Fall 2018 $\boxed{\#1}$

Let γ be a loop in X based at y_0.

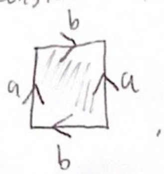
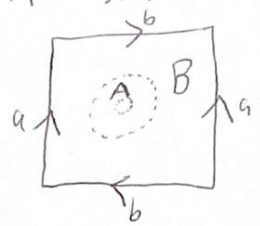
Let $\tilde{y}_0$ be some point in $p^{-1}(y_0)$.

Then γ lifts to a unique $\tilde{\gamma}$, a path in $\tilde{X}$ starting at $\tilde{y}_0$. This path ends at some $\tilde{y}_0'$ in $p^{-1}(y_0)$.

Since $p^{-1}(Y)$ is path connected, $\exists$ $\tilde{\delta}$ path in $p^{-1}(Y)$ from $\tilde{y}_0$ to $\tilde{y}_0'$. This is mapped to $\delta := p(\tilde{\delta})$ a loop in Y based at y_0. Since $\tilde{X}$ is a universal cover it is simply connected $\Rightarrow$ $\tilde{\delta} \simeq \tilde{\gamma}$ and this homotopy is mapped down to a homotopy in X $\delta \simeq \gamma$.

Hence, $i_*([\delta]) = [\gamma]$ for $i_* : \pi_1(Y; y_0) \longrightarrow \pi_1(X; y_0)$, $i : Y \hookrightarrow X$

That is, i_* surjective.

$\boxed{\#2}$ Consider the Klein bottle as $[0,1]\times[0,1]$ with sides identified as such:  . Then we can write A = small disc in the center, $B = K$ − smaller disc in center like so 

So then $A \simeq \{*\}$, $B \simeq S^1 \vee S^1$, $A \cap B \simeq S^1$, $A \cup B = K$.

Then by M.V. we have a LES: (with $\mathbb{Z}_4$ coeffs)

$$\cdots \longrightarrow H_n(A\cap B) \longrightarrow H_n(A)\oplus H_n(B) \longrightarrow H_n(K) \longrightarrow \cdots$$

$$\begin{array}{ccccc}
\mathbb{Z}_4 & \hookrightarrow & \mathbb{Z}_4 \oplus \mathbb{Z}_4 & \longrightarrow & H_0(K) \xrightarrow{=\mathbb{Z}_4} 0 \\
\mathbb{Z}_4 & \longrightarrow & \mathbb{Z}_4\oplus\mathbb{Z}_4 & \twoheadrightarrow & H_1(K) = \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\
0 & \longrightarrow & 0 & \longrightarrow & H_2(K) = \mathbb{Z}_2 \\
0 & \longrightarrow & 0 & \longrightarrow & 0 = H_3(K)
\end{array}$$

$\times 0$

Since K path connected, $H_0(K) = \mathbb{Z}_4$.

$$\begin{array}{l}
H_0(A\cap B) \longrightarrow H_0(A)\oplus H_0(B) \\
\mathbb{Z}_4 \longrightarrow \mathbb{Z}_4 \oplus \mathbb{Z}_4 \\
c \longmapsto (c, c)
\end{array}$$

$\Rightarrow$ kernel $= \{0\}$ $\Rightarrow$ $H_1(K) \xrightarrow{\times 0} H_0(A\cap B)$ $\Rightarrow$ $\mathbb{Z}_4\oplus\mathbb{Z}_4 \twoheadrightarrow H_1(K)$.

$$\begin{array}{l}
\underset{\mathbb{Z}_4}{H_1(A\cap B)} \longrightarrow \underset{0\ \oplus\ (\mathbb{Z}_4\oplus\mathbb{Z}_4)}{H_1(A)\oplus H_1(B)} \\
d \longmapsto (0, 2d)
\end{array}$$

$\Rightarrow$ kernel $= \{0, 2\} = \text{im } H_2(K) \hookrightarrow \mathbb{Z}_4 \Rightarrow H_2(K) = \mathbb{Z}_2$

$\Rightarrow$ im $= \{(0,0), (0,2)\} \Rightarrow H_1(K) = \mathbb{Z}_4 \oplus \mathbb{Z}_2$

since $\text{Ker}(\mathbb{Z}_4\oplus\mathbb{Z}_4 \twoheadrightarrow H_1(K)) = \{(0,0),(0,2)\}$

In all: $$H_n(K;\mathbb{Z}_4) = \begin{cases} \mathbb{Z}_4 & n=0 \\ \mathbb{Z}_4\oplus\mathbb{Z}_2 & n=1 \\ \mathbb{Z}_2 & n=2 \\ 0 & \text{else.} \end{cases}$$

#4

Consider the map $f: M \times M - \Delta \longrightarrow S^{n-1} \subset \mathbb{R}^n$ defined by

$f(P,Q) = \overrightarrow{PQ} / \|\overrightarrow{PQ}\|$, where $\Delta = \{(P,Q) \in M \times M, P = Q\}$.

Since $\dim(M \times M - \Delta) = 2m < \dim S^{n-1}$ df_P cannot be surjective anywhere, hence all values in $im(f)$ are critical, and therefore by Sards, $im(f)$ has measure 0 in S^{n-1}.

Thus, we can find a unit vector $V \in S^{n-1}$ s.t. no $P, Q \in M$ exist satisfying $\overrightarrow{PQ} / \|\overrightarrow{PQ}\| = V$.

The hyperplane H perpindicular to V must then have injective $\pi_H|_M$ where $\pi_H : \mathbb{R}^n \to H$ is the orthogonal projection.

#5 Consider inclusion $i: \partial M \hookrightarrow M$. Then $\partial M \xrightarrow{i} M \xrightarrow{F} \partial M$ the composition is the identity map on ∂M.

Let ω be a volume form on M.

Then $\int_M \omega \neq 0$.

~~Let ω be~~ Stokes gives

$$\int_{\partial M} \omega = \int_M d\omega = 0$$ since $d\omega \in \Omega^{n+1}(M)$ for M n-dim'l

But also $\int_{\partial M} \omega = \int_{\partial M} (F \circ i)^* \omega = \int_{\partial M} i^*(F^* \omega) = \int$

$\boxed{\#5}$

$\cancel{B}$ $F: B^3 \longrightarrow S^2$ s.t. $F|_{S^2} = id$

$f^*\omega|_{\partial M} = \omega$

$i: S^2 \hookrightarrow B^3$

$\int_M \omega = \int_{\partial M} \overset{\neq 0}{\omega} = \int_{\partial M} f^*\omega$

$= \int_M f^* d\omega$

$= \int_{\partial M} d\omega$

$= 0$

$\partial M \xrightarrow{i} M \xrightarrow{F} \partial M$, id

$S^2 \xrightarrow{i} B^3 \xrightarrow{F} S^2$, id

$F \circ i = id$

$H_2(S^2) \xrightarrow{i_*} H_2(B^3) \xrightarrow{F_*} H_2(S^2)$ id $\Rightarrow\Leftarrow$

$\mathbb{Z}$ $\quad 0 \quad$ $\mathbb{Z}$

$\int_{\partial M} \alpha = \int_M d\alpha$

$\int_{\partial M} (F\circ i)^* \alpha = \int_M (F\circ i)^* d\alpha$

$\int_{S^2} \overset{\text{2 form}}{i^*F^*\omega}$ $\quad \int_{B^3} d\omega \quad \int_{S^2} \omega$

$\int_{\partial M} \omega > 0$

$= \int_M d\omega$

$= \int_{\partial M} F^* d\omega$

$= \int_{\partial M} dF^*\omega = \int_{\partial\partial M} F^*\omega = 0$

$$H^{m-1}_{dR}(\partial M) \xleftarrow{i^*} H^{m-1}_{dR}(M) \xleftarrow{F^*} H^{m-1}_{dR}(\partial M)$$

id

$i^*F^*\omega$ $\quad F^*\omega \quad \omega$

must be non-exact

ω vol on ∂M

$\int_{\partial M} F^*\omega = \int$

$\int_{\partial M} i^*F^*\omega = \int_{\partial M} \omega \neq 0$

$= \int_M d\omega$

$= \int_{\partial M} i^* d\omega = \int_M d d i^*\omega = 0$

deg 0 b/c not surjective

$M \xrightarrow{F} \partial M \xrightarrow{i} M$

$\omega \in \Omega^{n-1}(\partial M)$ $\qquad f: M \longrightarrow \partial M$ $\qquad i^*f^*\omega \in \Omega^{n-1}(\partial M)$

$0 = d\omega \in \Omega^{n}(\partial M)$ $\qquad f^*\omega \in \Omega^{n-1}(M)$ $\qquad \overset{\shortparallel}{\omega}$

$i: \partial M \hookrightarrow M$ $\qquad f^*\omega|_{\partial M} \in \Omega^{n-1}(\partial M)$

$$(f^*\omega|_{\partial M})_p(v_1,\dots,v_{n-1}) = f^*\omega_p(v_1,\dots,v_{n-1})$$

$$(i^*f^*\omega)_p(v_1,\dots,v_{n-1}) = f^*\omega_p(v_1,\dots,v_{n-1})$$

$$0 < \int_{\partial M}\omega \cancel{= \int_{\partial M} f^*\omega|_{\partial M} = \int_M f^*\omega = \int}$$

$$\shortparallel$$

$$\int_{\partial M}(f\circ i)^*\omega = \int_{\partial M} \underbrace{i^*f^*(\omega)}_{\text{defined on } M} = \cancel{\int_{\partial M}} \int_M df^*(\omega) = \int_M f^*(d\omega) = 0 \Rightarrow \Leftarrow$$

$\boxed{\#6}$ ω closed because $d\omega \in \Omega^{n+1}(S^n)$ and all $(n+1)$-forms are zero for n-dim'l S^n.

If ω exact, then $\omega = d\alpha$ for $\alpha \in \Omega^{n-1}(\mathbb{R}^{n+1})$

Then by Stokes,

$$\int_{S^n} \omega = \int_{S^n} d\alpha = \int_{\partial S^n} \alpha = \int_{\emptyset} \alpha = 0.$$

But, also by Stokes,

$$\int_{S^n} \omega = \int_{\partial B^{n+1}} \omega = \int_{B^{n+1}} d\omega = \int_{B^{n+1}} dx_{n+1} \wedge dx_1 \wedge \cdots \wedge dx_n = \pm \operatorname{Vol}(B^{n+1}) \neq 0. \Rightarrow\Leftarrow$$

Hence, ω not exact.