i)$ where $i : S^1 \to B^2$ is the inclusion map. The chain of maps:

   $$S^1 \xrightarrow{i} B^2 \xrightarrow{g} S^2 \tag{0.7}$$

   gives rise to the homology sequence:

   $$\mathbb{Z} \to 0 \to \mathbb{Z} \tag{0.8}$$

   which gives us the desired contradiction. □

4. *Proof.* Consider a plane in $\mathbb{R}^3$. We can slide this plane such that it starts on one side of M and ends up on the other side. Since this process is continuous, by the intermediate value theorem, there must be a point where this plane contains a point $x \in M$ and M lies entirely on one side of this plane. It is clear that this plane is tangent to M at x. □

5. Is there a covering map $\mathbb{R}^2-2$ points $\to \mathbb{R}^2-$ 1 points?

   Proof. $\mathbb{R}^2-2$ points deformation retracts to a surface with genus 2 while $\to \mathbb{R}^2-$ 1 deformation retracts to a surface of genus 1. Thus, there is no such covering map. □

6. Let U be an open subset of $\mathbb{R}^n$. Show that U is homeomorphic to no open subset of $\mathbb{R}^p$ with $p < n$.

   Proof. Suppose such a homeomorphism $f^{-1} : U \to \mathbb{R}^p$ existed. We first consider the following maps:

$$p : \mathbb{R}^n \to \mathbb{R}^p \quad \text{projection map} \tag{0.9}$$

$$g : U \to \mathbb{R}^n \quad \text{a homeomorphism} \tag{0.10}$$

   We can now consider the function $\phi = p \circ g \circ f^{-1} : f(\mathbb{R}^p) \to \mathbb{R}^n$. the image of this map is open by the invariance of domain , but the map is also seen to be closed. Thus the image is either the entire $\mathbb{R}^n$ or it is empty.

   □

7. Recall that the tangent bundle TM of a manifold M consists of all pairs $(x, \overrightarrow{v})$ where $x \in M$ and $\overrightarrow{v}$ is the tangent space T_xM of M at x. Show that TM is an oriented manifold.

   Proof. An orientable manifold is one such that there exists a continuous vector field. If (ϕ, U) is a chart for M we can obtain a natural chart (Φ, TU) for the tangent bundle by defining $\Phi : TU \to \mathbb{R}^n \times \mathbb{R}^n$ by $\Phi(x, y) = (\phi(x), (D\phi_a^{-1})^{-1}y)$ where D is the derivative map. We just need to check the Jacobian of the transition maps have positive determinant. If (U, Φ) and (V, Ψ) where $\Psi(x, y) = (\psi(x), (D\psi_a^{-1})^{-1}(y))$ are two overlapping charts of TM then the transition maps is for $(x, y) \in \mathbb{R}^{2n}$.

$$\Phi \circ \Psi^{-1}(x, y) = (\phi \circ \psi^{-1}(x), (D\phi_a^{-1})^{-1} \circ D\psi_a^{-1}(y)) \tag{0.11}$$

   Thus, the Jacobian of the transition map $\mathcal{J}(\Phi \circ \Psi^{-1}(x, y))$

$$\begin{vmatrix} \mathcal{J}(\phi \circ \psi^{-1}(x)) & 0 \\ 0 & \mathcal{J}((D\phi_a^{-1})^{-1} \circ D\psi_a^{-1})(y)) \end{vmatrix} \tag{0.12}$$

   □