Fall 2019 $\boxed{\#1}$

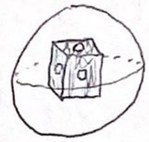
$X =$ . Compute $\pi_1(\mathbb{R}^3 - X)$.

we see that $\mathbb{R}^3 - X \simeq S^2 \cup$ coordinate axes inside B^3.

Then this $\simeq S^2 \vee (S^1 \vee S^1 \vee S^1 \vee S^1 \vee S^1)$

So $\pi_1(\mathbb{R}^3 - X) = \pi_1(S^2) * \pi_1(S^1)^{*5}$

$= \mathbb{Z}^{*5}$ = free group on 5 elts.

To see the homotopy equivalence, consider hollowing out the edges of the cube enough so that we have $B^3 - \{$punctured faces of the cube$\} =$ . Then hollow out these punctured faces inward toward the center, and outward toward S^2 to obtain $S^3 \cup$ axes $=$  which $\simeq S^2 \vee (\vee^5 S^1)$ by contracting several curves.

$\boxed{\#2}$

$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ induced by

$P(x) = x^7 + 5x^3 - 6x^2 + 1$

$\alpha \in \Omega^2(\mathbb{CP}^1) \qquad K := \int_{\mathbb{CP}^1} \alpha$

$$\int_{\mathbb{CP}^1} f^* \alpha = \deg(f) \int_{\mathbb{CP}^1} \alpha = \deg(f) K$$

Let $g: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ be induced by

$Q(x) = x^7$. We claim $f \simeq g$ and $\deg(g) = 7$

Then $\deg(f) = \deg(g) = 7$ and $\int_{\mathbb{CP}^1} f^* \alpha = \boxed{7K}$.

Let $H_t: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ be induced by

$$P_t(x) = x^7 + t(5x^3 - 6x^2 + 1)$$

Then $H_0 = g$, $H_1 = f$ and $H: \mathbb{CP}^1 \times I \longrightarrow \mathbb{CP}^1$

$$([x], t) \longmapsto H_t([x])$$

is continuous, so $f \simeq g$.

$\deg(g) = 7$ because g has 7 roots of unity all of which are orientation preserving, so

$\deg(g) = \sum_7 (+1) = 7$.

Fall 2019 $\boxed{\#3}$

Consider the good pair (Z, Y) which gives a LES:

$$\cdots \longrightarrow H_n(Y) \xrightarrow{i_*} H_n(Z) \longrightarrow H_n(Z, Y) \longrightarrow \cdots$$

for i_* the map induced by inclusion $i : Y \hookrightarrow Z$.

We see that $Z/Y = (X \times [0,1]) / (x,0) \sim (x,1)$

which is SX, the suspension, with tips identified

which is $\simeq$ to $SX \vee S^1$.

Then $\tilde{H}_n(Z/Y) \approx \tilde{H}_n(SX) \oplus \tilde{H}_n(S^1) = \begin{cases} \tilde{H}_n(SX) \oplus \mathbb{Z} & n=1 \\ \tilde{H}_n(SX) & \text{else} \end{cases}$

We also know $\tilde{H}_n(SX) = \tilde{H}_{n-1}(X)$ for $n \neq 1$

and $\tilde{H}_0(X) \approx \tilde{H}_1(SX) \oplus \mathbb{Z}$ for $n=1$ by M.V.

Hence $\tilde{H}_n(Z/Y) = \tilde{H}_{n-1}(X)$ $\forall\, n \geq 1$.

Thus we get a L.E.S.

$$\dashrightarrow H_n(X) \longrightarrow H_n(Y) \xrightarrow{i_*} H_n(Z) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(Y) \longrightarrow \cdots$$

Since $\tilde{H}_n(Z, Y) \approx \tilde{H}_n(Z/Y)$.

The homomorphism $H_n(Z) \longrightarrow H_{n-1}(X)$ is ~~induced~~ the boundary

map ∂. $H_n(X) \longrightarrow H_n(Y)$ sends an n-cell in X

to $f(n\text{-cell}) + g(n\text{-cell})$ in Y

$\boxed{\#4}$

$f : S^n \to S^n$ has no fixed points, then the straight line path from $-x$ to $f(x)$ does not go through the origin, so it may be mapped to S^n via the retraction $\frac{x}{|x|}$.

This creates a homotopy between the antipodal map $\alpha(x) = -x$ and f. Hence $\deg(f) = \deg(\alpha) = \boxed{(-1)^{n+1}}$.

The homotopy is continuous:

$$H : S^n \times I \longrightarrow S^n$$
$$(x, t) \longmapsto \frac{-x + t(f(x)+x)}{|-x + t(f(x)+x)|}$$

composed of continuous functions.

$\boxed{\#7}$

$M = f^{-1}(0)$ where $f : \mathbb{R}^5 \to \mathbb{R}$ defined by $f(\overbrace{x_1, x_2, x_3, x_4, x_5}^{x}) = x_1^2 + x_2^2 + x_3^2 - x_4^3 - x_5^3$.

If x has $x_4 \leq 0$, and $x_5 < 0$ then $f(x) > 0$ so $x \notin M$.

Hence no path γ in M may travel into $\{x \in \mathbb{R}^5 : x_4 < 0, x_5 < 0\}$

~~Then $(1,1,0,-1,-1) \notin T_0M$ and $(-1,1,0,-1,-1) \notin T_0M$,~~

~~b/c otherwise there would exist such γ.~~

Consider the path $\gamma(t) = (\sqrt{2}\, t^{3/2}, 0, 0, t, t) \in M$

we see $f(\gamma(t)) = 2t^3 - t^3 - t^3 = 0$ so $\gamma(t) \in M \ \forall t$.

Then $\gamma'(0) = (0,0,0,1,1) \in T_0M$, but $(0,0,0,-1,-1) = -\gamma'(0)$ cannot be in T_0M, since no path enters $\{x \in \mathbb{R}^5 : x_4 < 0, x_5 < 0\}$

Hence T_0M cannot be well defined as a vector space $\Rightarrow$ M cannot be a submfd. For if so, then T_0M subspace of $T_0\mathbb{R}^n$.

$\boxed{\#3}$ $Z = X \times [0,1] \sqcup Y / \begin{matrix}(x,0) \sim f(x) \\ (x,1) \sim g(x)\end{matrix}$

Want $\cdots \to H_{n+1}(Z) \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to \cdots$

We see relative homology LES: (since (Z,Y) good pair)

$\cdots \to H_n(Y) \xrightarrow{i_*} H_n(Z) \to H_n(Z,Y) \to \cdots$ is close.

Consider $q: X \times [0,1] \sqcup Y \to Z$ quotient map

We have another relative homology LES:

$\cdots \to H_n(X \times \partial[0,1]) \to H_n(X \times [0,1]) \to H_n(X \times [0,1], X \times \partial[0,1]) \to \cdots$

Then the restriction of $q: X \times [0,1] \to Z$ induces a map:

$(X \times [0,1], X \times \partial[0,1]) \to (Z, Y)$ which induces a chain map on homology

$$\begin{array}{ccccccc} \cdots \xrightarrow{\times 0} & \overset{H_n(X)}{H_{n+1}(X \times [0,1], X \times \partial[0,1])} & \xrightarrow{\partial} & \overset{H_n(X) \oplus H_n(X)}{H_n(X \times \partial[0,1])} & \xrightarrow{i_*} & \overset{H_n(X)}{H_n(X \times [0,1])} & \xrightarrow{\times 0} \cdots \\ & \downarrow q_* \approx & & \downarrow q_* & & \downarrow q_* & \\ \cdots \to & H_{n+1}(Z,Y) & \xrightarrow{f_* - g_*} & H_n(Y) & \xrightarrow{i_*} & H_n(Z) & \to \cdots \end{array}$$

$\text{im}(\text{top } i_*) = H_n(X \times [0,1])$ since $(\alpha, 0) \mapsto \alpha \in H_n(X)$ surjective

$\Rightarrow$ maps after i_* are 0-maps $\Rightarrow \partial$ injective.

$\ker(\overset{\text{top}}{i_*}) = \{(\alpha, -\alpha) \in H_n(X) \oplus H_n(X)\} \approx H_n(X)$

$\Rightarrow H_{n+1}(X \times [0,1], X \times \partial[0,1]) \approx \text{im}(\partial) \approx \ker(\text{top } i_*) \approx H_n(X)$

left q_* is an isomorphism because $(X \times [0,1], X \times \partial[0,1])$ and (Z,Y) are good pairs and q induces a homeomorphism on quotient spaces $X \times [0,1] / X \times \partial[0,1] \to Z/Y$. Then $(\alpha, -\alpha)$ maps to $f(\alpha) - g(\alpha)$ in $H_n(Y)$ by q_*

Fall 2019 $\boxed{\#5}$

Let $p \in \mathbb{R}[x_1, \dots, x_n]$ be a non-zero poly over $\mathbb{R}$ homogeneous of degree d.

Then $dP_x = \left[\frac{\partial p}{\partial x_1} \cdots \frac{\partial p}{\partial x_n} \right]$. If $\left.\frac{\partial p}{\partial x_1}\right|_x = \dots = \left.\frac{\partial p}{\partial x_n}\right|_x = 0$

then $P(x)$ is a critical value since dP_x is not surjective.

~~Otherwise dP_x is surjective and if this is true $\forall\, y \in P^{-1}(c)$~~

Claim: If $\left.\frac{\partial p}{\partial x_i}\right|_x = 0 \ \forall i$ then $P(x) = 0$.

Assume not: $P(x) \neq 0$.

Consider the path $\gamma(t) = P(tx) = t^d p(x)$

$\gamma(1) = P(x)$, $\gamma'(1) = d t^{d-1} P(x) +$

$= d p(x) \neq 0$

$\Rightarrow$ some $\left.\frac{\partial p}{\partial x_i}\right|_x \neq 0$.

Hence $P(x) \neq 0 \Rightarrow dP_x \neq 0 \Rightarrow x$ regular point of p

$\Rightarrow c \neq 0$ regular value of p $\Rightarrow$ by regular level set Thm

$p^{-1}(c)$ submfd of $\mathbb{R}^n$ if $c \neq 0$.

$\boxed{\#6}$

Consider $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$

$$d\omega = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}\,dx\wedge dy - \frac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\,dy\wedge dx$$

$$= \frac{2(x^2+y^2)-2x^2-2y^2}{(x^2+y^2)^2}\,dx\wedge dy = 0 \quad \Rightarrow \omega \text{ closed.}$$

ω also a 2-form on $\mathbb{R}^2 \setminus \{0\}$

Let π_z be projection onto the x,y plane then

$\pi_z{}^*\omega$ closed in $\mathbb{R}^3 \setminus \{z \text{ axis}\}$

And $\displaystyle\int_{\text{Cylinder}} \pi_z{}^*\omega = \int_{S^1} \omega = \int_{S^1} x\,dy - y\,dx \overset{\text{stokes}}{=} \int_{B^2} dx\wedge dy - dy\wedge dx$

$$= 2\int_{B^2} dx\,dy = 2\pi \neq 0$$

~~But if~~

$\Rightarrow \pi_z{}^*\omega$ not exact b/c if exact

then $\displaystyle\int_{\text{Cylinder}} \pi_z{}^*\omega = \int_{\partial\,\text{Cylinder}} \alpha = \int_{\emptyset} \alpha = 0$

for some α with $d\alpha = \pi_z{}^*\omega$.