Geometry and Topology Graduate Exam

Spring 2014

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

Sure. I will.

lifting ⇒ homeomorphic connectedness + compactness

Problem 1. Let X_n denotes the complement of n distinct points in the plane $\mathbb{R}^2$. Does there exist a covering map $X_2 \to X_1$? Explain.

homotope into good pairs

Problem 2. Let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ denote the unit disk, and choose a base point z_0 in the boundary $S^1 = \partial D = \{z \in \mathbb{C}; |z| = 1\}$. Let X be the space obtained from the union of D and $S^1 \times S^1$ by gluing each $z \in S^1 \subset D$ to the point $(z, z_0) \in S^1 \times S^1$. Compute all homology groups $H_k(X; \mathbb{Z})$.

use induced homology

Problem 3. Let $B^n = \{x \in \mathbb{R}^n; \|x\| \leq 1\}$ denote the n-dimensional closed unit ball, with boundary $S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$. Let $f\colon B^n \to \mathbb{R}^n$ be a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$. Show that the origin 0 is contained in the image $f(B^n)$. (Hint: otherwise, consider $S^{n-1} \to B^n \xrightarrow{f} \mathbb{R}^n - \{0\}$.) $\to S^{n-1}$

let $f_*: \frac{\partial}{\partial x} \mapsto X$, $\frac{\partial}{\partial y} \mapsto Y$

check Jacobian

solve ODE's + inverse function thm.

Problem 4. Consider the following vector fields defined in $\mathbb{R}^2$:

$$\mathbf{X} = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial y}.$$

Determine whether or not there exists a (locally defined) coordinate system (s, t) in a neighborhood of $(x, y) = (0, 1)$ such that

$$\mathbf{X} = \frac{\partial}{\partial s}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial t}.$$

induced atlas + check jacobian determinant

Problem 5. Let M be a differentiable (not necessarily orientable) manifold. Show that its cotangent bundle

$$T^*M = \{(x, u); x \in M \text{ and } u\colon T_xM \to \mathbb{R} \text{ linear}\}$$

is a manifold, and is orientable.

stokes thm.

Problem 6. Calculate the integral $\int_{S^2} \omega$ where S^2 is the standard unit sphere in $\mathbb{R}^3$ and where ω is the restriction of the differential 2–form

$$(x^2 + y^2 + z^2)(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)$$

sard's theorem.

Problem 7. Let M be a compact m-dimensional submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. Show that the space of points $x \in \mathbb{R}^m$ such that $M \cap \mathbb{R}^n$ is infinite has measure 0 in $\mathbb{R}^m$.

Spring 2014

① $X_n = \mathbb{R}^2 - \{n\text{-pts}\}$

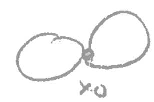
Does $\exists$ a covering map $X_2 \to X_1$? Explain.

Pf $\mathbb{R}^2 - \{pt\} = X_1 \simeq S^1$

$\mathbb{R}^2 - \{2pts\} = X_2 \simeq S^1 \vee S^1$

However $\pi_1(S^1) = \mathbb{Z}$ so if $p: S^1 \vee S^1 \to S^1$ is a cover

$\Rightarrow p_*(\pi_1(S^1 \vee S^1)) \leq \mathbb{Z}$, so $p_*(\pi_1(S^1 \vee S^1)) = m\mathbb{Z}$

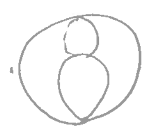
but $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$. Furthermore since $S^1 \vee S^1 \cong$ 

locally $\overset{x_0}{X}$ cannot be homeomorphic to any nbhd of S^1, so

$S^1 \vee S^1$ cannot be a cover of S^1.

② $D = \{|z| \leq 1\}$; X be the union of D & $S^1 \times S^1$ by

gluing each $z \in S^1 \subseteq D$ to $(z, z_0) \in S^1 \times S^1$. Find $H_k(X, \mathbb{Z})$.

Pf Notice that $X \simeq$  $\simeq$  $\simeq S^2 \vee S^1$

so $\tilde{H}_k(S^2 \vee S^1) \simeq \tilde{H}_k(S^2) \oplus \tilde{H}_k(S^1)$ since (S^1, pt) & (S^2, pt)

are good pairs. So $H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 1, 2 \\ 0 & \text{else.} \end{cases}$

(3) $B^n = \{x \in \mathbb{R}^n, \|x\| \le 1\}$, let $f: B^n \to \mathbb{R}^n$ be continuous such that $f(x) = x \ \forall x \in S^{n-1}$. Show $0 \in f(B^n)$.

Pf Suppose not. Then $S^{n-1} \xhookrightarrow{i} B^n \xrightarrow{f} \mathbb{R}^n - \{0\} \xrightarrow{r} S^{n-1}$

$x \longmapsto \frac{x}{|x|}$

then $r \circ f \circ i = id_{S^{n-1}} \Rightarrow (r \circ f \circ i)^* = id^*$

However $0 = H_n(S^{n-1}) \xrightarrow{i^*} H_n(B^n) \Rightarrow i^* \equiv 0$.

So $(r \circ f \circ i)^* = 0 \Rightarrow \Leftarrow$. Thus $0 \in f(B^n)$.

(4) Consider $X = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ & $Y = \frac{\partial}{\partial y}$

and determine whether or not $\exists$ a locally defined coordinate system (s,t) in a nbd of $(0,1)$ $\ni$ $X = \partial/\partial s$ & $Y = \frac{\partial}{\partial t}$.

Pf Let $f(x,y) = (s,t)$ be such a parametrization.

So $s = f_1$ & $t = f_2$. Then we have maps:

$$\begin{array}{ccc} & & \mathbb{R}^2 \ (x,y) \\ & & \downarrow id \\ \underset{(x,y)}{\mathbb{R}^2} & \xrightarrow{f} & \mathbb{R}^2 \end{array}$$

that induce maps on the tangent spaces so that $f_*: T_p\mathbb{R}^2 \to T_p\mathbb{R}^2$

$\Rightarrow f_*\left(\frac{\partial}{\partial x}\right) = X$

$f_*\left(\frac{\partial}{\partial y}\right) = Y$

where:

$$f_* = \begin{array}{ccc} & \partial/\partial x & \partial/\partial y \\ & \left[\begin{array}{c} \partial f_1/\partial x \\ \partial f_2/\partial x \end{array}\right. & \left.\begin{array}{c} \partial f_1/\partial y \\ \partial f_2/\partial y \end{array}\right] \end{array} \begin{array}{c} \partial/\partial x \\ \partial/\partial y \end{array}$$

So then we obtain a system of diff'l eqn's.

$$f_*\left(\frac{\partial}{\partial x}\right) = \frac{\partial f_1}{\partial x}\cdot\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial x}\cdot\frac{\partial}{\partial y} = X = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

& $$f_*\left(\frac{\partial}{\partial y}\right) = \frac{\partial f_1}{\partial y}\cdot\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial y}\cdot\frac{\partial}{\partial y} = Y = \frac{\partial}{\partial y}$$

$$\Rightarrow \quad \frac{\partial f_1}{\partial x} = 2 \qquad \frac{\partial f_2}{\partial x} = x \quad \Rightarrow \quad f_1 = 2x + C(y) = 2x + C$$

$$\frac{\partial f_1}{\partial y} = 0 \qquad \frac{\partial f_2}{\partial y} = 1 \quad \Rightarrow \quad f_2 = \frac{x^2}{2} + y + D$$

Hence if such a parametrization existed it would have to satisfy the equations above.

Hence around $(0,1)$ we see that if $0 = 2x + C \Rightarrow C = 0$, $x = 0$

& $1 = \frac{x^2}{2} + y + D = y + D \Rightarrow D = 0$ $y = 1$

Hence $\{f_1 = 2x, f_2 = \frac{x^2}{2} + y\}$ are candidates. To see if they form a local coordinate system we use the inverse function theorem:

$$\det(f_*) = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_2}{\partial x}\cdot\frac{\partial f_1}{\partial y} = 2 \neq 0 \quad \text{so by IFT}$$

f is a local diffeomorphism around $(0,1)$, & hence a local coordinate system.

(6) Calculate: $\int_{S^2} \omega$, $\omega = (x^2+y^2+z^2)(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge$

Pf $\int_{S^2} \omega = \int_{S^2} x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$

by stokes $= \int_{B^2} 3\,dx\wedge dy\wedge dz = 3\cdot vol(B^2)$

(7) M compact m-dim submanifold of $\mathbb{R}^m \times \mathbb{R}^n$.
Show the space of points $x \in \mathbb{R}^m \ni M \cap \mathbb{R}^n$ is infinite has measure 0 in $\mathbb{R}^m$.

Pf Consider

$$\begin{tikzcd} M \arrow[r, hook, "i"] \arrow[dr, "f = \pi\circ i"'] & \mathbb{R}^m \times \mathbb{R}^n \arrow[d, "\pi"] \\ & \mathbb{R}^m \end{tikzcd}$$

Then $f = \pi\circ i$ is smooth so the set of critical values has measure zero in $\mathbb{R}^m$. So then let $x \in \mathbb{R}^m \cap M \ni x$ is a regular value.
Then $f_{*x}: T_p M \to T_q \mathbb{R}^m$ is surjective & nonzero, so locally by the inverse function theorem, x is locally diffeomorphic to $f^{-1}(x)$.
So then if $y \in f^{-1}(x)$ $\exists$ $U_y \ni y \in U_y \simeq f(U) = W \ni x$
Now, $\{x\}$ is closed in $\mathbb{R}^m$, so $f^{-1}(x)$ is closed in M & hence compact since M is compact. However, $\{U_y\}_{y\in f^{-1}(x)}$ is an open cover for $f^{-1}(x)$ so by compactness, $\exists$ finite subcover, so $f^{-1}(x) \subseteq \bigcup_{i=1}^{k} U_{y_i}$

Now since each $\mathcal{U}_{y_i} \simeq W$, if $\mathcal{U}_{y_\alpha} \cap \mathcal{U}_{y_\beta} \neq \emptyset \Rightarrow \mathcal{U}_{y_\alpha} = \mathcal{U}_{y_\beta}$

Hence $\exists$ finitely many points in the fiber of x.

Thus $\{y : (x,y) \in M\}$ is finite.

In particular this implies the fibers of all regular points are finite so necessarily, all points w/ infinite fibers are critical, which by Sard's theorem, have measure zero.

Geometry/Topology Qualifying Exam - Fall 2014

$G(\tilde{X}) \simeq \pi_1(X)$

#sheets $= |\pi_1(X)|$

1. Show that if (X, x) is a pointed topological space whose universal cover exists and is compact, then the fundamental group $\pi_1(X, x)$ is a finite group.

2. Recall that if (X, x) and (Y, y) are pointed topological spaces, then the wedge sum (or 1-point union) $X \vee Y$ is the space obtained from the disjoint union of X and Y by identifying x and y. Show that T^2 (the 2-torus $S^1 \times S^1$) and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups, but are not homeomorphic.

3. Suppose S^n is the standard unit sphere in Euclidean space and that $f : S^n \to S^n$ is a continuous map.

   construct homotopy

   degree $\Rightarrow \pm 1 \Rightarrow \varepsilon$

   i) Show that if f has no fixed points, then f is homotopic to the antipodal map.

   ii) Show that if $n = 2m$, then there exists a point $x \in S^{2m}$ such that either $f(x) = x$ or $f(x) = -x$.

Kunneth formula.

4. If M is a smooth manifold of dimension d, using basic properties of de Rham cohomology, describe the de Rham cohomology groups $H^*_{dR}(S^1 \times M)$ in terms of the groups $H^*_{dR}(M)$ (along the way, please explain, quickly and briefly, how to compute $H^*_{dR}(S^1)$).

Gauss-Bonnet

5. Show that if $X \subset \mathbb{R}^3$ is a closed (i.e., compact and without boundary) submanifold that is homeomorphic to a sphere with $g > 1$ handles attached, then there is a non-empty open subset on which the Gaussian curvature K is negative.

stokes + cartan

6. Suppose M is a (non-empty) closed oriented manifold of dimension d. Show that if ω is a differential d-form, and X is a (smooth) vector field on X, then the differential form $\mathfrak{L}_X \omega$ necessarily vanishes at some point of M.

atlas

7. Let V be a 2-dimensional complex vector space, and write $\mathbb{CP}^1$ for the set of complex 1-dimensional subspaces of V. By explicit construction of an atlas, show that $\mathbb{CP}^1$ can be equipped with the structure of an oriented manifold.

Fall 2014

1) (X,x) is a pointed topological space whose universal cover exists & is compact then $\pi_1(X,x) < \infty$.

Pf Suppose $\tilde{X} \xrightarrow{p} X$ is the universal cover.

Then $\pi_1(\tilde{X}) = 0$ so $p_*(\pi_1(\tilde{x})) = 0$ is normal in $\pi_1(X)$.

So then $G(\tilde{X}) \simeq \pi_1(X)$. Now $G(\tilde{x})$ acts freely on $\tilde{X}$ & on the fiber of any $x \in X$, hence the number of sheets in the cover is precisely the number of elements in $G(\tilde{x})$ (since it acts transitively on the fiber of $x \in X$).

But $\tilde{X}$ is compact and the map $\tilde{X} \to X$ surjective so X is also compact, hence the cover is finitely sheeted $\Rightarrow \pi_1(X) < \infty$.

Lemma: $\tilde{X}$ compact & X compact $\Rightarrow$ cover is finitely sheeted.

Pf Suppose
$$\begin{tikzcd} \tilde{X} \arrow[d] & \tilde{x}_i \arrow[d] \cdots \\ X & x \end{tikzcd}$$
, then for any x, $\exists$ nbh such that s/t $\tilde{U}^x_\alpha = p^{-1}(U) \simeq U$. Now each α corresponds to a distinct sheet of the cover.

In particular, $\bigcup \tilde{U}^x_\alpha$ form a cover for $\tilde{X}$, so $\exists$ a finite subcover, by compactness of $\tilde{X}$. $\Rightarrow$ there are finitely many $\tilde{U}^x_\alpha$ so there are finitely many sheets.

② Show that $S^1 \times S^1$ & $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups but are not homeomorphic.

Pf. $H_k(S^1 \times S^1) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}\times\mathbb{Z} & k=1 \\ \mathbb{Z} & k=2 \\ 0 & \text{else} \end{cases}$ (Recall $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$)

can show w/ Mayer Vietoris
let $A = S^1 \times S^1 \setminus$ north pole $\simeq S^1$
$B = S^1 \times S^1 \setminus$ south pole $\simeq S^1$

& $\tilde{H}_k(S^1 \vee S^1 \vee S^2) = \tilde{H}_k(S^1) \oplus \tilde{H}_k(S_1) \oplus \tilde{H}_k(S^2)$

(since all good pairs) $\Rightarrow$ $\tilde{H}_k(S^1 \vee S^1 \vee S^2) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}\times\mathbb{Z} & k=1 \\ \mathbb{Z} & k=2 \\ 0 & \text{else.} \end{cases}$

However $\pi_1(S^1 \times S^1) = \mathbb{Z}\times\mathbb{Z}$ but $\pi_1(S^1 \vee S^1 \vee S^2) = \mathbb{Z} * \mathbb{Z}$

so not homeomorphic since $\mathbb{Z}\times\mathbb{Z}$ is abelian but $\mathbb{Z} * \mathbb{Z}$ is not.

③ Show that if $f: S^n \to S^n$ is a continuous map.

(i) if $f(x) \neq x \ \forall x \Rightarrow f \simeq \alpha : x \mapsto -x$

(ii) if $n = 2m$ then ... $\exists \ x \in S^{2m} \ni f(x) = x$ or $f(x) = -x$

Pf (i) $f(x) \neq x \Rightarrow h(t,x) = \dfrac{-xt + (1-t)f(x)}{|-xt + (1-t)f(x)|}$ is a homotopy between f & α, since denominator is never zero.

(ii) if $f(x) \neq x$ & $f(x) \neq -x$ then f & $-f$ are both homotopic to antipodal map $\Rightarrow \deg f = (-1)^{2m+1}$ & $\deg(-f) = (-1)^{2m+1}$

$\Rightarrow \deg f = -1$ and $\deg f = 1$ $\Rightarrow \leftarrow$.

So one of these must occur.

④ M - dim d, smooth.

Describe $H^*_{dR}(S^1 \times M)$ in terms of $H^*_{dR}(M)$, & how to obtain $H^*_{dR}(S^1)$.

Pf: Recall the Künneth formula:

$$H^*_{dR}(S^1 \times M) \cong H^*_{dR}(S^1) \otimes_{\mathbb{R}} H^*_{dR}(M)$$

but $H^*_{dR}(S^1 \times M)$ is graded so:

$$H^n_{dR}(S^1 \times M) = \bigoplus_{i+j=n} \left(H^i_{dR}(S^1) \otimes_{\mathbb{R}} H^j_{dR}(M) \right)$$

However $H^i_{dR}(S^1) = \begin{cases} \mathbb{R} & i = 0,1 \\ 0 & \text{else} \end{cases}$

so $$H^n_{dR}(S^1 \times M) \simeq \left(\mathbb{R} \otimes_{\mathbb{R}} H^n_{dR}(M) \right) \oplus \left(\mathbb{R} \otimes_{\mathbb{R}} H^{n-1}_{dR}(M) \right)$$

$$H^n_{dR}(S^1 \times M) \simeq H^n_{dR}(M) \oplus H^{n-1}_{dR}(M)$$

Now for S^1 note that $H^0_{dR}(S^1) = \ker d : \Omega^0 \to \Omega^1$

i.e. $\{f;\ df = 0\} \Rightarrow \{f = \text{constant}\} = \mathbb{R}$.

and if $w = d\theta$ - an orientation on S^1, then $\int_{S^1} d\theta = 2\pi$

so then for any other $\eta \in \Omega^1(S^1)$, we have that if $c = \frac{1}{2\pi} \int_{S^1} \eta$

then $\int_{S^1} (\eta - c \cdot w) = 0 \Rightarrow [\eta - c w] \equiv 0 \Rightarrow [\eta] = c \cdot [w]$.

That is $[w]$ generates $H^1_{dR}(S^1)$. So $H^1_{dR}(S^1) = \mathbb{R}$.

(5) $X \subseteq \mathbb{R}^3$ closed (compact w/o boundary) submanifold homeomorphic to sphere w/ $g > 1$ handles attached, then $\exists$ nonempty open subset on which gaussian curvature is neg.

Pf $X \simeq$ surface of genus g.

so by Gauss-Bonnet theorem:

$$\int_X K(g) = 2\pi\,\chi(X) = 2\pi(2-2g) < 0 \cdot \text{ since } g < 1$$

so then the set on which $K(g) < 0$ has nonempty interior

in particular, $\exists\, p$ & $B_{\epsilon,p} \ni p \ni K(x) < 0 \quad \forall x \in B_{\epsilon,p}$.

(6) M - nonempty closed oriented manifold of dimension d

Show if ω is a d-form & X - smooth v.f. on X then $\mathcal{L}_X\omega$ necessarily vanishes somewhere.

Pf Recall cartan's formula & stoke's theorem:

$$\int_M \mathcal{L}_X \omega = \int_M i_X(d\omega) + d\, i_X(\omega) = \int_M d\, i_X(\omega)$$

$d\omega = 0$ since M is d-dim so $\Omega^{n+1}(M) = 0$ & ω is a d-form

$$\text{so by stokes} = \int_{\partial M} i_X(\omega) = \int_{\emptyset} i_X\omega = 0.$$

Since $M \neq \emptyset$ then $\mathcal{L}_X\omega = 0$ somewhere by continuity of the Lie derivative.

$\mathbb{C}P^1 = \{1$ Dimensional subspaces of $\mathbb{C}^2\}$. = lines in $\mathbb{C}^2$

each line has equation $az_1 + bz_2 = 0$; $a, b \in \mathbb{C}$, not a or $b \neq 0$.

So each line is representable by $[a,b]$ where $[a,b] \sim [\lambda a, \lambda \beta]$

$\forall \ \lambda \in \mathbb{C} \setminus \{0\}$.

So consider $\mathcal{U}_1 = \{[a,b] \ni a \neq 0\}$, $\mathcal{U}_2 = \{[a,b] ; b \neq 0\}$.

Then, $\mathbb{C}^2_{\setminus\{0\}} \xrightarrow{\pi} \mathbb{C}P^1$ then $\pi^{-1}(\mathcal{U}_1) = \mathbb{C}^2 - \mathbb{C} \simeq \mathbb{C}$ is open

$(a,b) \longmapsto [a,b]$ $\quad \pi^{-1}(\mathcal{U}_2) \cong \mathbb{C}$ also open.

Since projections are open maps $\Rightarrow \mathcal{U}_1, \mathcal{U}_2$ are open & cover $\mathbb{C}P^1$.

Now, consider $\varphi_1 : \mathcal{U}_1 \xrightarrow{b \neq 0} \mathbb{C}$ then φ_1 is invertible since

$[a,b] \longrightarrow \frac{a}{b}$ for any $c \in \mathbb{C} \xrightarrow{\varphi_1^{-1}} [c, 1]$

Likewise $\varphi_2 : \mathcal{U}_2 \xrightarrow{a \neq 0} \mathbb{C}$ likewise invertible. $\varphi_2^{-1}(d) = [1, d]$

$[a,b] \longrightarrow \frac{b}{a}$

So then $\varphi_2^{-1} \circ \varphi_1 : \mathcal{U}_1 \cap \mathcal{U}_2 \longrightarrow \mathcal{U}_2$; so $a, b \neq 0$

$\varphi_2^{-1} \varphi_1([a,b]) = \varphi_2^{-1}\left(\frac{a}{b}\right) = [1, \frac{a}{b}] = [b, a] \in \mathcal{U}_2$ ✓

$\varphi_1^{-1} \varphi_2([a,b]) = \varphi_1^{-1}\left(\frac{b}{a}\right) = [b/a, 1] = [b, a] \in \mathcal{U}_1$ ✓

smooth anywhere on $\mathcal{U}_1 \cap \mathcal{U}_2$.

Now by the quotient topology $\Rightarrow \mathbb{C}P^1$ is Hausdorff & 2nd countable,

as well as orientable.

GEOMETRY TOPOLOGY QUALITFYING EXAM
SPRING 2013

Solve all of the problems that you can. Partial credit will be given for partial solutions.

(1) Consider the form
$$\omega = (x^2 + x + y)dy \wedge dz$$
on $\mathbb{R}^3$. Let $S^2 = \{x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$ be the unit sphere, and $i\colon S^2 \to \mathbb{R}^3$ the inclusion.

stokes. (a) Calculate $\int_{S^2} \omega$.

stokes (b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

inverse fnl. thm.

(2) Find all points in $\mathbb{R}^3$ in a neighborhood in which the functions $x, x^2+y^2+z^2-1, z$ can serve as a local coordinate system.

chart

(3) Prove that the real projective space $\mathbb{R}P^n$ is a smooth manifold of dimension n.

$H^1_{dR}(S^n) = 0$

$\mathbb{R}P^n \hookleftarrow S^n \twoheadrightarrow \mathbb{R}P^n$

(4) (a) Show that every closed 1-form on S^n, $n > 1$ is exact.
(b) Use this to show that every closed 1-form on $\mathbb{R}P^n$, $n > 1$ is exact.

homotope S/S'

(5) Let X be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes. Calculate $\pi_1(X)$ and $H_*(X)$.

draw picture

(6) Let $X = T^2 - \{p, q\}$, $p \neq q$ be the twice punctured 2-dimensional torus.
(a) Compute the homology groups $H_*(X, \mathbb{Z})$.
(b) Compute the fundamental group of X.

$\mathbb{Z}/2\mathbb{Z} \hookrightarrow S^2$

(7) (a) Find all of the 2-sheeted covering spaces of $S^1 \times S^1$.
(b) Show that if a path-connected, locally path connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.

covering map

(8) (a) Show that if $f\colon S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.
(b) Show that if X has S^{2n} as universal covering space then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

recall $\pi_1(X) \simeq G(S^{2n})$
and G acts freely on S^{2n}

Date: February 1, 2013.

Spring 2013

① $w = (x^2+x+y)\, dy \wedge dz$ on $\mathbb{R}^3$

Let $i : S^2 \hookrightarrow \mathbb{R}^3$

(a) Calculate $\int_{S^2} \omega$

(b) Construct a closed form α on $\mathbb{R}^3 \ni i^*\alpha = i^*\omega$ or show that such a form α does not exist

Pf (a)
$$\int_{S^2} (x^2+x+y)\, dy \wedge dz \overset{\text{Stokes}}{=} \int_{B^2} d(x^2+x+y\, dy \wedge dz)$$

$$= \int_{B^2} 2x\, dx \wedge dy \wedge dz + \int_{B^2} dx \wedge dy \wedge dz$$

Now B^2 is a symmetric domain & x an antisymmetric function

so $\int_{B^2} x\, dx \wedge dy \wedge dz = 0$ & $\int_{B^2} dx \wedge dy \wedge dz = \mathrm{Vol}(B^2)$.

(b) if it did then $d\alpha = 0$, $i^* : H^k_{dR}(\mathbb{R}^3) \to H^k_{dR}(S^2)$

but if $\int_{S^2} i^*\alpha = \int_{S^2} i^*\omega = \int_{S^2} \omega \neq 0$

However $\int_{S^2} i^*\alpha = \int_B d(i^*\alpha) = \int_B i^*(d\alpha) = \int_B 0 = 0$

$\Rightarrow \Leftarrow$. So it cannot exist.

(2) Let $p(x,y,z) = (x, x^2+y^2+z^2-1, z)$
Where can it be a local coordinate system?

Pf consider $p_* = \begin{pmatrix} 1 & 0 & 0 \\ 2x & 2y & 2z \\ 0 & 0 & 1 \end{pmatrix}$

so $\det p_* = 2y = 0$ iff $y=0$.

so it works anywhere on $\mathbb{R}^3 - \{y \text{ axis}\}$.

(4) (a) Show every closed 1-form on S^n, $n>1$ is exact
(b) use this to show that every closed 1 form on $\mathbb{R}P^n$ $n>1$ is exact.

Pf (a) Since $H^1_{dR}(S^n) = 0$ then if $d\eta = 0$ for $\eta \in \Omega^1(S^1)$

then $\eta \in H^1_{dR}(S^n)$. But then $[\eta] = 0 \Rightarrow \eta \in \operatorname{im} d: \Omega^0 \to \Omega^1$

so $\exists \beta \in \Omega^0 \ni d\beta = \eta$, so η is exact.

(b) consider $\mathbb{R}P^n \xhookrightarrow{p} S^n \xrightarrow{q} \mathbb{R}P^n$. Then $q \cdot p = id_{\mathbb{R}P^n}$

$[x] \longmapsto x \longmapsto [x]$

so $(q \cdot p)^* = id^*$. However $p^*: H^1_{dR}(S^n) \to H^1_{dR}(\mathbb{R}P^n)$

& $H^1(S^1) = 0 \Rightarrow p^* = 0$

But then $(q \circ p)^*: H^1_{dR}(\mathbb{R}P^n) \to H^1_{dR}(\mathbb{R}P^n)$ & $(q \circ p)^* = id$

so $H^1_{dR}(\mathbb{R}P^n) = 0$. Thus all closed 1 forms are exact.

Spring 2013

(3) Prove $\mathbb{R}P^n$ is smooth manifold of dim n.

Consider $\mathbb{R}^{n+1} - \{0\} \xrightarrow{\pi} \mathbb{R}P^n$. Then this map is continuous
$x \longmapsto [x]$ and open so it induces

a quotient topology on $\mathbb{R}P^n$.

Now, let $\tilde{U}_i \subseteq \mathbb{R}^{n+1} - \{0\} \ni x_i \neq 0$, & let $U_i = \pi(\tilde{U}_i)$

Then U_i is open and $\bigcup U_i \supseteq \mathbb{R}P^n$.

Now consider $\varphi_i : U_i \to \mathbb{R}^n \ni [x_1 \ldots x_{n+1}] \to \left(\frac{x_1}{x_i}, \ldots, \hat{x_i}, \ldots, \frac{x_n}{x_i}\right)$

then this map is cont's & invertible w/

$\varphi^{-1}(x_1, \ldots, x_n) \to [x_1, \ldots, 1, \ldots, x_n]$

and smooth composition on intersections $U_i \cap U_j$.

So $\{U_i, \varphi_i\}$ are a chart to $\mathbb{R}^n$.

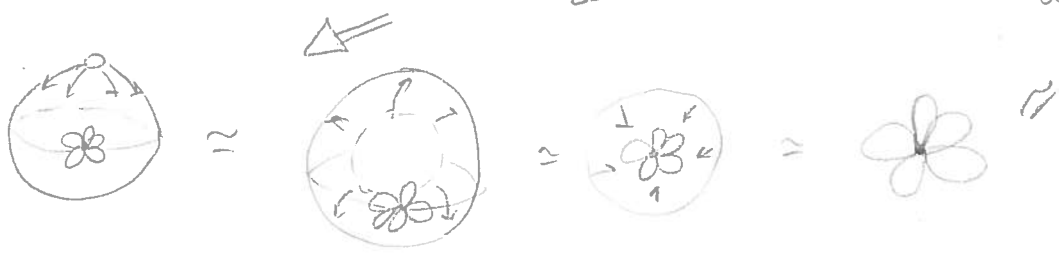
Lastly since $\mathbb{R}^{n+1}$ is Hausdorff & second countable via π

$\Rightarrow$ $\mathbb{R}P^n$ is also so $\mathbb{R}P^n$ is a n-dim smooth manifold.

⑤ Let $X = \mathbb{R}^3 -$ all axis. Find $\pi_1(X)$ & $H_*(X)$

Pf $\mathbb{R}^3 -$ 3axis $\Rightarrow$  $= S^1 - 6$ pts $= \overset{5}{\vee} S^1$

wedge of 5 circles.



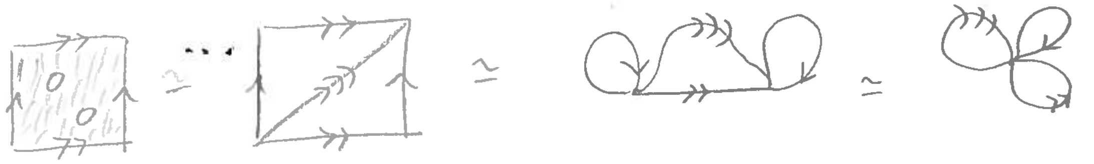
So $\pi_1(X) = \overset{5}{*} \mathbb{Z}$

& $H_*(X) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^5 & k=1 \\ 0 & \text{else.} \end{cases}$

⑥ Let $X = T^2 - \{p,q\}$ $p \neq q$.

(a) find $H_*(X, \mathbb{Z})$

(b) $\pi_1(X)$.

Pf $X =$  $\simeq \overset{3}{\vee} S^1$

So $H_*(\overset{3}{\vee} S^1) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^3 & k=1 \\ 0 & \text{else} \end{cases}$ $\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

⑦ (a) Find all twosheeted covering spaces of $S^1 \times S^1$.
(b) Show if $\pi_1(X)$ is finite then $\forall f: X \to S^1$ is nullhomotopic.

Pf (a) want homomorphisms from $\mathbb{Z} \times \mathbb{Z} \to S_2$
$\Rightarrow \exists$ 4 such maps. $\Rightarrow$ 4 two sheeted connected covers.

(b) if $\pi_1(X)$ is finite then since $f_*(\pi_1(X)) \leq \pi_1(S^1) = \mathbb{Z}$
& the only finite subgroup of $\mathbb{Z}$ is $\{0\} \Rightarrow \pi_1(X) = 0$.

Thus $\exists$ lift
$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & S^1 \end{tikzcd} \Rightarrow p \circ \tilde{f} = f.$$

But $\mathbb{R}$ is contractible so $\tilde{f} \simeq$ homotopic constant map $\Rightarrow$ $p \circ \tilde{f}$ homotopic to constant map. $\Rightarrow f \simeq e$.

⑧ (a) if $f: S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$
(b) Show if X has S^{2n} as a universal cover then $\pi_1(X) = \{1\}$ or $\mathbb{Z}/2$.

Pf (a) if $f(x) \neq x \Rightarrow f \simeq \alpha: x \mapsto -x \Rightarrow \deg f = \deg \alpha = (-1)^{n+1}$
(b) if S^{2n} is the universal cover then $\pi_1(X) \simeq G(S^{2n})$ the deck transformations of S^{2n}. However $G(S^{2n})$ acts freely on $S^{2n} \Rightarrow G(S^{2n}) = \mathbb{Z}/2\mathbb{Z}$ or $\{1\}$. $\Rightarrow \pi_1(X) = \{1\}$ or $\mathbb{Z}/2\mathbb{Z}$

To see why consider $\gamma \in G(S^{2n})$ so $\gamma: S^{2n} \to S^{2n}$. Then $\deg \gamma = \pm 1$ so $\exists$ homomorphism $G(S^{2n}) \to \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$. Now each γ has no fixed points so $\deg \gamma = (-1)^{2n+1} = -1 \Rightarrow G(S^{2n}) = \mathbb{Z}/2\mathbb{Z}$ (unless S^{2n} consists of only one point).

incomplete

Geometry/Topology Qualifying Exam

Fall 2013

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

$S^2 \vee S^1$ cellular homology cellular chain complex

1. (15 pts) Let X denote S^2 with the north and south poles identified.
   (a) (5 pts) Describe a cell decomposition of X and use it to compute $H_i(X)$ for all $i \geq 0$.
   (b) (5 pts) Compute $\pi_1(X)$.
   (c) (5 pts) Describe (i.e., draw a picture of) the universal cover of X and all other connected covering spaces of X.

open covers. + inv. fct thm.

2. (10 pts) Show that if M is compact and N is connected, then every submersion $f : M \to N$ is surjective.

regular val thm.

3. (10 pts) Show that the orthogonal group $O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = id\}$ is a smooth manifold. Here $M_n(\mathbb{R})$ is the set of $n \times n$ real matrices.

do generating $H^1_{dR}(S^1)$

4. (10 pts) Compute the de Rham cohomology of $S^1 = \mathbb{R}/\mathbb{Z}$ from the definition.

Hatcher

5. (10 pts) Let X, Y be topological spaces and $f, g : X \to Y$ two continuous maps. Consider the space Z obtained from the disjoint union $(X \times [0,1]) \sqcup Y$ by identifying $(x, 0) \sim f(x)$ and $(x, 1) \sim g(x)$ for all $x \in X$. Show that there is a long exact sequence of the form:
$$\cdots \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to \dots$$

?

6. (10 pts) A lens space $L(p, q)$ is the quotient of $S^3 \subset \mathbb{C}^2$ by the $\mathbb{Z}/p\mathbb{Z}$-action generated by $(z_1, z_2) \mapsto (e^{2\pi i/p}z_1, e^{2\pi iq/p}z_2)$ for coprime p, q.
   (a) (5 pts) Compute $\pi_1(L(p,q))$.
   (b) (5 pts) Show that any continuous map $L(p,q) \to T^2$ is null-homotopic.

embed in $\mathbb{RP}^2$

7. (10 pts) Consider the space of all straight lines in $\mathbb{R}^2$ (not necessarily those passing through the origin). Explain how to give it the structure of a smooth manifold. Is it orientable?

Fall 2013

(1) $X = S^2/\sim$ north pole $\sim$ s. pole.

(a) Describe a cell decomposition of X & compute $H_i(X)$ $\forall i \geq 0$.

(b) $\pi_1(X)$

(c) Draw univ. cover of X & all other connected covering spaces.

Pf a) $X \simeq$  $\simeq S^2 \vee S^1$ 

so $\Rightarrow$
$\Delta_0 = 1$ zero cell $= \langle v \rangle$
$\Delta_1 = 1$ one cell $\Rightarrow \langle a \rangle$ - glued along v
$\Delta_2 = 1$ two cell $\Rightarrow \langle B \rangle$ - glued along S^1

so $0 \xrightarrow{\partial_3} \Delta_2 \xrightarrow{\partial_2} \Delta_1 \xrightarrow{\partial_1} \Delta_0 \xrightarrow{\partial_0} 0$

$\ker \partial_0 = \langle v \rangle$ $\quad$ $\ker \partial_1 = \langle a \rangle$ $\quad$ $\ker \partial_2 = \langle B \rangle$ $\quad$ $\ker \partial_3 = 0$

$\operatorname{Im} \partial_0 = 0$ $\quad$ $\operatorname{Im} \partial_1 = 0$ $\quad$ $\operatorname{Im} \partial_2 = 0$ $\quad$ $\operatorname{Im} \partial_3 = 0$

so $H_0 = \dfrac{\ker \partial_0}{\operatorname{Im} \partial_1} = \mathbb{Z}$ ... $\quad H_1 = \dfrac{\ker \partial_1}{\operatorname{Im} \partial_2} = \dfrac{\langle a \rangle}{0} = \mathbb{Z}$

$H_2 = \dfrac{\ker \partial_2}{\operatorname{Im} \partial_3} = \dfrac{\langle B \rangle}{0} = \mathbb{Z}$ $\Rightarrow$ $H_k(X) = \begin{cases} \mathbb{Z} & \text{for } k = 0,1,2 \\ 0 & \text{else.} \end{cases}$

(b) $\pi_1(X) = \pi_1(S^2) * \pi_1(S^1) = \mathbb{Z}$

(c) the universal cover is an infinite wedge of S^2 & $\mathbb{R}$

all other connected subspace must be compact $\Rightarrow$ bouquets of S^2 & $\mathbb{R}$

$\Rightarrow$  n sheeted $\Rightarrow$ n times since all correspond to $n\mathbb{Z} \leq \mathbb{Z} = \pi_1(X)$.

(2) M compact, N is connected then $f: M \to N$ is surjective. for $\forall f$ - submersions.

Pf If f is a submersion $\Rightarrow$ $f_*: T_pM \to T_qN$ is surjective. so then all points in M are regular points so then $f_{*,p} \neq 0$ $\Rightarrow$ by the inverse function theorem $\forall\ p \in M\ \exists$ nbd $\ni$ f is a local diffeomorphism. But then let U_p be such nbds $\ni$ $f(U_p) \simeq U_p$. Then M is compact so $f(M)$ is compact $\Rightarrow$ closed. However by compactness $\exists$ finite subcover of points p_i $\ni$ $M = \bigcup^k U_i$ $\Rightarrow$ $f(M) = \bigcup^k f(U_i)$ is open in N since $f(U_i) \simeq U_i$ for each i. $\Rightarrow$ f(M) is both open & closed in N. Since N is connected $\Rightarrow$ $f(M) = \phi$ or $f(M) = N$. Since $M \neq \phi$ $\Rightarrow$ $f(M) = N$ $\Rightarrow$ f is surjective

(4) Compute de Rham Cohomology of S^1 from defn:

Pf Recall $S^1 \simeq \mathbb{R}^2 - \{pt\}$

so then $H^k_{dR}(S^1) = \dfrac{\ker d: \Omega^k \to \Omega^{k+1}}{\operatorname{Im} d: \Omega^{k-1} \to \Omega^k}$

so $\Omega^0 = \{f \in C^\infty(S^1)\} \Rightarrow \ker d: \Omega^0 \to \Omega^1 = \{f,\ df = 0\}$.

But locally $S^1 \simeq \mathbb{R}$ so $\Rightarrow f' = 0 \Rightarrow f$ is constant.

Thus $\ker d_0 = \{c;\ c \in \mathbb{R}\} = \mathbb{R}. \Rightarrow H^0_{dR}(S^1) = \mathbb{R}$.

Now, let $w \in \Omega^1(\mathbb{R}^2 - \{p\})$. Then on S^1 $w|_{S^1} = d\theta$ is an orientation form so then $\int_{S^1} w = \int_{S^1} d\theta = 2\pi$.

So then if $\eta \in \Omega^1(\mathbb{R}^2 - \{p\})$ then if $\eta \neq 0 \Rightarrow$ let $c = \frac{1}{2\pi} \int_{S^1} \eta$.

then $\int_{S^1} \eta - cw = 0 \Rightarrow [\eta - cw] = 0 \Rightarrow [\eta] = c[w]$

i.e. $[w]$ generates $H^1_{dR}(S^1) \Rightarrow H^1_{dR}(S^1) = \mathbb{R}$.

...

(5) In Hatcher

(6) ??

(7) Consider all lines in $\mathbb{R}^2$

Explain how to give it the structure of a smooth manifold.

Is it orientable?

~~Pf~~ any line has the form $ax+by+c=0$ for $a,b,c \in \mathbb{R}$.

Since $\lambda(ax+by+c)=0$ represents the same line $\exists$ natural

map from this set: $S \longrightarrow \mathbb{R}P^2$

$ax+by+c \longmapsto [a,b,c]$

In particular, we can consider $\mathcal{U}_a = \{[a,b,c] ; a \neq 0\}$

$\mathcal{U}_b = \{[a,b,c] ; b \neq 0\}$ & $\mathcal{U}_c = \{[a,b,c] ; c \neq 0\}$.

then $\mathcal{U}_a = \{[1, b/a, c/a] ; a \neq 0\}$

Now, $\mathcal{U}_a^c = \{[0,b,c]\}$ which is closed in $\mathbb{R}P^2$ by quotient topology.

So $\mathcal{U}_a$ is open.

Now let $\varphi_a : \mathcal{U}_a \longrightarrow \mathbb{R}^2$ w/ inverse $\varphi_a^{-1}(b,c) \to [1,b,c]$

$[1, b/a, c/a] \longrightarrow [\frac{b}{a}, \frac{c}{a}]$

then $\{(\varphi_i, \mathcal{U}_i)\}$ are a chart for S, hence S is a smooth

manifold. (Clearly 2nd countable & Hausdorff are inherited

from $\mathbb{R}^2$ or $\mathbb{R}P^2$)

Now $\varphi_a^{-1}\varphi_b([1,1,c]) = \varphi_a^{-1}((1,c)) = [1,1,c]$

$\varphi_a^{-1}\varphi_c([1,b,1]) = \varphi_a^{-1}((1,b)) = [1,1,b]$

$\varphi_c^{-1}\varphi_a([1,b,1]) = \varphi_c^{-1}((b,1)) = [b,1,1]$

so not orientable? $\Rightarrow$ compute transition matrix & find determinant.

Geometry-Topology Qualitfying exam
Fall 2012

Solve all of the problems. Partial credit will be given for partial answers.

1. Denote by $S^1 \subset \mathbf{R}^2$ the unit circle and consider the torus $T^2 = S^1 \times S^1$. Now, define $A \subset T^2 = S^1 \times S^1$ by

$$A = \{(x, y, z, w) \in T^2 \mid (x, y) = (0, 1) \text{ or } (z, w) = (0, 1)\ \}.$$

Compute $H^*(T^2, A)$. *Here we regard S^1 as a subset of the plane, hence we indicate points on S^1 as ordered pairs.*

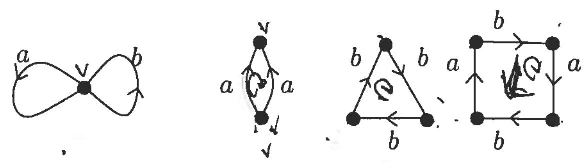
wrong! mistake?

2. Denote by S^1 and S^2 the circle and sphere respectively. Recall that the definition of the smash product $X \wedge Y$ of two pointed spaces is the quotient of $X \times Y$ by $(x, y_0) \sim (x_0, y)$.

Show that $S^1 \times S^1$ and $S^1 \wedge S^1 \wedge S^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not.

3. Let X be a CW-complex with one vertex, two one cells and 3 two cells whose attaching maps are indicated below.

Cellular homology



1-skeleton 2-skeleton

(a) Compute the homology of X.

(b) Present the fundamental group of X and prove its nonabelian.

(*Justify your work.*)

$\mathbb{R}P^2$ compact + $\mathbb{R}^2$ connected + d_x surj $\Rightarrow \phi$ is surjective

4. Does there exist a smooth embedding of the projective plane $\mathbf{R}P^2$ into $\mathbf{R}^2$? Justify your answer.

Consider $x \xrightarrow{\phi_f} xf$

5. Let M be a manifold, and let $C^\infty(M)$ be the algebra of C^∞ functions $M \to \mathbf{R}$. Explain the relationship between vector fields on M and $C^\infty(M)$. If we consider the vector fields X and Y as maps $C^\infty(M) \to C^\infty(M)$ is the composition map XY also a vector field? What about $[X, Y] = XY - YX$? Explain.

$X = \partial/\partial x$ $\quad Y = \partial/\partial y$
$f = x$ $\quad g = y$

Stokes.

6. Let S be the unit sphere defined by $x^2 + y^2 + z^2 + w^2 = 1$ in $\mathbf{R}^4$. Compute $\int_S \omega$ where $\omega = (w + w^2) dx \wedge dy \wedge dz$.

Regular value thm.

7. Does the equation $x^2 = y^3$ define a smooth submanifold in $\mathbf{R}^3$? Prove your claim.

Fall 2012

① ??

② Mistake? $S^1 \wedge S^1 \wedge S^2 \simeq S^4$

so $H_n(S^4) = \begin{cases} 0 & n \neq 0,4 \\ \mathbb{Z} & n = 0,4 \end{cases}$ but $H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\times\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{else} \end{cases}$

③ Let X be a CW complex w/ one vertex, two one cells & 3 two cells w/

(a) Compute homology of X

(b) Present the fundament gp & show it is nonabelian

Pf $0 \to \Delta_2 \xrightarrow{\partial_2} \Delta_1 \xrightarrow{\partial_1} \Delta_0 \xrightarrow{\partial_0} 0$ $\quad$ $\ker \partial_0 = V$ $\quad$ $\operatorname{im} \partial_0 = 0$

$0 \to \langle U, F, L \rangle \to \langle a, b \rangle \to \langle v \rangle$

$\partial_1(a) = v - v = 0$ $\Rightarrow$ $\operatorname{im} \partial_1 = 0$
$\partial_1(b) = v - v = 0$ $\quad \ldots \ker \partial_1 = \langle a, b \rangle$

$\partial_2(U) = a - a = 0$ $\quad$ $\ker \partial_2 = U$
$\partial_2(F) = b + a + b + a = 2a + 2b$ $\quad$ $\operatorname{im} \partial_2 = \langle 2a+2b, 3b \rangle$
$\partial_2(L) = 3b$

$$H_0 = \frac{\ker \partial_0}{\operatorname{Im} \partial_1} = \langle v \rangle = \mathbb{Z}$$

$$H_1 = \frac{\ker \partial_1}{\operatorname{im} \partial_2} = \frac{\langle a, b \rangle}{\langle 2(a+b), 3b \rangle} = \langle a+b, b \mid 2(a+b) = 3b = 0 \rangle = \langle c, b \mid c^2 = b^3 = 0 \rangle$$
$$= \mathbb{Z}_2 \times \mathbb{Z}_3$$

$$H_2 = \frac{\ker \partial_2}{\operatorname{im} \partial_3} = \frac{\langle u \rangle}{0} = \mathbb{Z}$$

(b) $\pi_1(X) = \langle a, b \mid aa^{-1} = e,\ b^3 = e,\ baba = e \rangle$
$= \langle a, b \mid b^3 = e, (ba)^2 = e \rangle$
$= \langle c, b \mid b^3 = c^2 = e \rangle = \mathbb{F}_2 * \mathbb{F}_3$.

(4) Does there $\exists$ a smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^2$?

Pf Suppose yes, then if $\phi: \mathbb{RP}^2 \hookrightarrow \mathbb{R}^2$ is a smooth embedding

$\Rightarrow$ $\phi_*: T_p\mathbb{RP}^2 \longrightarrow T_q\mathbb{R}^2$ is injective.

But the dimensions are both $= 2$ $\Rightarrow$ ϕ_* is an isomorphism.

So ϕ_* is surjective. $\Rightarrow$ ϕ is a local diffeomorphism.

However $\mathbb{RP}^2$ is compact, and ϕ_* surjective & $\mathbb{R}^2$ is connected $\Rightarrow$ ϕ is surjective.

$\Rightarrow$ $\phi(\mathbb{RP}^2) = \mathbb{R}^2$ $\Rightarrow$ $\mathbb{R}^2$ is compact $\Rightarrow \in$.

So such a map cannot exist.

for justification see #2 Fall 2013

(5) Let M- manifold, $C^\infty(M)$ be the algebra of C^∞ functions $M \to \mathbb{R}$.
Explain the relationship btw vector fields on M & $C^\infty(M)$.
If we consider v.f. as maps btw $C^\infty(M) \to C^\infty(M)$ is XY also a v.f? $[X,Y]$?

Pf Given any vector field on M, X, $\phi_X: f \longrightarrow Xf$ defines a smooth vector field on $C^\infty(M)$ iff X is smooth

Where $Xf(p) = X_p f = \left(\sum x_i(p) \frac{\partial}{\partial x_i}\Big|_p\right) f$.

So given XY we want to show it is a derivation.

However $XY(fg) = fXYg + gXYf$

also $XY(fg) = X(Yfg) = Yf\,Xg + g\,XYf$

and $fXYg \neq Yf \cdot Xg$ normally. (see Lee for counterexample)

Now $[X,Y](fg) = (XY - YX)(fg) = XY(fg) - YX(fg)$

$= X(fYg + gYf) - Y(fXg + gXf)$

$= XfYg + XgYf - YfXg - YgXf$

$= fYXg + gX\cancel{fY} + gYXf + \cancel{fXgY} - fXYg - \cancel{gYfX}$

$- gXYf - \cancel{fYgX}$

$= fYXg - fXYg + gYXf - gXYf$

$= f[Y,X]g + g[Y,X]f$ Signs?

(6) Compute $\int_S \omega = \int (w+w^2)\, dx \wedge dy \wedge dz$

Pf By stokes.

$$\int_{S^3} \omega = \int_{B^4} d\omega = \int_{B^4} (2w+1)\, dw \wedge dx \wedge dy \wedge dz$$

$$= -\int_{B^4} 2w\, dx \wedge dy \wedge dz \wedge dw - \int_{B^4} dx \wedge dy \wedge dz \wedge dw$$

$$= 0 - vol(B^4)$$

since B^4 is symmetric & $2w$ is odd $\Rightarrow 2\int_{B^4} \omega = 0$

(7) is $\{x^2 = y^3\}$ smooth submanifold of $\mathbb{R}^3$?

Pf Consider $S = \{x^2 - y^3 = 0\}$ & let $\phi: \mathbb{R}^3 \longrightarrow \mathbb{R}$
$(x,y,z) \longmapsto x^2 - y^3$

then $S = \phi^{-1}(0)$

Now $\phi_*: T_p\mathbb{R}^3 \longrightarrow T_{f}\mathbb{R}$ so $\phi_* = [\partial\phi/\partial x \; \partial\phi/\partial y \; \partial\phi/\partial z] = [2x \; -3y^2 \; 0]$

but $\phi_* = 0$ iff $x = y = 0$ so ϕ_* is not surjective at $(0,0)$

since $(0,0) \in S$, then S cannot be a smooth submanifold.

Geometry/Topology Qualifying Exam

Spring 2012

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

Inv. fn. thm compactness 1. (10 pts) Prove that a compact smooth manifold of dimension n cannot be immersed in $\mathbb{R}^n$.

2. (10 pts) Let $\Sigma_{1,1}$ be the compact oriented surface with boundary, obtained from $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x, y) by removing a small disk $\{(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{100}\}$.
   $\Sigma_{1,1} \simeq S^1 \vee S^1$ (a) Compute the homology of $\Sigma_{1,1}$.
   Mayer Vietoris (b) Let Σ_2 denote a closed oriented surface of genus 2. Use your answer from (a) to compute the homology of Σ_2.

3. (10 pts) Let S be an oriented embedded surface in $\mathbb{R}^3$ and ω be an area form on S which satisfies $\omega(p)(e_1, e_2) = 1$ for all $p \in S$ and any orthonormal basis (e_1, e_2) of T_pS with respect to the standard Euclidean metric on $\mathbb{R}^3$. If (n_1, n_2, n_3) is the unit normal vector field of S, then prove that

recall $e_1 \times e_2 = \vec{n}$

$$\omega = n_1 dy \wedge dz - n_2 dx \wedge dz + n_3 dx \wedge dy,$$

where (x, y, z) are the standard Euclidean coordinates on $\mathbb{R}^3$.

4. (10 pts) Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a *Möbius band* is the quotient space $([-1, 1] \times [-1, 1])/((1, y) \sim (-1, -y))$.
   van kampen (a) Determine the fundamental group of X.
   picture. (b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

5. (10 pts) Determine all the connected covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$.

6. (10 pts) Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that $f^*(\mathcal{L}_Y\omega) = \mathcal{L}_X(f^*\omega)$, where ω is a 1-form on N. Here $\mathcal{L}$ denotes the Lie derivative.

cartan

7. (10 pts) Consider the linearly independent vector fields on $\mathbb{R}^4 - \{0\}$ given by:

frobenius thm.

$$X(x_1, x_2, x_3, x_4) = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}$$

$$Y(x_1, x_2, x_3, x_4) = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}.$$

Is the rank 2 distribution orthogonal to these two vector fields integrable? Here orthogonality is measured with respect to the standard Euclidean metric on $\mathbb{R}^4$.

Spring 2012

1) Prove a compact smooth manifold of dim n cannot be immersed in $\mathbb{R}^n$.

Pf M - compact smooth w/ dim n.

so if $\varphi: M \longrightarrow \mathbb{R}^n$ an immersion $\Rightarrow \varphi_* . T_pM \to T_q\mathbb{R}^n$ is injective.

$\Rightarrow$ dim $M = n$ $\Rightarrow$ dim $T_pM = n$ $\Rightarrow$ φ_* is an isomorphism.

Hence φ is a local diffeomorphism since $\varphi_{*p} \neq 0$ $\forall p \in T_pM$ so we can apply the inverse function theorem on $\varphi(M)$.

Since M is compact and φ is continuous $\varphi(M)$ is compact in $\mathbb{R}^n$. Thus to each $p \in \varphi(M)$ $\exists$ $U_i \ni p \in U_i \simeq \varphi^{-1}(U_i) = W_i$.

Now each W_i is open by continuity of φ. Now $\cup W_i$ is an open cover of M so by compactness $\exists$ finite subcover, so that $\varphi(M) = \varphi(\overset{k}{\cup} W_i) = \overset{k}{\cup}\varphi(W_i) = \overset{k}{\cup} U_i$

However each U_i is open $\Rightarrow$ $\overset{k}{\cup} U_i$ is open in $\mathbb{R}^n$.

But M is compact $\Rightarrow$ $\varphi(M)$ is compact. so $\varphi(M)$ is both closed & open $\Rightarrow$ $\varphi(M) = \phi$ or $\varphi(M) = \mathbb{R}^n$.

if $\varphi(M) = \phi \Rightarrow M = \phi \Rightarrow \varphi_* : \phi \to T_q\mathbb{R}^n \Rightarrow T_q\mathbb{R}^n \simeq \phi \Rightarrow \in$.

if $\varphi(M) = \mathbb{R}^n$ $\Rightarrow$ $\mathbb{R}^n$ is compact $\Rightarrow \in$.

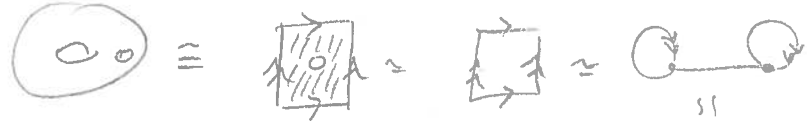
Thus such a map is impossible.

② $\Sigma_{1,1}$ is T^2 - small disk $= \{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{100}\}$, $T^2 = \mathbb{R}^2/\mathbb{Z}^2$

(a) compute the homology of $\Sigma_{1,1}$

(b) Let Σ_2 denote the closed oriented surface of genus 2. Compute $H_*(\Sigma_2)$

Pf since $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ & $\{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{100}\} \cap \mathbb{Z}^2 = \emptyset$

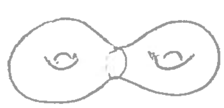
then $\Sigma_{1,1} \simeq T^2 - pt.$ $\Rightarrow$ 

$S^1 \vee S^1$

so $\Sigma_{1,1} \simeq S^1 \vee S^1$

(a) $\tilde{H}_k(S^1 \vee S^1) \simeq \tilde{H}_k(S^1) \oplus \tilde{H}_k(S^1)$ (since all are good pairs)

$$\Rightarrow \quad H_k(\Sigma_{1,1}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z} \oplus \mathbb{Z} & k=1 \\ 0 & \text{else} \end{cases}$$

(b) $\Sigma_2 =$  so let $A = T^2 - pt$ $\quad B = T^2 - pt$

$A \cup B = \Sigma_2 \quad A \cap B \simeq S^1$

$\Rightarrow \cdots \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(A \cup B) \to H_{k-1}(A \cap B) \to \cdots$

so $k \geq 3 \Rightarrow H_k(\Sigma_2) = 0$

$k \leq 2 \Rightarrow$ $0 \xrightarrow{j_2} H_2(\Sigma_2) \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{i_1} \mathbb{Z}^2 \oplus \mathbb{Z}^2 \xrightarrow{j_1} H_1(\Sigma_2) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{i_0} \mathbb{Z}$

$$\begin{array}{c} \mathbb{Z} \\ \downarrow j_0 \\ H_0(\Sigma_2) \\ \downarrow \partial_0 \\ 0 \end{array}$$

$H_0(\Sigma_2) = \mathbb{Z}$ by path conn.

$\ker j_2 = 0$, $\operatorname{im} j_2 = 0 = \ker \partial_2$

$\Rightarrow H_2(\Sigma_2) = \operatorname{im} \partial_2$

$\ker \partial_0 = H_0(\Sigma_2) = \mathbb{Z} \Rightarrow \operatorname{Im} j_0 = \mathbb{Z}$, $\ker j_0 = 0 \Rightarrow \operatorname{Im} i_0 = \mathbb{Z}$

$\Rightarrow \ker i_0 = 0 \Rightarrow \operatorname{Im} \partial_1 = 0$ so $H_1(\Sigma_2) = \ker \partial_1 = \operatorname{Im} j_1$

$\operatorname{Im} i_1 = 0 \Rightarrow \ker j_1 = 0 \Rightarrow \operatorname{Im} j_1 = \mathbb{Z}^4 \Rightarrow H_1(\Sigma_2) = \mathbb{Z}^4$

$H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ $\qquad H_2(\Sigma_2) = \operatorname{im} \partial_2 = \ker i_1 = \mathbb{Z}$

$\alpha \longmapsto 0, 0$

$\ker i_1 = \alpha = \mathbb{Z}$

$\operatorname{Im} i_1 = 0$

$$\Rightarrow H_k(\Sigma_2) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^4 & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & \text{else} \end{cases}$$

(3) S - oriented embedded surface in $\mathbb{R}^3$, ω - area form on S

$\Rightarrow \omega(p)(e_1, e_2) = 1$ $\forall p$ w/ e_1, e_2 of T_pS

if (n_1, n_2, n_3) is the unit normal v. field of S prove.

$\omega = n_1 dy \wedge dz - n_2 dx \wedge dz + n_3 dx \wedge dy$.

~~Pf~~ let $\omega = \omega_1 dy \wedge dz - \omega_2 dx \wedge dz + \omega_3 dx \wedge dz$. Since S is a surface in $\mathbb{R}^3$, it has dim 2, so T_pS has dim 2

hence ω must be a two form representable as above.

Now, if e_1, e_2 are an O.N.B for $T_pS \Rightarrow$ $e_1 \times e_2 = \vec{n} = (n_1, n_2, n_3)$

so if $e_1 = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$ & $e_2 = b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} + b_3 \frac{\partial}{\partial z}$

$\Rightarrow (a_2 b_3 - a_3 b_2) = n_1$, $-(a_1 b_3 - a_3 b_1) = n_2$, $(a_1 b_2 - a_2 b_1) = n_3$

so then if $\omega_p(e_1, e_2) = 1$

$$w_p(e_1,e_2) = w_p\left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z},\ b_1\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y},\ b_3\frac{\partial}{\partial z}\right)$$

$$= w_1(p)\, dy\wedge dz\,(e_1,e_2) + w_2(p)\, dx\wedge dz\,(e_1,e_2) + w_3(p)\, dx\wedge dz$$

$$= w_1(p)\left[dy(e_1)\,dz(e_2) - dy(e_2)\,dz(e_1)\right] + w_2(p)\left[dx(e_1)\,dz(e_2) - dx(e_2)\,dz(e_1)\right]$$
$$+ w_3(p)\left[dx(e_1)\,dz(e_2) - dx(e_2)\,dz(e_1)\right]$$

$$= w_1(p)\left[a_2b_3 - b_2a_3\right] + w_2(p)\left[a_1b_3 - a_3b_1\right] + w_3(p)\left[a_1b_2 - a_2b_1\right]$$

$$= n_1 w_1(p) - n_2 w_2(p) + n_3 w_3(p) = 1.$$

However $n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow w_1(p) = n_1 \quad w_2(p) = -n_2 \quad w_3(p) = n_3$

□

(4) $X = M_1 \cup M_2$ w/ $M_1 \cap M_2 = \partial M_1 = \partial M_2$

(a) find $\pi_1(X)$

(b) is X homotopy equivalent to a compact orientable surface of genus g?

Pf a) $A = M_1 \quad B = M_2 \quad A\cap B \cong S^1 \quad A \cup B = X$, By van Kampen:

$$\begin{tikzcd}
 & \pi_1(A) \arrow[dr] & \\
\pi_1(A\cap B) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi_1(A) * \pi_1(B)/N \\
 & \pi_1(B) \arrow[ur] &
\end{tikzcd}$$

where $A \simeq S^1$, $B \simeq S^1$

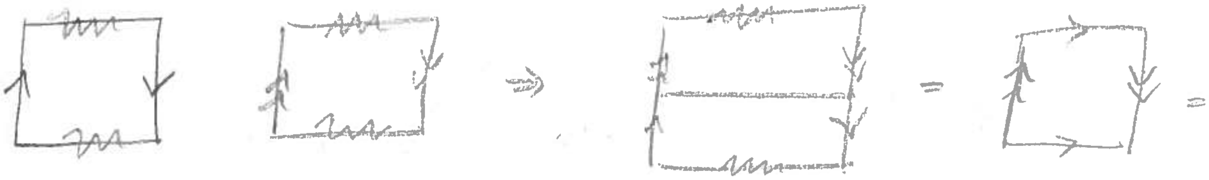
so let $\pi_1(A) = \langle a\rangle$, $\pi_1(B) = \langle b\rangle$

$i_1 : \langle\gamma\rangle \longrightarrow \langle a\rangle$
$\gamma \longmapsto a^2$

so $\pi_1(X) = \dfrac{\langle a,b\rangle}{\langle a^2b^{-2}\rangle} = \langle a,b \mid a^2 = b^2\rangle$

$i_2 : \langle\gamma\rangle \longrightarrow \langle b\rangle$
$\gamma \longmapsto b^2$

(b)

No;  $= K$

so X is $\simeq$ klein bottle, which is non orientable.

⑤ Determine all covering spaces of $\mathbb{R}P^{14} \vee \mathbb{R}P^{15}$

~~Pf~~ since $\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$ then its universal cover is S^n.

Now, for any $X \vee Y$ the universal cover is an alternating ∞ wedge of both so for $\mathbb{R}P^{14} \vee \mathbb{R}P^{15}$ the universal cover is an infinite wedge of S^{14} & S^{15} $\Rightarrow$ 

Intermediate covers are in bijection w/ subgroups of $\pi_1(X) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ i.e. subgroups of $\langle a, b \mid a^2 = b^2 = e \rangle$, so all words of the form $a^i b a b \dots a b^j$ w/ $j, i = 1, 0, 1, 0.$ $\overset{bij}{\Longleftrightarrow} \mathbb{N}$.

Clearly these correspond to alternating wedges of S^{14} & S^{15} of finite length.

+) Given X & Y linearly indep on $\mathbb{R}^4 - \{0\}$ is the rank 2 distribution orthogonal to these two vector fields integrable

Pf Using Frobenius' theorem we compute $[X,Y]$.

$$[X,Y] = \left[x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}, -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}\right]$$

$$= \cancel{x_1 \frac{\partial}{\partial x_2}} - \cancel{x_2 \frac{\partial}{\partial x_1}} + \cancel{x_3 \frac{\partial}{\partial x_4}} - \cancel{x_4 \frac{\partial}{\partial x_3}}$$

$$-\left(\cancel{-x_2 \frac{\partial}{\partial x_1}} + \cancel{x_1 \frac{\partial}{\partial x_2}} - \cancel{x_4 \frac{\partial}{\partial x_3}} + \cancel{x_3 \frac{\partial}{\partial x_4}}\right) = 0$$

Thus $[X,Y] = 0 \Rightarrow \langle [X,Y], X\rangle = \langle [X,Y], Y\rangle = 0$

So by Frobenius $[X,Y]$ is orthogonal to X, Y $\Rightarrow$ it is a linear combination of span $\langle X, Y\rangle^\perp$, & hence integrable.