Spring 2019 $\boxed{\#1}$

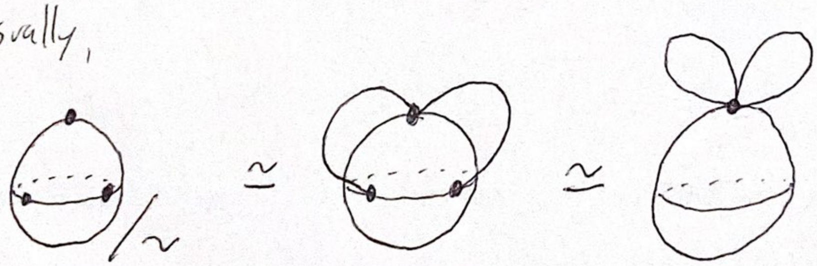
$X = S^2 / \{p_1, p_2, p_3\}$

We see $X \simeq S^2 \vee S^1 \vee S^1 \Rightarrow \pi_1(X; x_0) = \pi_1(S^2) * \pi_1(S^1) * \pi_1(S^1)$

$$= \mathbb{Z} * \mathbb{Z}$$

To see the homotopy equivalence, note that X is homotopy equivalent to the sphere with two curves glued to it: one curve is glued with endpoints on p_1 and p_2 and the other is glued at p_2 and p_3. These two curves together are contractible and form a good pair with the space, so we may contract them and obtain X. Then, similarly there are two curves in the sphere between p_1 and p_2 and between p_2 and p_3. This is contractible and a good pair, so we may contract ~~these~~ it and obtain $S^2 \vee S^1 \vee S^1$.

Visually,



#4

Conside

By

Hence

Then

Sub,

=>

#5

(a)

(b)

#2

(a)

$\tilde{X}$

(b) Hence $H_2(\tilde{X};\mathbb{Z}) \approx \mathbb{Z}^{\oplus\infty}$ since $\tilde{X}$ is homotopy equivalent to the infinite wedge sum of spheres S^2.

and $\tilde{H}_n(\tilde{X};\mathbb{Z}) = \tilde{H}_n(S^2;\mathbb{Z})^{\oplus\infty}$.

Consider $\mathbb{Z}^{\oplus\infty} = \{$ functions $f: \mathbb{N} \to \mathbb{Z}\}$ with group law $(f+g)(n) = f(n) + g(n)$. This satisfies the group axioms of associativity, identity, and inverses.

$\boxed{\#3}$

(a) $f(z) = 2z^3 + 3z^2 \Rightarrow f(a,b) = (2a^3 - 6ab^2 + 3a^2 - 3b^2,\ 6a^2b - 2b^3 + 6ab)$

as a map $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Then $$df_p = \begin{pmatrix} 6a^2-6b^2+6a & -12ab-6b \\ 12ab+6b & 6a^2-6b^2+6a \end{pmatrix} = 6\begin{pmatrix} a^2-b^2+a & -2ab-b \\ 2ab+b & a^2-b^2+a \end{pmatrix}$$

Hence $\frac{1}{6}\det(df_p) = (a^2-b^2+a)^2 + (2ab+b)^2 = 0$

when $a^2-b^2+a=0$ and $2ab+b=0$.

if $b \neq 0$ then $2a+1=0 \Rightarrow a = -\frac{1}{2} \Rightarrow (-\frac{1}{2})^2 - b^2 + (-\frac{1}{2}) = 0$

$\Rightarrow b^2 = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} \Rightarrow\Leftarrow$

if $b=0$ then $a^2+a=0 \Rightarrow a = 0$ or -1

So, the only critical points of f are 0 and $-1 \in \mathbb{C}$.

Hence, the restriction $g: \mathbb{C} - \{-\frac{3}{2}, -1, 0, \frac{1}{2}\} \to \mathbb{C} - \{0,1\}$

has dg_p surjective $\forall\, p \in \mathbb{C} - \{-\frac{3}{2}, -1, 0, \frac{1}{2}\}$ so g is a

submersion and by dimension an immersion, so g is a local diffeo.

Now let $\tilde{X}$ be a covering space of $\mathbb{C} - \{0,1\}$ such that

(*) $p_*(\pi_1(\tilde{X})) = g_*(\pi_1(\mathbb{C} - \{-\frac{3}{2}, -1, 0, \frac{1}{2}\}))$ where $p: \tilde{X} \to \mathbb{C} - \{0,1\}$.

Then $\exists$ continuous $\tilde{g}: \mathbb{C} - \{-\frac{3}{2}, -1, 0, \frac{1}{2}\} \longrightarrow \tilde{X}$ s.t.

$p \circ \tilde{g} = g$ is a local diffeomorphism and in particular is open

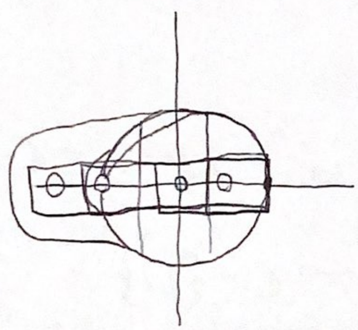
$\Rightarrow$ for U open in $\mathbb{C} - \{-\frac{3}{2}, -1, 0, \frac{1}{2}\}$, $(p\circ\tilde{g})(U)$ open in $\mathbb{C} - \{0,1\}$ and since

p continuous $p^{-1}(p\circ\tilde{g})(U)$ open in $\tilde{X} \Rightarrow \tilde{g}$ open map. And $\tilde{g}$

bijective b/c (*) so $\tilde{g}$ homeomorphism

i.e. g covering map.

(b) This is a two-sheeted cover so the index is 2.



$\boxed{\#4}$

Consider $f: M \to \mathbb{R}$ defined by $f(x_1,\dots,x_n) = \sum_{i=1}^n x_i^2$.

By Sard's theorem, the set of critical values has measure 0 in $\mathbb{R}$.

Hence $\forall \varepsilon > 0$, $\exists r \in [1-\varepsilon, 1+\varepsilon]$ s.t. r is a regular value of f.

Then $f^{-1}(r) = \{(x_1,\dots,x_n) \in M ; \sum_{i=1}^n x_i^2 = r^2\} = M \cap S_r$ is a submfd by regular level set theorem. And it's codimension $=1$

$\Rightarrow$ dimension $= m-1$.

$\boxed{\#5}$

(a) Let $g: \mathbb{R}^4 \to \mathbb{R}$ be defined by $g(x,y,z,w) = x^2+y^2+z^2-w^4$.

Then $M = f^{-1}(-1)$. $dg_p = [2x \;\; 2y \;\; 2z \;\; -4w^3] \neq 0$ for $p \in M$

because $x^2+y^2+z^2-w^4 = -1 \Rightarrow w \neq 0$. $\Rightarrow$ dg_p surjective $\forall p \in M$

$\Rightarrow -1$ regular value of g $\Rightarrow$ by reg. level set thm M submfd of $\mathbb{R}^4$.

(b) $f: \mathbb{R}^4 \to \mathbb{R}$, $f(x,y,z,w) = w$ $\quad f|_M : M \to \mathbb{R}$

$T_pM = \ker dg_p$ $\qquad d(f|_M)_p = df_p|_{T_pM} = df_p|_{\ker dg_p}$

$df_p = [0 \;\; 0 \;\; 0 \;\; 1]$, $\ker dg_p = \{(v_1,v_2,v_3,v_4) \in \mathbb{R}^4 : 2xv_1 + 2yv_2 + 2zv_3 - 4w^3v_4 = 0\}$

$= \{(v_1, v_2, v_3, \frac{xv_1+yv_2+zv_3}{2w^3}) \in \mathbb{R}^4\}$

If $v \in \ker dg_p$ then $df_p(v) = \frac{xv_1+yv_2+zv_3}{2w^3}$ surjective if x or y or $z \neq 0$

$\Rightarrow$ p critical point if $x=y=z=0 \Rightarrow w = \pm 1$ since $p \in M$.

if $w = \pm 1$ then $x^2+y^2+z^2 = 0 \Rightarrow x=y=z=0 \Rightarrow f|_M^{-1}(1) = \{(0,0,0,1)\}$

and $f|_M^{-1}(-1) = \{(0,0,0,-1)\}$ $\Rightarrow$ ± 1 are the critical values of $f|_M$

$\boxed{\#6}$ $Z(x,y) = -y\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial x}$ By Cartan's magic formula,

$$\mathcal{L}_Z(dx\wedge dy) = Z \lrcorner (\cancel{d(dx\wedge dy)}^{0}) + d(Z\lrcorner(dx\wedge dy))$$

$$= d(Z \lrcorner (dx\wedge dy))$$

$$(Z\lrcorner(dx\wedge dy))_p(v) = (dx\wedge dy)_p(Z_p, v)$$

$$= \det\begin{pmatrix} dx(Z_p) & dy(Z_p) \\ dx(v) & dy(v)\end{pmatrix}$$

$$= \det\begin{pmatrix} 2x & -y \\ v_x & v_y\end{pmatrix} = 2x v_y + y v_x$$

$$= 2x\,dy + y\,dx$$

$$\Rightarrow \mathcal{L}_Z(dx\wedge dy) = d(2x\,dy + y\,dx)$$

$$= 2\,dx\wedge dy + dy\wedge dx$$

$$= \boxed{dx\wedge dy}$$