Fall 2020 $\boxed{\#1}$

Let $\tilde{X}$ be the universal cover of X with $\begin{array}{c}\tilde{X}\\ \downarrow p\\ X\end{array}$.

$\tilde{X}$ path connected and $\pi_1(\tilde{X})$ trivial.

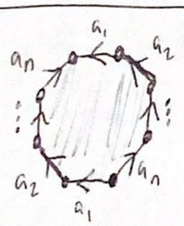
Assume $\tilde{X}$ compact.

Then $\tilde{X}$ must be a finitely sheeted covering space, for if not, then for some $x \in X$, $p^{-1}(U) \subset \tilde{X}$ is an infinite disjoint union of open sets in $\tilde{X}$ for $x \in U$, open nbd. Then $\tilde{X} - p^{-1}(x) \cup p^{-1}(U)$ is an open cover of $\tilde{X}$ which must have a finite subcover. But the removal of any sheet in $p^{-1}(U)$ results in a non-cover, hence there must only be finitely many sheets.

Then $p_*(\pi_1(\tilde{X}))$ has finite index in $\pi_1(X)$

But $p_*(\pi_1(\tilde{X})) =$ identity elt in $\pi_1(X) \Rightarrow \pi_1(X)$ finite.

$\boxed{\#2}$

$X =$ 

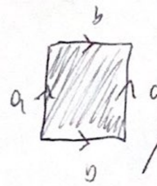
$$\Rightarrow \pi_1(X) = \langle a_1, \dots, a_n \mid a_1 \cdots a_n a_1^{-1} \cdots a_n^{-1} = 1 \rangle$$

$$= \langle a_1, \dots, a_n \mid a_1 \cdots a_n = a_n \cdots a_1 \rangle$$

#3

We see that $X \cong$ , the torus with two discs attached one inside the torus and one inside the circle made by collapsing the torus.

~~Then $X \simeq$~~  we may collapse these two discs to the point (x_0, x_0) since their union is contractible and a good pair with X.

Then $X \simeq$ 

$/\{a \cup b\}$ with sides identified to a point.

I.e. $X \simeq S^2$ the sphere.

Hence $\boxed{H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ 0 & \text{else} \end{cases}}$

$\boxed{\#4}$

$T_{Id} SL_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) : tr(A) = 0 \}$.

<u>Pf</u>: $SL_n(\mathbb{R}) = (\det)^{-1}(1)$ for $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$

Since $SL_n(\mathbb{R})$ is a level set, the tangent space is equal to the kernel of the differential

$$T_{id} SL_n(\mathbb{R}) = \ker \, d(\det)_{id} : M_n(\mathbb{R}) \longrightarrow \mathbb{R}.$$

$$\ker \, d(\det)_{id} = \{ A \in M_n(\mathbb{R}) \text{ s.t. } \tfrac{d}{dt}\det(I + tA)\big|_{t=0} = 0 \}$$

$$\det(I + tA) = \det \begin{pmatrix} 1 + ta_{11} & & & ta_{1n} \\ & 1 + ta_{22} & & \\ & & \ddots & \\ ta_{n1} & & & 1 + ta_{nn} \end{pmatrix}$$

$$\cancel{(1 + ta_{11}) \det(I + tA)_{\hat{11}} + ta_{12} \det(I + tA)_{\hat{12}} + \dots + ta_{1n} \det(I + tA)_{\hat{1n}}}$$

$$= (1 + ta_{11}) \det(I + tA)_{\hat{11}} + (1 + ta_{22}) \det(I + tA)_{\hat{22}} + \dots + (1 + ta_{nn}) \det(I + tA)_{\hat{nn}}$$

$$= (1 + ta_{11})(1 + t\,tr(A_{\hat{11}}) + O(t^2)) + (1 + ta_{22})(1 + t\,tr(A_{\hat{22}}) + O(t^2)) + \dots + (1 + ta_{nn})(1 + t\,tr(A_{\hat{nn}}) + O(t^2))$$

$$= 1 + t\,tr(A_{\hat{11}}) + ta_{11} +$$

$$\det(I + tA) = \sum_{\sigma \in S_n} sgn(\sigma) (I + tA)_{1\sigma(1)} \cdots (I + tA)_{n\sigma(n)}$$

$$\det(I+tA) = 1 + t\,tr(A) + O(t^2) \qquad \text{claim}$$

$$\det(I+tA) = \det\begin{pmatrix} 1+ta_{11} & ta_{12} & \cdots & ta_{1n} \\ ta_{21} & 1+ta_{22} & & ta_{2n} \\ \vdots & & \ddots & \vdots \\ ta_{n1} & ta_{n2} & \cdots & 1+ta_{nn} \end{pmatrix}$$

$$= (1+ta_{11})\underbrace{\det(I+tA_{\hat{11}})}_{1+t\,tr(A_{\hat{11}})+O(t^2)} - ta_{12}\underbrace{\det(I+tA)_{\hat{12}}}_{O(t)} + \cdots \pm (1+ta_{1n})\underbrace{\det(I+tA)_{\hat{1n}}}_{O(t)}$$

$O(t)$ b/c $\det\begin{pmatrix} ta_{21} & ta_{23} & ta_{24} & \cdots & ta_{2n} \\ & & (*) & & \end{pmatrix}$

$$= 1 + t\underbrace{(tr(A_{\hat{11}})+a_{11})}_{tr(A)} + O(t^2)$$

$$= 1 + t\,tr(A) + O(t^2)$$

$$\frac{d}{dt}\left(\det(I+tA)\right) = tr(A) + O(t)$$

So $\frac{d}{dt}\left(\det(I+tA)\right)\Big|_{t=0} = tr(A)$

So $\ker d(\det)_I = \{A \in M_n(\mathbb{R}) : tr(A) = 0\}$

#4

$SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A = 1\}$

$T_A SL_n(\mathbb{R}) \subset M_n(\mathbb{R})$

$\boxed{T_A SL_n(\mathbb{R}) = \{A : \frac{d(\det)}{dx_{ij}} = 0 \ \forall\, i,j\}}$

$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$

$ij \ x_{ij} \overset{?}{\in}$

$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, x_{1\sigma(1)} \cdots x_{n\sigma(n)}$

$x_{i\sigma(i)}$

$j = \sigma(k)$ $\quad x_{k,\sigma(k)}$ $\quad \sigma(i) = j$

$\frac{df}{dx_{ij}} = \sum_{\substack{\sigma \in S_n \\ : \sigma(i) = j}} \operatorname{sgn}(\sigma)\, x_{1\sigma(1)} \cdots \widehat{x_{i(\sigma(i))}} \cdots x_{n\sigma(n)}$

$\sigma^{-1}(j)$

$\# = |S_{n-1}|$

$\frac{df}{dx_{ij}} = \det(A_{\hat{ij}})(-1)^{i+j}$

$\gamma(0) = I_n$

$\gamma : [-1,1] \longrightarrow M_n(\mathbb{R})$

$\det(\gamma(t)) = 1 \quad \forall\, t$

$\frac{d}{dt}\det(\gamma(t)) = 0 = \frac{d(\det)}{dx_{ij}} \frac{dx_{ij}}{dt}$

Say $f(z_1, z_2) = (1, 1)$ $(e^{i\theta_1})^{-1} e^{i\theta_2} = 1$ $e^{i(\theta_2 - \theta_1)} = 1$

Then $z_1^2 z_2 = 1$, $z_1^{-1} z_2 = 1$

$z_1 = z_2$

$z_1^3 = 1 \Rightarrow z_1 = \frac{2\pi}{3}, \frac{4\pi}{3}, 0$

$z^2 = (a+bi)^2 = (a^2 - b^2) + 2abi$

$z^{-1} = a - bi$

$f(a,b,c,d) = \left(((a^2-b^2)+2abi)(c+di), (a-bi)(c+di)\right)$

$= \left(\underline{(a^2-b^2)c} + (a^2-b^2)di + 2abci + \underline{2abd(-1)}, ac+bd+(ad-bc)i\right)$

$= \left((a^2-b^2)c - 2abd, (a^2-b^2)d + 2abc, ac+bd, ad-bc\right)$

$\{ (x,y,z,w) : x^2+y^2 = z^2+w^2 = 1$ or $((x,y) = \underset{\in S^1}{(x_0,y_0)}$ and $z^2+w^2 \leq 1)$

or $(x^2+y^2 \leq 1, (z,w) = \underset{\in S^1}{(x_0,y_0)}) \}$

$\boxed{\#5}$

$\lambda \in \Omega^1(M)$, $\omega = d\lambda \in \Omega^2(M)$. $L \subset M$ submfd. $i : L \hookrightarrow M$ inclusion.

$i^*(\lambda) \in \Omega^1(L)$ exact $\Rightarrow \exists\ \varphi : L \to \mathbb{R}$ s.t. $i^*(\lambda) = d\varphi$

Claim: For any smooth map $f : D^2 \to M$ with $f(\partial D^2) = L$

$$\int_{D^2} f^*(\omega) = 0.$$

Well, $\int_{D^2} f^*(\omega) = \int_{D^2} f^*(d\lambda) = \int_{D^2} d f^*(\lambda) \overset{\text{Stokes}}{=} \int_{\partial D^2} j^* f^*(\lambda)$

where $j : \partial D^2 \longrightarrow D^2$ inclusion.

L must be compact since ∂D^2 compact and $L = f(\partial D^2)$.

~~Then~~ $\cancel{\int_{\partial D^2} j^* f^*(\lambda) = \int_{\partial D^2} i^*(\lambda) = \int_{\partial D^2} i^*(d\varphi) = \int_{\partial D^2} \ldots}$

$j^* f^* : \partial D^2 \longrightarrow L$

$f|_{\partial D^2} : \partial D^2 \longrightarrow L$

has a degree

$$\int_{\partial D^2} f|_{\partial D^2}^*(\alpha) = k \int_L \alpha$$

$\cancel{= \int_{\partial D^2} j^* f^*(\lambda)}$

$\cancel{0 \neq \int_\emptyset \varphi = \int_L d\varphi}$

$\cancel{\lambda = i^*(\lambda) \text{ on } L?}$

$$= k \int_L \lambda = k \int_L i^*(\lambda) = i \int_L d\varphi =$$

$\boxed{\#6}$

$$f(z_1,z_2) = (z_1^2 z_2, z_1^{-1} z_2) = (1,1) \Rightarrow z_1^2 z_2 = 1, \; z_1^{-1} z_2 = 1$$

$$\Rightarrow z_1 = z_2 \Rightarrow z_1^3 = 1 \Rightarrow z_1 = 0, \tfrac{2\pi}{3}, \tfrac{4\pi}{3} \text{ in radians} = 1, -\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i, -\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i \text{ in } \mathbb{C}$$

Hence there are 3 points in $f^{-1}(1,1)$.

As a map from $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$, f is

$$f(a,b,c,d) = \left((a^2-b^2)c - 2abd, \; (a^2-b^2)d + 2abc, \; ac+bd, \; ad-bc\right)$$

So

$$df_p = \begin{pmatrix} 2ac-2bd & -2bc-2ad & a^2-b^2 & -2ab \\ 2ad+2bc & -2bd+2ac & 2ab & a^2-b^2 \\ c & d & a & b \\ d & -c & -b & a \end{pmatrix}$$

$$\det = (2ac-2bd)\left[(-2bd+2ac)(a^2+b^2) - 2ab(ad+bc) + (a^2-b^2)(ac-bd)\right]$$

$$= (2ac-2bd)\left[-2a^2bd + 2a^3c - 2b^3d + \cancel{2ab^2c} - 2a^2bd - \cancel{2ab^2c} + a^3c - a^2bd - ab^2c + b^3d\right]$$

$$= (2ac-2bd)\left[3a^3c - 5a^2bd - ab^2c - b^3d\right]$$

$$= 6a^4c^2 - 10a^3bcd - 2a^2b^2c^2 - \cancel{2ab^3cd} - 6a^3bcd + 10a^2b^2d^2 + \cancel{2ab^3cd} + 2b^4d^2$$

Then

$$df_{(1,1)} = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, \quad \boxed{\det df_{(1,1)} = 2\left(2(1) + 1(1)\right) = 6 > 0}$$

$$df_{\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right)} = \begin{pmatrix} \frac{1}{2} - \frac{3}{2} & -2\frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right) - 2\left(-\frac{1}{2}\right)\frac{\sqrt{3}}{2} & \frac{1}{4} - \frac{3}{4} & -2\left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) \\ 2\left(-\frac{1}{2}\right)\frac{\sqrt{3}}{2} + 2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) & -2\left(\frac{3}{4}\right) + 2\left(\frac{1}{4}\right) & -2\frac{\sqrt{3}}{4} & \frac{1}{4} - \frac{3}{4} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\det df_{\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}\right)} = -1\left(-1\left(\frac{1}{4}+\frac{3}{4}\right) + 2\frac{\sqrt{3}}{4}\left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) - 1\left(-\frac{3}{4} + \frac{1}{4}\right)\right)$$

$$= -\left(-1 - \frac{3}{4} + \frac{1}{2}\right) > 0$$

#6 Cont.

Then $$df_{\left(-\frac{1}{2},-\frac{\sqrt{3}}{2},-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)} = 6\left(-\tfrac{1}{2}\right)^4\left(-\tfrac{1}{2}\right)^2 - 10\left(-\tfrac{1}{2}\right)^3\left(-\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{\sqrt{3}}{2}\right)$$
$$-2\left(-\tfrac{1}{2}\right)^2\left(-\tfrac{\sqrt{3}}{2}\right)^2\left(-\tfrac{1}{2}\right)^2 - 6\left(-\tfrac{1}{2}\right)^3\left(-\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{\sqrt{3}}{2}\right)$$
$$+10\left(-\tfrac{1}{2}\right)^2\left(-\tfrac{\sqrt{3}}{2}\right)^2\left(-\tfrac{\sqrt{3}}{2}\right)^2 + 2\left(-\tfrac{\sqrt{3}}{2}\right)^4\left(-\tfrac{\sqrt{3}}{2}\right)^2$$

$$= \frac{6}{2^6} - 10\left(\frac{3}{2^6}\right) - 2\frac{3}{2^6} - 6\left(\frac{3}{2^6}\right)$$
$$+10\left(\frac{9}{2^6}\right) + 2\left(\frac{27}{2^6}\right) = \frac{6-30-6-18+90+54}{2^6} > 0$$

So each of these is orientation preserving $\Rightarrow$ f has degree 3. $\Rightarrow$ $\int_{T^2} f^*\omega = 3\int_{T^2}\omega$

$$\Rightarrow \quad H^2(f) : H^2_{dR}(T^2) \longrightarrow H^2_{dR}(T^2)$$
$$\mathbb{R} \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto [f^*\omega]$$
$$\int_{T^2}\omega \longmapsto \frac{1}{3}\int_{T^2} f^*\omega$$

So the induced map is multiplication by $\frac{1}{3}$.

$\boxed{\#7}$

$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \in \Omega^2(\mathbb{R}^4)$

$Z = 3x_1 \partial_{x_1} + 3x_2 \partial_{x_2} + 3x_3 \partial_{x_3} + 3x_4 \partial_{x_4}$.

$(\varphi_t)_{t \in \mathbb{R}}$ flow defined by Z. The flow exists $\forall t$.

Calculate $(\varphi_t)^* \omega$. Hint look at the diff. eq. that $(\varphi_t)^*\omega$ satisfies.

$\varphi_t : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$

$x \longmapsto \gamma_x(t)$ where $\gamma_x : \mathbb{R} \longrightarrow \mathbb{R}^4$

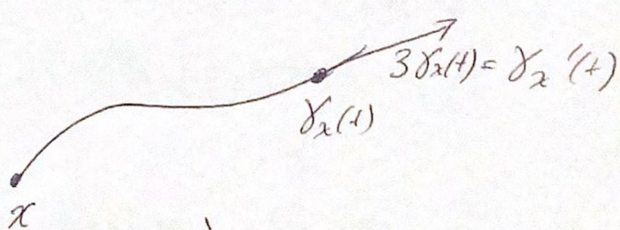
$t \longmapsto \cancel{3t} \; e^{3t}x$

~~Then $\gamma_x(0) = x$ and $\gamma_x'(t) = 3x$~~

~~But need $\gamma_x'(t) = 3\gamma_x(t)$~~ Then $\gamma_x(0) = x$ and

~~and $\gamma_x(t) = e^{3t}x$ works b/c~~ $\gamma_x'(t) = 3e^{3t}x = 3\gamma_x(t)$ ✓.

$(\varphi_t^* \omega)_x(v) = \omega_{e^{-3t}x}$



$\varphi_t(x) = e^{3t}x$

$= (e^{3t}x_1, e^{3t}x_2, e^{3t}x_3, e^{3t}x_4)$

$d(e^{3t}x_1) \wedge d(e^{3t}x_2) + d(e^{3t}x_3) \wedge d(e^{3t}x_4)$

$e^{6t}dx_1 \wedge dx_2 + e^{6t} dx_3 \wedge dx_4$

$\boxed{e^{6t}\omega}$