

Spring 2017 #1

Let $\beta = \underbrace{\omega \wedge \omega \wedge \dots \wedge \omega}_{n-1 \text{ } \omega\text{'s}} \in \Omega^{2n-2}(M)$.

Then $\beta \wedge \alpha \in \Omega^{2n-1}(M)$.

And $d(\beta \wedge \alpha) = \overset{0}{d\beta} \wedge \alpha + (-1)^{2n-2} \beta \wedge d\alpha$ since $d\beta = d(\omega \wedge \dots \wedge \omega) = d(\omega \wedge \dots \wedge \omega)$
 $= \beta \wedge d\alpha$
 $= \underbrace{\omega \wedge \dots \wedge \omega}_{n \text{ } \omega\text{'s}} \in \Omega^{2n}(M)$

Hence, $\omega \wedge \dots \wedge \omega = \omega^n$ exact.

#2

$U \cap V = S^1 \times S^1$ in X , and this is a neighborhood

deformation retract by taking a small sliver inside of each of the two solid tori. Hence, we may apply

SVK: $\pi_1(X) \approx \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) = \mathbb{Z} *_{\mathbb{Z} \times \mathbb{Z}} \mathbb{Z}$ since U, V are

homotopy equivalent to S^1 . Let a and b represent the generators of $\pi_1(U \cap V) = \mathbb{Z} \times \mathbb{Z}$. The inclusion of a in U is

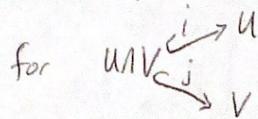
a meridional loop while its inclusion in V is a longitudinal one.

Similarly b is longitudinal in U and meridional in V . The meridional

loops are trivial in U and V . Hence if $\pi_1(U) = \mathbb{Z} \langle c \rangle$, $\pi_1(V) = \mathbb{Z} \langle d \rangle$

Then we have $\pi_1(X) = \langle c, d \mid i_*(a) j_*(a)^{-1} = i_*(b) j_*(b)^{-1} = 1 \rangle$

$= \langle c, d \mid c^{-1} = d = 1 \rangle$



which is the trivial group.

$\pi_1(X) = 1$

#3 We compute Homology using M.V.

$$\dots \rightarrow H_n(U \cup V) \rightarrow H_n(U) \oplus H_n(V) \rightarrow H_n(X) \rightarrow \dots$$

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{\quad} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\quad} & \begin{array}{c} H_0(X) \\ \mathbb{Z} \end{array} & \rightarrow & 0 \\ & & & & \swarrow & & \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{\times 0} & H_1(X) = 0 & & \\ & & & \swarrow \times 0 & & & \\ \mathbb{Z} & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & H_2(X) = 0 & & \\ & & & \swarrow & & & \\ 0 & \xrightarrow{\quad} & 0 & \xrightarrow{\quad} & H_3(X) = \mathbb{Z} & & \end{array}$$

$$H_n(X) = 0 \text{ for } n \geq 4$$

$$0 \rightarrow H_3(X) \rightarrow \mathbb{Z} \rightarrow 0 \Rightarrow H_3(X) = \mathbb{Z}$$

$$H_n(U \cup V) \rightarrow H_n(U) \oplus H_n(V)$$

$$a \mapsto c \oplus 1$$

$$b \mapsto 1 \oplus d$$

isomorphism $\Rightarrow H_2(X) = 0$ since $0 \rightarrow H_2(X)$

$$\mathbb{Z} \xrightarrow{\quad} \mathbb{Z} \oplus \mathbb{Z}$$

$$a \mapsto a \oplus a$$

injective $\Rightarrow \text{im}(H_1(X) \rightarrow \mathbb{Z}) = 0$

$$\text{and } \times 0 \rightarrow H_1(X) \Rightarrow H_1(X) \hookrightarrow \mathbb{Z}$$

$$\Rightarrow H_1(X) = 0$$

So

$$H_n(X) = \begin{cases} \mathbb{Z} & n=0, 3 \\ 0 & \text{else} \end{cases}$$

Spring 2017 #4

Define $f: M \times M - \Delta \rightarrow S^{n-1}$ by $f(x,y) = \frac{x-y}{|x-y|}$ where $\Delta = \{(x,x) \in M \times M\}$.

Then $df_{(x,y)}$ can never be surjective because $\dim(M \times M - \Delta) \leq n-2$

and $\dim(S^{n-1}) = n-1$. Hence all $v \in \text{im}(f)$ are critical values.

So by Sard's, $\text{im}(f)$ has measure 0 in S^{n-1} .

$$\text{im}(f) = \left\{ v \in S^{n-1} : \frac{x-y}{|x-y|} = v \text{ for some } x \neq y \text{ pair in } M \right\}$$

$$= \left\{ v \in S^{n-1} : \exists (x,y) \in M \times M - \Delta \text{ s.t. } \pi_v(x) = \pi_v(y) \right\}$$

$$= \left\{ v \in S^{n-1} : \pi_v|_M \text{ not injective} \right\}$$

So for almost every $v \in S^{n-1}$, $\pi_v|_M$ is injective.

#5 Let \cdot denote concatenation of loops and gh denote multiplication.

We need to show $\gamma \cdot \gamma' \simeq \gamma' \cdot \gamma$ for any loops γ, γ' with basepoint e .

Consider $H: I \times I \rightarrow G$ defined by ~~$H(s,t) = \gamma(s) \cdot \gamma'(t) \cdot \gamma(s)^{-1} \cdot \gamma'(t)^{-1}$~~

~~$H(s,0) = \gamma(s) \cdot \gamma'(0) \cdot \gamma(s)^{-1} \cdot \gamma'(0)^{-1} = e$, $H(s,1) = \gamma(s) \cdot \gamma'(1) \cdot \gamma(s)^{-1} \cdot \gamma'(1)^{-1} = e$~~

~~$H(1,t) = \gamma(1) \cdot \gamma'(t) \cdot \gamma(1)^{-1} \cdot \gamma'(t)^{-1} = e$~~

$$H(s,t) = \begin{cases} \gamma(s) \gamma'(t) \gamma(s)^{-1} \gamma'(t)^{-1} & s \in [0, \frac{1}{2}] \\ \gamma'(t) \gamma(s) \gamma'(t)^{-1} \gamma(s)^{-1} & s \in [\frac{1}{2}, 1] \end{cases}$$

$\gamma \gamma' \simeq \gamma' \gamma$

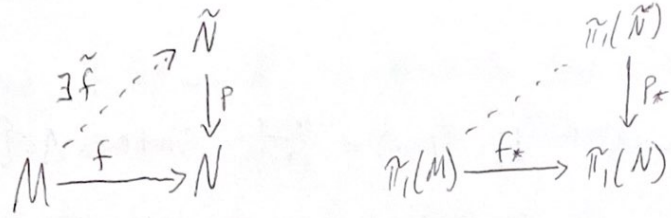
~~$H(0,t) = \gamma(0) \gamma'(t) \gamma(0)^{-1} \gamma'(t)^{-1} = e$, $H(1,t) = \gamma(1) \gamma'(t) \gamma(1)^{-1} \gamma'(t)^{-1} = e$~~

and $\gamma \cdot \gamma' \simeq \gamma \gamma'$

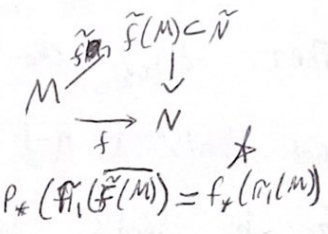
$$H(s,t) = \begin{cases} \gamma(s) \gamma'(t) \gamma(s)^{-1} \gamma'(t)^{-1} \\ \gamma'(t) \gamma(s) \gamma'(t)^{-1} \gamma(s)^{-1} \end{cases}$$

$\gamma' \cdot \gamma \simeq \gamma' \gamma$

#6



$\deg(f) = k$
 $\Rightarrow f$ surjective
 $\Rightarrow p \circ \tilde{f}$ surjective



Let \tilde{N} be a covering space of N s.t.

~~$p_*(\tilde{N}) = f_*(M) \subset \pi_1(N)$~~

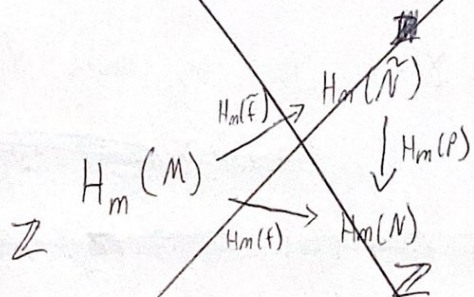
$p_*(\pi_1(\tilde{N})) = f_*(\pi_1(M)) \subset \pi_1(N)$

Such an \tilde{N} exists by classification of ^{connected} covering spaces by subgroups of $\pi_1(N)$.

Assume this subgroup

does not have finite index in $\pi_1(N)$. $\exists \tilde{f}: M \rightarrow \tilde{N}$ s.t. $p \circ \tilde{f} = f$.

Then \tilde{N} is an infinitely sheeted covering space of N and hence \tilde{N} non compact. In homology, we have



$\text{im}(\tilde{f})$ compact in \tilde{N}
 $\Rightarrow \tilde{f}$ non-surjective
 $\Rightarrow \deg(\tilde{f}) = 0$

But this is no degree ∞ map $\mathbb{Z} \rightarrow \mathbb{Z}$
 So $\deg \tilde{f}$ must be 0 and therefore $\deg(f)$ must be 0 as well.

$\Rightarrow \deg(f) = \deg(\tilde{f}) \deg(p) = 0$