Spring 2017 $\boxed{\#1}$

Let $\beta = \underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n-1\ \omega\text{'s}} \in \Omega^{2n-2}(M)$.

Then $\beta \wedge \alpha \in \Omega^{2n-1}(M)$.

And $d(\beta \wedge \alpha) = \overset{0}{\cancel{d\beta \wedge \alpha}} + (-1)^{2n-2} \beta \wedge d\alpha$ since $d\beta = d(\omega \wedge \cdots \wedge \omega) = d(d\alpha \wedge \cdots \wedge d\alpha)$

$= \beta \wedge d\alpha$

$= \underbrace{\omega \wedge \cdots \wedge \omega}_{n\ \omega\text{'s}} \in \Omega^{2n}(M)$

Hence, $\omega \wedge \cdots \wedge \omega = \omega^n$ exact.

$\boxed{\#2}$ $U \cap V = S^1 \times S^1$ in X, and this is a neighborhood deformation retract by taking a small sliver inside of each of the two solid tori. Hence, we may apply SVK: $\pi_1(X) \approx \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) = \mathbb{Z} *_{\mathbb{Z}\times\mathbb{Z}} \mathbb{Z}$ since U, V are homotopy equivalent to S^1. Let a and b represent the generators of $\pi_1(U \cap V) = \mathbb{Z} \times \mathbb{Z}$. The inclusion of a in U is a meridional loop while it's inclusion in V is a longitudinal one. Similarly b is longitudinal in U and meridional in V. The meridional loops are trivial in U and V. Hence if $\pi_1(U) = \mathbb{Z}\langle c \rangle$, $\pi_1(V) = \mathbb{Z}\langle d \rangle$ Then we have $\pi_1(X) = \langle c, d \mid i_*(a) j_*(a)^{-1} = i_*(b) j_*(b)^{-1} = 1 \rangle$

$= \langle c, d \mid c^{-1} = d = 1 \rangle$ for $U \cap V \overset{i}{\hookrightarrow} U$, $U \cap V \overset{j}{\hookrightarrow} V$

which is the trivial group.

$$\boxed{\pi_1(X) = 1}$$

$\boxed{\#3}$ We compute Homology using M.V.

$$\cdots \longrightarrow H_n(U\cap V) \longrightarrow H_n(U)\oplus H_n(V) \longrightarrow H_n(X) \longrightarrow \cdots$$

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, hook] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, two heads] & H_0(X) \arrow[r] & 0 \\
& & \mathbb{Z} & \\
\mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "\times 0"] & H_1(X)=0 \arrow[uull] & \\
\mathbb{Z} \arrow[r] & 0 \arrow[r] & H_2(X)=0 \arrow[ull, "\times 0"'] & \\
0 \arrow[r] & 0 \arrow[r] & H_3(X)=\mathbb{Z} \arrow[ull] &
\end{tikzcd}$$

$H_n(X)=0$ for $n\geq 4$

$0 \to H_3(X) \to \mathbb{Z} \to 0 \Rightarrow H_3(X)=\mathbb{Z}$

$$\begin{array}{ccc} H_n(U\cap V) & \longrightarrow & H_n(U)\oplus H_n(V) \\ a & \longmapsto & c\oplus 1 \\ b & \longmapsto & 1\oplus d \end{array}$$

isomorphism $\Rightarrow H_2(X)=0$ since $0 \to H_2(X)$

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}\oplus\mathbb{Z} \\ a & \longmapsto & a\oplus a \end{array}$$

injective $\Rightarrow \mathrm{im}(H_1(X)\to\mathbb{Z})=0$

and $\xrightarrow{\times 0} H_1(X) \Rightarrow H_1(X) \hookrightarrow \mathbb{Z}$ $\Rightarrow H_1(X)=0$

So $\boxed{H_n(X)=\begin{cases}\mathbb{Z} & n=0,3\\ 0 & \text{else}\end{cases}}$

Spring 2017 $\boxed{\#4}$

Define $f: M\times M - \Delta \longrightarrow S^{n-1}$ by $f(x,y) = \frac{x-y}{|x-y|}$ where $\Delta = \{(x,x) \in M\times M\}$.

Then $df_{(x,y)}$ can never be surjective because $\dim(M\times M - \Delta) \leq n-2$ and $\dim(S^{n-1}) = n-1$. Hence all $v \in \operatorname{im}(f)$ are critical values.

So by Sard's, $\operatorname{im}(f)$ has measure 0 in S^{n-1}.

$$\operatorname{im}(f) = \left\{ v \in S^{n-1} : \frac{x-y}{|x-y|} = v \text{ for some } x\neq y \text{ pair in } M \right\}$$

$$= \{ v \in S^{n-1} : \exists\, (x,y) \in M\times M - \Delta \text{ s.t. } \pi_v(x) = \pi_v(y) \}$$

$$= \{ v \in S^{n-1} : \pi_v|_M \text{ not injective} \}$$

So for almost every $v \in S^{n-1}$, $\pi_v|_M$ is injective.

$\boxed{\#5}$ Let $\cdot$ denote concatenation of loops and gh denote multiplication.

We need to show $\gamma\cdot\gamma' \simeq \gamma'\cdot\gamma$ for any loops γ, γ' with basepoint e.

Consider $H: I\times I \longrightarrow G$ defined by ~~$H(s,t) = \gamma(s(1-t))\gamma'(st) \cdot \gamma(st)\gamma'(s(1-t))$~~

~~Then $H(s,0) = \gamma(s)\cdot\gamma'(s)$, $H(s,1) = \gamma'(s)\cdot\gamma(s)$, $H(0,t) = \gamma(0)\gamma'(0) = \gamma(0)\gamma(0) = e$~~

~~$H(1,t) = \gamma(1-t)\gamma'(t)\cdot\gamma$~~

$H(s,t) = \begin{cases} \cancel{\gamma(2s(1-t))\gamma'(2st)} & \cancel{s\in[0,1/2]} \\ \cancel{\gamma((2s-1)t)\gamma'((2s-1)(1-t))} & \cancel{s\in[\tfrac12,1]} \end{cases}$

$\gamma\gamma' \simeq \gamma'\gamma$

~~Then $H(0,t) = \gamma(0)\gamma'(0) = e$, $H(1,t) = \gamma($~~

and $\gamma\cdot\gamma' \simeq \gamma\gamma'$

$\gamma'\cdot\gamma \simeq \gamma'\gamma$

~~$H(s,t) = \{ \gamma(s)\gamma(s(1-t))^{-1}$~~

$\boxed{\#6}$

$$\begin{tikzcd} & \tilde N \arrow[d, "p"] \\ M \arrow[r, "f"] \arrow[ur, dashed, "\exists \tilde f"] & N \end{tikzcd} \qquad \begin{tikzcd} & \pi_1(\tilde N) \arrow[d, "p_*"] \\ \pi_1(M) \arrow[r, "f_*"] \arrow[ur, dotted] & \pi_1(N) \end{tikzcd}$$

$\deg(f) \overset{=k}{\neq} 0$

$\Rightarrow f$ surjective

$\Rightarrow p \circ \tilde f$ surjective

$$\begin{tikzcd} & \tilde f(M) \subset \tilde N \arrow[d] \\ M \arrow[r, "f"] \arrow[ur, "\tilde f"] & N \end{tikzcd}$$

$p_*(\pi_1(\tilde f(M))) = f_*(\pi_1(M))$

Let $\tilde N$ be a covering space of N s.t.

~~$p_*(\tilde N) = f_*(M) \subset \pi_1($~~

$p_*(\pi_1(\tilde N)) = f_*(\pi_1(M)) \subset \pi_1(N)$. Such an $\tilde N$ exists by classification of connected covering spaces by subgps of $\pi_1(N)$.

Assume this subgp does not have finite index in $\pi_1(N)$. $\exists \tilde f : M \to \tilde N$ s.t. $p \circ \tilde f = f$.

Then $\tilde N$ is an infinitely sheeted covering space of N and hence $\tilde N$ non compact. ~~and $\deg(p) = \infty$.~~ In homology, we have

$$\begin{tikzcd} & H_m(\tilde N) \arrow[d, "H_m(p)"] \\ H_m(M) \arrow[r, "H_m(f)"'] \arrow[ur, "H_m(\tilde f)"] & H_m(N) \end{tikzcd}$$

$\mathbb{Z}$ $\qquad$ $\mathbb{Z}$

$\operatorname{im}(\tilde f)$ compact in $\tilde N$

$\Rightarrow \tilde f$ non-surjective

$\Rightarrow \deg(\tilde f) = 0$

~~But there is no degree ∞ map $\mathbb{Z} \to \mathbb{Z}$~~

~~So $\deg \tilde f$ must be 0 and therefore $\deg(f)$ must be 0 as well.~~

$\Rightarrow \deg(f) = \deg(\tilde f)\deg(p) = 0$