Spring 2020 $\boxed{\#1}$

$X = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 ; \; x_1^4 + x_2^4 = 1 + x_3^2 + x_4^2 + x_5^2\} \subset \mathbb{R}^5$

Let $f: \mathbb{R}^5 \to \mathbb{R}$ be defined by $f(x) = x_1^4 + x_2^4 - 1 - x_3^2 - x_4^2 - x_5^2$

Then $X = f^{-1}(0)$.

$$df_x = \begin{bmatrix} 4x_1^3 & 4x_2^3 & -2x_3 & -2x_4 & -2x_5 \end{bmatrix}$$

If $f(x) = 0$ then $x_1^4 + x_2^4 - x_3^2 - x_4^2 - x_5^2 = 1$

$\Rightarrow$ some $x_i \neq 0$ $\Rightarrow$ df_x surjective

Hence 0 is a regular value of f, and by reg. level set Thm, X is a submfd with codim $= \dim(\mathbb{R}) = 1$

$\Rightarrow \dim(X) = 4$.

Assume $x \in X$, $\overset{\neq 0}{\lambda} \in \mathbb{R}$. Is $\lambda x \in X$? Assume so, then

$$(\lambda x_1)^4 + (\lambda x_2)^4 = 1 + (\lambda x_3)^2 + (\lambda x_4)^2 + (\lambda x_5)^2$$

$$\lambda^4 (x_1^4 + x_2^4) = 1 + \lambda^2 (x_3^2 + x_4^2 + x_5^2)$$

$$\lambda^4 (1 + x_3^2 + x_4^2 + x_5^2) = 1 + \lambda^2 (x_3^2 + x_4^2 + x_5^2)$$

$$\lambda^4 - 1 = (\lambda^2 - \lambda^4)(x_3^2 + x_4^2 + x_5^2)$$

$$\frac{\lambda^4 - 1}{\lambda^2 - \lambda^4} = x_3^2 + x_4^2 + x_5^2 \qquad \underline{\text{if } \lambda \neq \pm 1}$$

$$\frac{(\lambda^2+1)(\lambda^2-1)}{\lambda^2(1-\lambda^2)} = x_3^2 + x_4^2 + x_5^2$$

$$-\frac{(\lambda^2+1)}{\lambda^2} = x_3^2 + x_4^2 + x_5^2$$

$\Rightarrow \lightning$ since LHS<0, RHS≥ 0

So if $x \in X$, $-x$ is the only other λx in X. Hence, we may map $X \to S^4$, and since S^4 is oriented, we may pull back its orientation form to an orientation form on X.

#2

Assume $f: M \to \partial M$ differentiable with $f(x) = x \ \forall x \in \partial M$, with M compact, orientable. Let α be a volume form on ∂M which is guranteed to exist because M is orientable. If $i: \partial M \hookrightarrow M$ is the inclusion map, then $f \circ i: \partial M \to \partial M$ is the identity map.

Then $(f \circ i)^* \alpha = \alpha$

Then $\int_{\partial M} \alpha = \int_{\partial M} (f \circ i)^* \alpha = \int_{\partial M} i^*(f^* \alpha) \overset{\text{Stokes}}{=} \int_M d f^* \alpha = \int_M f^* d\alpha = \int_M f^* 0 = 0$

But this is a contradiction because $\int_{\partial M} \alpha \neq 0$ since α is a volume form.

#3 Define $f: M \times M - \Delta \to S^{2n+1}$ where $\Delta = \{(x,x) \in M \times M\}$
$(x,y) \mapsto \frac{x-y}{|x-y|}$

Since $M \times M - \Delta$ has dim $2n < \dim(S^{2n+1}) = 2n+1$

$df_{(x,y)}$ is not surjective for any $(x,y) \in M \times M - \Delta$.

Then all points in $\operatorname{im}(f)$ are critical values and therefore have measure 0 (by Sard's thm) in S^{2n+1}.

Hence, $\exists \ v \in S^{2n+1}$ s.t. $\frac{x-y}{|x-y|} \neq v \quad \forall (x,y) \in M \times M - \Delta$

Let H be the hyperplane with normal vector v.

Let π_H denote the orthogonal projection $\mathbb{R}^n \to H$.

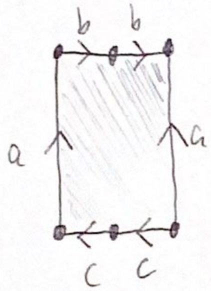
If $x, y \in M$, and $\pi_H(x) = \pi_H(y)$, then $\lambda_1 v + x = \lambda_2 v + y$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$, i.e. $x - y = (\lambda_2 - \lambda_1) v \Rightarrow x = y$. That is $\pi_H|_M$ injective.

#4

Let $Y = [0,2\pi] \times [0,1] \subset \mathbb{R}^2$.

Then $X = Y/\sim$ where
$$(\theta,0) \sim (\theta+\pi, 0) \quad \text{for } \theta \in [0,\pi]$$
$$(\theta,1) \sim (\theta+\pi, 1) \quad \text{for } \theta \in [0,\pi]$$
$$(0,t) \sim (2\pi, t) \quad \forall\, t \in [0,1]$$

Visually we have



which means $\pi_1(X) = \langle a,b,c \mid ab^2a^{-1}c^2 = 1 \rangle$

#5 First, note that $X \times \{x_0\}$ is a retraction of $X \times S^n$, so we get a splitting
$$H_p(X \times S^n) \approx H_p(X \times \{x_0\}) \oplus H_p(X \times S^n, X \times \{x_0\}) \approx H_p(X) \oplus H_p(X \times S^n, X \times \{x_0\})$$
WLOG take $x_0 \neq N$ or S poles of S^n. Then let
$(A,C) = (X \times (S^n \setminus \{N\}), X \times \{x_0\})$, $(B,D) = (X \times (S^n \setminus \{S\}), X \times \{x_0\})$ Then we get Relative M.V. LES:
$$\cdots \to H_p(X \times S^{n-1}, X \times \{x_0\}) \to H_p(X \times \{x_0\}, X \times \{x_0\}) \oplus H_p(X \times \{x_0\}, X \times \{x_0\}) \to H_p(X \times S^n, X \times \{x_0\}) \to \cdots$$
Since $H_p(X \times \{x_0\}, X \times \{x_0\}) = 0 \ \forall p$, we get
$$H_p(X \times S^n, X \times \{x_0\}) \approx H_{p-1}(X \times S^{n-1}, X \times \{x_0\})$$
By induction,
$$\begin{aligned} H_p(X \times S^n, X \times \{x_0\}) &\approx H_{p-n}(X \times S^0, X \times \{x_0\}) \\ &\approx H_{p-n}(X \sqcup X, X \times \{x_0\}) \\ &\approx H_{p-n}(X) \end{aligned}$$

Hence $H_p(X \times S^n) = H_p(X) \oplus H_{p-n}(X)$

$\boxed{\#5}$

Let $A = X \times (S^n \setminus N)$, $B = X \times (S^n \setminus S)$ for N, S the north, south poles.

Then $A \simeq X \times \{*\} \cong X$, $B \simeq X \times \{*\} \cong X$,

$A \cap B \simeq X \times S^{n-1}$, $A \cup B = X \times S^n$

Then M.V. gives LES

$$\cdots \longrightarrow H_p(X \times S^{n-1}) \longrightarrow H_p(X) \oplus H_p(X) \longrightarrow H_p(X \times S^n) \longrightarrow \cdots$$

If $n=0$ then $H_p(X \times S^0) = H_p(X) \oplus H_p(X)$ $\forall p$ since $X \times S^0 = X \sqcup X$.

If $n=1$ we have

$$\cdots \longrightarrow H_p(X \times S^0) \longrightarrow H_p(X) \oplus H_p(X) \longrightarrow H_p(X \times S^1) \xrightarrow{\times 0} \cdots$$

$$H_p(X) \oplus H_p(X)$$

$$(\alpha, \beta) \longmapsto (\alpha+\beta, \alpha+\beta)$$

$\mathrm{Ker}\left(H_p(X \times S^0) \longrightarrow H_p(X) \oplus H_p(X)\right) = \{(\alpha, -\alpha) \in H_p(X \times S^0)\} \approx H_p(X)$

$\mathrm{im}\left(\qquad " \qquad \right) = \{(\alpha+\beta, \alpha+\beta) \in H_p(X) \oplus H_p(X)\} \approx H_p(X)$

Then we have $H_p(X) \oplus H_p(X) \longrightarrow H_p(X \times S^1) \longrightarrow H_{p-1}(X) \oplus H_{p-1}(X)$

$\ker = H_p(X)$ $\qquad$ $\mathrm{im} = H_{p-1}(X)$

$\mathrm{im} = H_p(X)$ $\Rightarrow$ $\ker = H_p(X)$

$\Rightarrow H_p(X \times S^1) = H_p(X) \oplus H_{p-1}(X)$ $\qquad$ By induction we get

$$\boxed{H_p(X \times S^n) = H_p(X) \oplus \cdots \oplus H_{p-n}(X)}$$

Indeed,

$$\cdots \longrightarrow H_p(X) \oplus H_p(X) \longrightarrow H_p(X \times S^n) \longrightarrow H_{p-1}(X \times S^{n-1}) \longrightarrow H_{p-1}(X) \oplus H_{p-1}(X) \longrightarrow \cdots$$

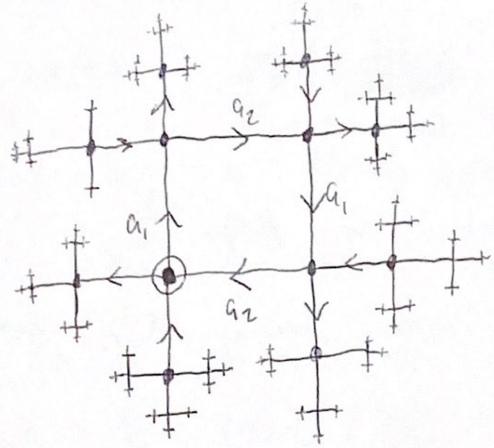
$\mathrm{im} = H_p(X)$ $\qquad$ $\ker = H_p(X)$

$\mathrm{im} = H_{p-1}(X \times S^{n-1})$

$$\Rightarrow H_p(X \times S^n) \approx H_p(X) \oplus H_{p-1}(X \times S^{n-1})$$

$$= H_p(X) \oplus \left(H_{p-1}(X) \oplus \cdots \oplus H_{\underbrace{p-1-(n-1)}_{p-n}}(X)\right)$$

#6 Spring 2020



$\boxed{\text{i} \mid \#5}$

$\boxed{\#7}$

$i : S^n \to \mathbb{R}^{n+1} - \{0\}$ inclusion, $\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$ closed.

M compact oriented n-dim'l mfd w/ $\partial M = \emptyset$

$f : M \to \mathbb{R}^{n+1} - \{0\}$, $f^*(\omega) \in \Omega^n(M)$

Claim: $\int_M f^*(\omega) = k \int_{S^n} i^*(\omega)$ for some $k \in \mathbb{Z}$.

Consider $p : \mathbb{R}^{n+1} - \{0\} \to S^n$ defined by $p(x) = \frac{x}{|x|}$.

Then $p \circ f : M \to S^n$ has a degree because M, S^n both compact oriented n-dim'l mfds.

$\Rightarrow \exists\, k \in \mathbb{Z}$ s.t. $\int_M (p \circ f)^* \alpha = k \int_{S^n} \alpha \quad \forall\, \alpha \in \Omega^n(S^n)$

Also $p \circ i : S^n \to S^n$ has a degree, so

$\Rightarrow \exists\, l \in \mathbb{Z}$ s.t. $\int_{S^n} (p \circ i)^* \alpha = l \int_{S^n} \alpha \quad \forall\, \alpha \in \Omega^n(S^n)$

we see that $p \circ i$ has degree 1, so $l = 1$

Then $\int_M (p \circ f)^* \alpha = k \int_{S^n} \alpha = k \int_{S^n} (p \circ i)^* \alpha$

$$\int_M f^*(p^* \alpha) = k \int_{S^n} i^*(p^* \alpha).$$

So we are done if $\exists\, \alpha \in \Omega^n(S^n)$ s.t. $p^* \alpha = \omega$.

We see that ~~works because~~ $i^*(\omega) \in \Omega^n(S^n)$

$\cancel{(p^*\omega|_{S^n})_x(v_1, \dots, v_n) = (\omega|_{S^n})_{\frac{x}{|x|}}(dp_x v_1, \dots, dp_x v_n)}$, $\cancel{\in T_x \mathbb{R}^{n+1} - \{0\}}$

?

and $p^*(i^*(\omega)) = (i \circ p)^* \omega = \omega$