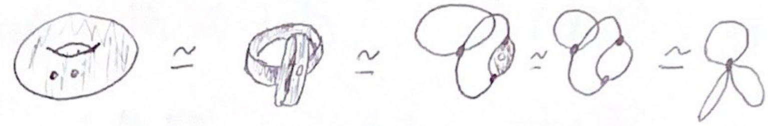
Spring 2021 $\boxed{\#1}$

$X = S^1 \times S^1 - \{p, q\}$, $p \neq q$ = 

(1) $X \simeq S^1 \vee S^1 \vee S^1$

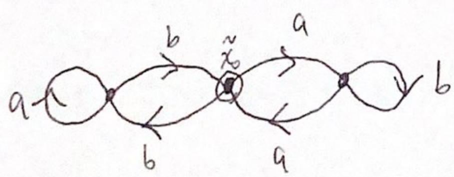
$$\Rightarrow \quad \tilde{H}_n(X) \approx \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1)$$

$$\Rightarrow \quad H_n(X) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

(2) $\pi_1(S^1 \vee S^1 \vee S^1) = \pi_1(S^1)^{*3} = \mathbb{Z}^{*3} = \boxed{F_3}$, the free group on 3 elements.

$\boxed{\#2}$

$\tilde{X}$



$p_*(\pi_1(\tilde{X}; \tilde{x}_0))$ is not normal because, for example,

The path $b(aba)b^{-1}$ is not a loop in $\tilde{X}$.

Hence $p_*(b)\, p_*(aba)\, p_*(b)^{-1}$ is not in $p_*(\pi_1(\tilde{X}))$

That is, $p_*(b)$ outside the normalizer of in $\pi_1(X)$

$\Rightarrow$ $p_*(\pi_1(\tilde{X}))$ not normal.

$\boxed{\#3}$

Let (U, ϕ) be a chart in M. $\phi : U \xrightarrow{\cong} V \subset \mathbb{R}^n$, open.

Define $(\tilde{U}, \tilde{\phi})$ a chart for T^*M:

$\tilde{U} = \{(x, u) : x \in U,\ u : T_xM \longrightarrow \mathbb{R} \text{ linear}\}$

$\tilde{\phi} : \tilde{U} \xrightarrow{\cong} \tilde{V} \subset \mathbb{R}^{2n}$, $\tilde{\phi}(x,u) = \left(\phi(x), u\left(\frac{\partial}{\partial x^1}\right), \ldots, u\left(\frac{\partial}{\partial x^n}\right)\right)$

Then $\tilde{\phi}^{-1} : \tilde{V} \longrightarrow \tilde{U}$ defined by $\tilde{\phi}^{-1}(y, w_1, \ldots, w_n) = (\phi^{-1}(y), w_1 dx^1 + w_2 dx^2 + \cdots + w_n dx^n)$

#4

$f: S^n \longrightarrow S^n$ no fixed points.

Then $\forall\ x \in S^n$, $\exists$ straight line from $f(x)$ to $-x$ which does not pass through the origin. Call this

$\gamma_x : I \longrightarrow \cancel{S^n}\ \mathbb{R}^{n+1}$, $\gamma_x(t) = f(x) + t(-x - f(x))$

Then there is a homotopy b/w the antipodal map α, $(\alpha(x) = -x)$ and f: Define $H(x,t) = \dfrac{\gamma_x(t)}{|\gamma_x(t)|}$ which is well defined b/c none of the straight lines go through the origin. $H(x,0) = f(x)$, $H(x,1) = -x$ and H is continuous.

Hence $\deg(f) = \deg(\alpha) = (-1)^{n+1}$.

#5

$$\int_T x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy = \int_D 3\, dx \wedge dy \wedge dz = 3 \operatorname{vol}(D) = 3(\pi)(4\pi)$$
$$= \boxed{12\pi^2}.$$

where D is the solid torus whose boundry $\partial D = T$.

$\boxed{\#6}$

$f: M \to N$, M, N connected cpct orientable n-dim'l.

$\Rightarrow \deg(f)$ well-defined, $\deg(f) = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(x)$ for reg. value $y \in N$.

$\exists$ open $U \subset N$ s.t. $f^{-1}(U) = U_1 \sqcup U_2 \sqcup U_3$ for which

$f|_{U_i} : U_i \to U$ diffeomorphism. Let $y \in U$, Then $f^{-1}(y)$

consists of 3 points $x_1 \in U_1$, $x_2 \in U_2$, $x_3 \in U_3$.

y is a regular value b/c df_{x_i} surjective $\forall i$ since

$f|_{U_i}$ is a diffeomorphism. Hence $\deg(f) = \operatorname{sgn}(x_1) + \operatorname{sgn}(x_2) + \operatorname{sgn}(x_3)$

and since $\operatorname{sgn}(x_i) = \pm 1$, $\deg(f) \neq 0$.

$\Rightarrow \int_M f^* \omega = k \int_N \omega \neq 0 \quad \forall$ volume forms ω on N.

However if f not surjective, then we can take ω to

be compactly supported on $N \setminus f(M)$. Then $\int_M f^* \omega = 0 \Rightarrow \Leftarrow$

So f must be surjective.

$\boxed{\#7}$ $\omega = \frac{dx \wedge dy}{x^2+y^2}$ on $X = \mathbb{R}^2 - \{0\}$,

$Y =$ Unit circle in X.

$f: D^2 \to X$ $\quad f|_{\partial D^2}$ ~~...~~ maps ∂D^2 to Y

Claim: $\boxed{\int_{D^2} f^*(\omega) = 0}$

$\omega = \frac{dx \wedge dy}{x^2+y^2} = dx \wedge dy$ on Y since $x^2+y^2=1$

$\Rightarrow \int_Y \omega = \int_Y dx \wedge dy = \int_{D^2} d(dx \wedge dy) = \int_{D^2} 0 = 0$

since $dx \wedge dy$ defined on $\mathbb{R}^2$ and by Stokes.

Then, if ω exact, we have $\int_{D^2} f^*(\omega)$

$= \int_{D^2} f^*(d\alpha) = \int_{D^2} df^*(\alpha) = \int_Y f^*(\alpha)$ for some α.

~~$\int_{D^2}$ ...~~ $\overset{?}{}$ If ω exact then ~~$\int_D \omega = \int_D d\alpha = \int_Y \alpha$~~

$\int_Y \omega = \int_Y d\alpha = \int_{\emptyset} \alpha = 0$