Fall 2017 $\boxed{\#1}$

M compact $\Rightarrow$ $f(M)$ compact $\Rightarrow$ f not surjective $\Rightarrow$ f has degree 0 when considered as a map from M to some m-dim'l ball in $\mathbb{R}^m$ containing $f(M)$. Then for all regular values $y \in \mathbb{R}^m$,

$\sum_{x \in f^{-1}(y)} \operatorname{sgn}(x) = \deg(f) = 0$, where $\operatorname{sgn}(x) = \pm 1$ based on whether f is orientaton preserving or reversing at x. Hence,

$\{ y \in \mathbb{R}^m : f^{-1}(y)$ has an odd # of points$\}$ has measure 0 since such points cannot be regular since $\sum_{x \in f^{-1}(y)} \operatorname{sgn}(x) \neq 0$. So they are critical values and Sard's theorem finishes.

$\boxed{\#2}$

$$\pi_1(X) = \langle a \mid a^2 a^{-1} a a^{-4} = 1 \rangle = \langle a \mid a^{-2} = 1 \rangle = \langle a \mid a^2 = 1 \rangle = \boxed{\mathbb{Z}/2}.$$

$\boxed{\#3}$

All connected covers of X correspond to subgps of $\pi_1(X; x_0) \cong \mathbb{Z}/5$. The only subgps of $\mathbb{Z}/5$ are the trivial subgp and all of $\mathbb{Z}/5$ since 5 is prime and $\langle 1 \rangle = \langle 2 \rangle = \langle 3 \rangle = \langle 4 \rangle = \mathbb{Z}/5$ where $\langle n \rangle$ is the subgp generated by the element n in $\mathbb{Z}/5$. Since the trivial subgp has index 5 in $\mathbb{Z}/5$ and $\mathbb{Z}/5$ has index 1, the only connected covers of X are either one-sheeted or 5-sheeted. So if $p^{-1}(x_0)$ has 4 points, then it is a 4-sheeted cover and must consist of 4 disjoint one-sheeted covers and is hence a trivial cover.

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$M = f^{-1}(0)$ where $f: \mathbb{R}^4 \to \mathbb{R}$ is defined by $f(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2$.

Hence, if it is a mfd it will be 3-dim'l since the codimension would be equal to the dimension of the codomain $\mathbb{R}$.

However, the four lines $x_1 = x_3$, $x_1 = -x_3$, $x_2 = x_4$, $x_2 = -x_4$ are all contained in M, and their tangent vectors at 0 are $(1,0,1,0)$, $(1,0,-1,0)$, $(0,1,0,1)$, and $(0,1,0,-1)$ respectively.

These are 4 linearly independent vectors since

$c_1 + c_2 = 0$, $c_1 - c_2 = 0$, $c_3 + c_4 = 0$, $c_3 - c_4 = 0 \Rightarrow c_1 = c_2 = c_3 = c_4 = 0$

So the tangent space at 0 is 4-dim'l a contradiction.

$\boxed{\#6}$

First, see that X deformation retracts to $\{x \in X \subset \mathbb{R}^3 : |x| \leq 2\}$ by bringing the sides of the cylinders in. Then notice that the end of the smaller cylinder is contractible

i.e. $\{(x,y,z) \in \mathbb{R}^3 : x^2 + z^2 - (\frac{1}{2})^2 = 0 \text{ and } x^2 + y^2 \geq 1 \text{ and } y < 0\}$

union $\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 - 1 = 0 \text{ and } x^2 + z^2 \leq (\frac{1}{2})^2 \text{ and } y < 0\}$.

This and the other similar construction with $y > 0$ are contractible and they make a good pair with X, so we may collapse them to a point. Then we have a large cylinder with a sphere glued in the inside at two points . The cylinder deformation retracts down onto the circle , and then we get finally $X \simeq S^2 \vee S^1 \vee S^1$. Hence $\tilde{H}_n(X) \approx \tilde{H}_n(S^2) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1)$

$$\Rightarrow \quad H_n(X) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

#7

$$d\sigma = \frac{(x^2+y^2+z^2)^{3/2} - x\left(\frac{3}{2}\right)(x^2+y^2+z^2)^{1/2}(2x)}{(x^2+y^2+z^2)^3}\, dx \wedge dy \wedge dz$$

$$- \frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\, dy \wedge dx \wedge dz$$

$$+ \frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}\, dz \wedge dx \wedge dy$$

$$= \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{1/2}(x^2+y^2+z^2)}{(x^2+y^2+z^2)^3}\, dx \wedge dy \wedge dz$$

$$= 0 \quad \Rightarrow \quad \sigma \text{ closed.}$$

If σ exact, then $\sigma = d\alpha$ for some $\alpha \in \Omega^1(\mathbb{R}^3 \setminus \{(0,0,0)\})$

Then by Stokes $\int_{S^2} \sigma = \int_{S^2} d\alpha = \int_{\partial S^2} \alpha = \int_{\emptyset} \alpha = 0$.

But $\int_{S^2} \sigma = \int_{S^2} x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy$ since $x^2+y^2+z^2 = 1$ on S^2

Then $\beta := x\, dy \wedge dz - y\, dx \wedge dz + z\, dx \wedge dy$ is a 2-form on all of $\mathbb{R}^3$

and we can apply Stokes again to get

$$\int_{S^2} \sigma = \int_{S^2} \beta = \int_{\partial B^3} \beta = \int_{B^3} d\beta = \int_{B^3} 3\, dx \wedge dy \wedge dz = 3\,\mathrm{vol}(B^3) \neq 0$$

a contradiction. Hence σ not exact.