Spring 2018 $\boxed{\#1}$

(a) $dF_p = \begin{pmatrix} 2x_1 & 2x_2 & 2x_3 & 2x_4 \\ 2x_1 & 2x_2 & -2x_3 & -2x_4 \end{pmatrix}$

$F(x_1,x_2,x_3,x_4)=(1,0) \Rightarrow x_1$ or x_2 or x_3 or $x_4 \neq 0$

If x_1 or $x_2 \neq 0$ then x_3 or $x_4 \neq 0$ since $x_1^2+x_2^2-x_3^2-x_4^2=0$

WLOG say it's x_1 and $x_3 \neq 0$. Then $\det\begin{pmatrix} 2x_1 & 2x_3 \\ 2x_1 & -2x_3 \end{pmatrix} = -8x_1x_3 \neq 0$

Similarly if x_3 or $x_4 \neq 0$ then x_1 or $x_2 \neq 0$ and we get the same nonsingular submatrix $\Rightarrow dF_p$ surjective for $p \in F^{-1}(1,0)$.

By regular level set theorem $\Rightarrow F^{-1}(1,0)$ smooth submfd of $\mathbb{R}^4$.

(b) $dF_p(x_2,-x_1,0,0) = \begin{pmatrix} 2x_1x_2 - 2x_1x_2 \\ 2x_1x_2 - 2x_1x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$dF_p(0,0,x_4,-x_3) = \begin{pmatrix} 2x_3x_4 - 2x_3x_4 \\ -2x_3x_4 + 2x_3x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

and $(x_2,-x_1,0,0)\cdot(0,0,x_4,-x_3) = 0 \Rightarrow$ these vectors span the kernel of dF_p, which is the tangent space at x.

(c) $G: \mathbb{R}^4 \to \mathbb{R}$ smooth, $g = G|_M$.

If x is a critical point of g, then dg_x not surjective

$\Rightarrow dg_x : T_pM \to T_p\mathbb{R}$ trivial
$\quad\quad\quad \mathbb{R}^2 \to \mathbb{R}$

(*)
$$\begin{tikzcd} & T_p\mathbb{R}^4 \arrow[dr, "dG_x"] & \\ T_pM \arrow[ur, "di_x"] \arrow[rr, "dg_x"'] & & T_p\mathbb{R} \end{tikzcd}$$

if $v \in \ker dF_x$ then $v \in T_pM \Rightarrow dg_x(v) = 0 \overset{(*)}{\Rightarrow} dG_x(v) = 0 \Rightarrow v \in \ker dG_x$.

So $\ker dF_x \subset \ker dG_x$. Conversely, assume $\ker dF_x \subset \ker dG_x$.

$dg_x : T_pM \to T_p\mathbb{R}$ sends $v \in T_pM = \ker dF_x \subset \ker dG_x$ to 0 in $T_p\mathbb{R}$ (*)

$\Rightarrow dg_x$ not surjective $\Rightarrow x$ critical point.

(d) $dG_x = (1010)$. $\ker dG_x = \{(v_1,v_2,v_3,v_4) \in T_x\mathbb{R}^4 : v_1 = -v_3\}$

$\ker dF_x = T_xM = \{(c_1x_2, -c_1x_1, c_2x_4, -c_2x_3) \in T_x\mathbb{R}^4 : c_1, c_2 \in \mathbb{R}\} \subset \ker dG_x$

iff $c_1x_2 = -c_2x_4 \ \forall c_1,c_2 \in \mathbb{R}$ i.e. if $\boxed{x_2 = x_4 = 0}$ such points are the crit. points by (c).

$\boxed{\#2}$

$SX := X \times [0,1]/\sim$, $\{(x,t) \sim (y,s)$ if $s=t=0$ or $s=t=1$ or $(x,t)=(y,s)\}$.

Let $A := X \times [0, 3/4]/\sim$, $B := X \times [\frac{1}{4}, 1]/\sim$.

Then $A \simeq \{*\}$, $B \simeq \{*\}$, $A \cap B \simeq X$, $A \cup B = SX$.

Then M.V. gives LES

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(A \cup B) \longrightarrow \cdots$$

$$\cdots \longrightarrow H_n(X) \longrightarrow H_n(\{*\}) \oplus H_n(\{*\}) \longrightarrow H_n(SX) \longrightarrow \cdots$$

$$\begin{array}{lllllll}
H_0(X) & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & H_0(SX) & \longrightarrow & 0 \\
H_1(X) & \longrightarrow & 0 & \longrightarrow & H_1(SX) & & \\
H_2(X) & \longrightarrow & 0 & \longrightarrow & H_2(SX) & & \\
H_3(X) & \longrightarrow & 0 & \longrightarrow & H_3(SX) & &
\end{array}$$

For $n \geq 2$ we have $0 \to H_n(SX) \to H_{n-1}(X) \to 0$

$\Rightarrow H_n(SX) \approx H_{n-1}(X) \ \forall n \geq 2$

Then we have

$$0 \to H_1(SX) \to H_0(X) \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(SX) \to 0$$

Let $H_0(X) = \mathbb{Z}^{\oplus m}$ for m path connected components of X.

Then $\sum_{i=1}^{m} a_i V_i \in H_0(X)$ maps to $(\sum_{i=1}^m a_i, \sum_{i=1}^m a_i)$ for $a_i \in \mathbb{Z}$, V_i represents each path component of X. Then kernel is $\{\sum_{i=1}^m a_i V_i : \sum_{i=1}^m a_i = 0\}$ which is isomorphic to $\mathbb{Z}^{\oplus m-1}$ since there are $m-1$ degrees of freedom.

Then $0 \to H_1(SX) \hookrightarrow H_0(X)$ shows that $H_1(SX) \approx \overset{\text{this}}{\text{kernel}} \approx \mathbb{Z}^{\oplus m-1}$. Finally $H_0(SX) = \mathbb{Z}$ since SX is path connected even if X is not.

In sum,

$$\boxed{H_n(SX) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}^{\oplus m-1} & n=1 \\ H_{n-1}(X) & n \geq 2 \end{cases}}$$

Spring 2018 $\boxed{\#3}$

Let $X = (0, -z, y)$, $Y = (-z, 0, x)$

$$= -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \qquad = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$

Then $[X,Y] = XY - YX = \cancel{-z\left(-z\partial_{xy} + x\partial_{yz}\right)} + y\left(-\frac{\partial}{\partial x} \cancel{- z\partial_{xz} + x\partial_{zz}}\right)$

$$-\left[\cancel{-z\left(-z\partial_{xy} + y\,\partial_{xz}\right)} + x\left(-\partial_y \cancel{- z\partial_{yz} + y\partial_{zz}}\right)\right]$$

$$= -y\,\partial_x + x\,\partial_y$$

$$= \boxed{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}} \quad \text{rotation about the } z\text{-axis.}$$

X and Y are as desired b/c:

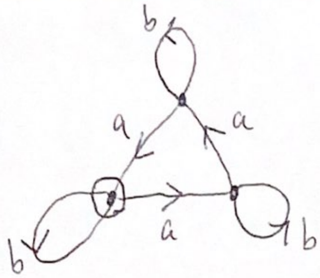
$$(0, -z, y) \times (0, y, z) = \det\begin{pmatrix} 0 & -z & y \\ 0 & y & z \\ i & j & k \end{pmatrix} = (-z^2 - y^2, 0, 0)$$

and $$(-z, 0, x) \times (x, 0, z) = \det\begin{pmatrix} -z & 0 & x \\ x & 0 & z \\ i & j & k \end{pmatrix} = (0, x^2 + z^2, 0)$$

where $(0, y, z)$ and $(x, 0, z)$ are the outward-pointing normal vectors on the cylinder centered at the appropriate axis.

#4

(1)



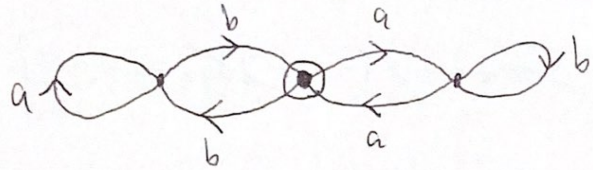
$\boxed{\text{normal}}$ b/c the paths

aa^3a^{-1}, aba^{-1}, a^2ba^{-2}, b,

ba^3b^{-1}, $baba^{-1}b^{-1}$, $ba^{-1}bab^{-1}$ all

end at the base point.

(2)



$\boxed{\text{not normal}}$ b/c for example

$a(aba)a^{-1}$ does not

end at the base point.

This means a is not in

the normalizer of the subgp

in $\mathbb{Z} * \mathbb{Z}$. Hence, the subgp

is not normal.

(1) Hence a, b and therefore all elements in $\mathbb{Z} * \mathbb{Z}$ are in

the normalizer of the subgp, so it is normal.

Spring 2018 $\boxed{\#5}$

(b) f not surjective $\Rightarrow \deg(f)=0 \Rightarrow$ for regular values $y \in N^p$,

$\sum_{x \in f^{-1}(y)} \operatorname{sgn}(x) = 0$, but if $f^{-1}(y) = \{x_1, x_2, x_3\}$

then $\operatorname{sgn}(x_1) + \operatorname{sgn}(x_2) + \operatorname{sgn}(x_3) = -3, -1, 1,$ or 3 not 0.

So f must be surjective.

(a) Let $\tilde{T}$ be a 2-sheeted covering of T^2. Then

$$\begin{tikzcd} & \tilde{T} \arrow[d, "p"] \\ S^2 \arrow[r, "f"'] \arrow[ur, dashed, "\exists \tilde{f}"] & T^2 \end{tikzcd}$$

Since $\pi_1(S^2)$ trivial and therefore in $p_*(\pi_1(\tilde{T}))$, $\exists\ \tilde{f} : S^2 \to \tilde{T}$

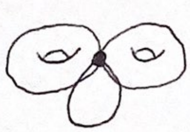
s.t. $p \circ \tilde{f} = f$. Then $\deg(p)\deg(\tilde{f}) = \deg(f) = 1$

But $\deg(p) = 2 \Rightarrow \deg(\tilde{f}) = \frac{1}{2}$, a contradiction.

So $\nexists$ degree 1 map $S^2 \to T^2$.

$\boxed{\#6}$ $T = S^1 \times S^1$ $x, y \in T$ distinct. $Y = T \times \{1,2\}/\sim$ $(x,1) \sim (x,2)$, $(y,1) \sim (y,2)$.

$Y \overset{①}{\cong}$ 

$\overset{②}{\cong}$  $= T \vee T \vee S^1$

$\Rightarrow \pi_1(Y, \bar{x}) = \pi_1(T) * \pi_1(T) * \pi_1(S^1)$

$= \boxed{(\mathbb{Z} \times \mathbb{Z}) * (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}}$ $= \langle a, b, c, d, e \mid aba^{-1}b^{-1} = cdc^{-1}d^{-1} = 1 \rangle$

We may do step 1 because the line from y to y is contractible and a good pair with Y.

We may do step 2 because there is a contractible curve on the tori going from y to $\bar{x}$ to the other y and this is a good pair as well.

$\boxed{\#7}$

(a) $\int_{S^2} 2xyz\, dx\wedge dy + (yz+xy^2)\, dx\wedge dz + xz\, dy\wedge dz$

$= \int_{B^3} 2xy\, dz\wedge dx\wedge dy + (z+2xy)\, dy\wedge dx\wedge dz + z\, dx\wedge dy\wedge dz$

$= \int_{B^3} (2xy - z - 2xy + z)\, dx\wedge dy\wedge dz = \int_{B^3} 0 = \boxed{0}$

(b) $d\omega = \dfrac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}\, dx\wedge dy - \dfrac{(x^2+y^2)-2y^2}{(x^2+y^2)^2}\, dy\wedge dx$

$= \dfrac{2(x^2+y^2)-2x^2-2y^2}{(x^2+y^2)^2}\, dx\wedge dy = \dfrac{0}{x^2+y^2}\, dx\wedge dy = 0 \Rightarrow \omega$ Closed.

If ω exact, then ~~...~~ $\omega = df$ for some $f \in \Omega^0(\mathbb{R}^2-\{0\})$

Then by stokes, $\int_{S^2} \omega = \int_{S^2} df = \int_{\partial S^2} f = \int_{\emptyset} f = 0$

However on S^2 $\omega = xdy - ydx$ since $x^2+y^2=1$.

$xdy-ydx$ is defined on all of $\mathbb{R}^2$, so by Stokes

$\int_{S^2} xdy - ydx = \int_{\partial B^3} xdy-ydx = \int_{B^3} dx\wedge dy - dy\wedge dx = 2\int_{B^3} dxdy = 2\,\mathrm{vol}(B^3) \neq 0$

a contradiction. So ω not exact.