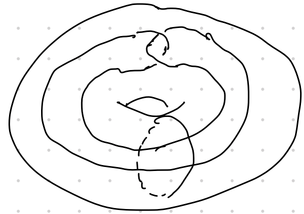
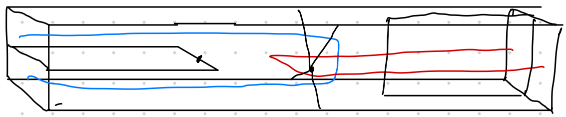
PRE-2012

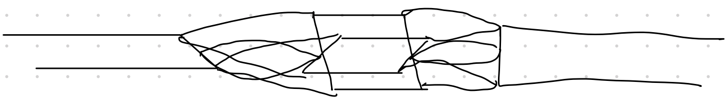
Spring 2005 $\boxed{\#7}$

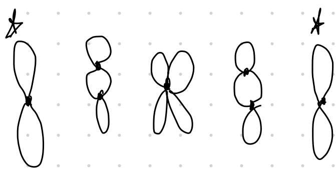
Compute Homology of

$T \setminus K$

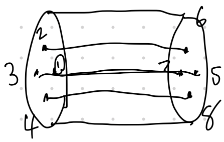







$\simeq \bigvee_1^{12} S^1$

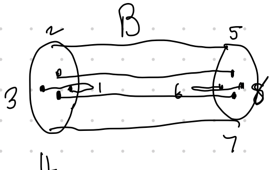
A

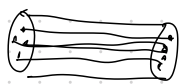


$\simeq D^2 - \{4 \text{ pts}\} \simeq \bigvee_1^4 S^1$

$A \cap B \simeq \bigvee_1^4 S^1 \sqcup \bigvee_1^4 S^1$

B



$\simeq$

$\simeq \bigvee_1^4 S^1$

$$\cdots \longrightarrow \tilde{H}_n(A\cap B) \longrightarrow \tilde{H}_n(A) \oplus \tilde{H}_n(B) \longrightarrow \tilde{H}_n(X) \longrightarrow \cdots$$

$n\geq 3$	0	0	0	
$n=2$	0	0		
$n=1$	$\mathbb{Z}^8$	$\mathbb{Z}^8$		
$n=0$	0	0	0	$\longrightarrow 0$

$$\tilde{H}_n(A\cap B) \xrightarrow{i_*} \tilde{H}_n(A) \oplus \tilde{H}_n(B)$$

γ_1	$\longmapsto$	α_1	β_1
γ_2		α_2	β_2
γ_3		α_3	β_1
γ_4		α_4	β_3
γ_5		α_1	β_2
γ_6		α_2	β_4
γ_7		α_3	β_3
γ_8		α_4	β_4

Fall 2005 $\boxed{\#1}$

Claim: complement of a finite set of points in $\mathbb{R}^n$ is simply connected if $n \geq 3$.

$$\mathbb{R}^n - \{p_1, \dots, p_m\} \simeq B^n - \{p_1, \dots, p_m\} \simeq \bigvee_{i=1}^{m} B^n - \{p_i\} \simeq \bigvee_{i=1}^{m} S^{n-1}$$

Thus, $\pi_1(\mathbb{R}^n - \{p_1, \dots, p_m\}) = *_{i=1}^{m} \pi_1(S^{n-1}) = 0$ if $n \geq 3$.

Fall 2005 $\boxed{\#5}$

Claim: for any space X, we have

$$H_i(X \times S^1) \approx H_i(X) \oplus H_{i-1}(X)$$

Let $X \times S^1 = X \times [0,1] / \sim$ where $(x,0) \sim (x,1)$

Let $A := X \times [\frac{1}{4}, \frac{3}{4}]$, $B := X \times ([0, \frac{1}{3}] \cup [\frac{2}{3}, 1]) / \sim$

Then A and B deformation retract to X and

$A \cap B = X \times ([\frac{1}{4}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{3}{4}])$ deformation retracts to $X \sqcup X$.

Then, MV gives

$$\cdots \longrightarrow \underset{H_i(X) \oplus H_i(X)}{H_i(A) \oplus H_i(B)} \longrightarrow H_i(X \times S^1) \longrightarrow \underset{H_{i-1}(X) \oplus H_{i-1}(X)}{H_{i-1}(A \cap B)} \longrightarrow \cdots$$

Consider $H_{i-1}(A \cap B) \xrightarrow{i_*} H_{i-1}(A) \oplus H_{i-1}(B)$.

$$[\alpha], [\beta] \longmapsto [\alpha + \beta], [\alpha + \beta]$$

this has kernel $\{([\alpha], [-\alpha])\} \approx H_{i-1}(X)$

So image of $H_i(X \times S^1) \longrightarrow H_{i-1}(A \cap B)$ is $H_{i-1}(X)$. ②

And the image of $H_i(A \cap B) \xrightarrow{i_*} H_i(A) \oplus H_i(B)$ is $\{([\alpha], [\alpha])\} \approx H_i(X)$.

So the kernel of $H_i(A) \oplus H_i(B) \longrightarrow H_i(X \times S^1)$ is $H_i(X)$.

? $\Rightarrow$ image of $H_i(A) \oplus H_i(B) \longrightarrow H_i(X \times S^1)$ is the remaining $H_i(X)$. ③

$\Rightarrow$ kernel of $H_i(X \times S^1) \longrightarrow H_{i-1}(A \cap B)$ is $H_i(X)$. ①

① + ② $\overset{?}{\Rightarrow}$ $H_i(X \times S^1) \approx H_i(X) \oplus H_{i-1}(X)$

$$0 \longrightarrow H_i(X) \xrightarrow{i} H_i(X \times S^1) \longrightarrow H_{i-1}(X) \longrightarrow 0$$

splits because

$$A \xrightarrow{\varphi} B$$

$$\ker(\varphi) \oplus \mathrm{Im}(\varphi)$$

$$\begin{tikzcd}
0 \arrow[r] & \ker(\varphi) \arrow[r, "i"] & A \arrow[r, "\varphi"] \arrow[l, bend right, dashed] & \mathrm{Im}(\varphi) \arrow[r] \arrow[l, bend right, dashed] & 0
\end{tikzcd}$$

splits?

Spring 2006 $\boxed{\#3}$

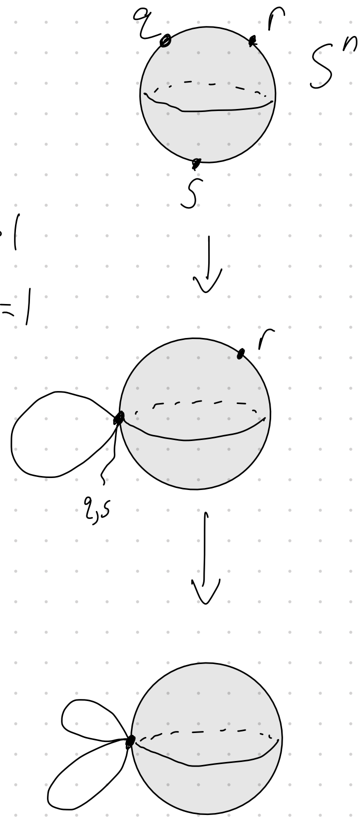
$\pi_1(X)$ where $X = S^n / \{q, r, s\}$

$X \simeq S^n \vee S^1 \vee S^1$

So $\pi_1(X) = \pi_1(S^n) * \pi_1(S^1) * \pi_1(S^1)$

$$= \begin{cases} \pi_1(S^1) * \pi_1(S^1) & \text{if } n > 1 \\ \pi_1(S^1) * \pi_1(S^1) * \pi_1(S^1) & n = 1 \end{cases}$$

$$= \boxed{\begin{cases} \mathbb{Z} * \mathbb{Z} & n > 1 \\ \mathbb{Z} * \mathbb{Z} * \mathbb{Z} & n = 1 \end{cases}}$$



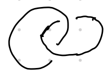
Spring 2006 $\boxed{\#6}$

unlink := two unknotted circles in S^3 contained in two disjoint 3-d balls in S^3.

Hopf link := two unknotted disjoint circles in S^3 where each circle meets a disk bounding the other in one point.

Drawn here in $\mathbb{R}^3 = S^3 - \{\infty\}$

unlink



link

$U := S^3 -$ unlink, $H := S^3 -$ Hopf link. Compute Homology of U & H.

Let $A :=$ the two disjoint 3-d balls containing the unlink.

$$S^3/A \simeq S^3/\{2 \text{ points}\} \simeq S^3 \vee S^1$$

So $H_n(S^3, A) \approx \tilde{H}_n(S^3/A) \approx \tilde{H}_n(S^3 \vee S^1) \approx \tilde{H}_n(S^3) \oplus \tilde{H}_n(S^1)$.

$\tilde{H}_n(S^3) = \begin{cases} \mathbb{Z} & n=3 \\ 0 & \text{else} \end{cases}$, $\tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$.

So $H_n(S^3, A) = \begin{cases} \mathbb{Z} & n=1,3 \\ 0 & \text{else} \end{cases}$

So $H_n(U, A - \text{unlink}) = \begin{cases} \mathbb{Z} & n=1,3 \\ 0 & \text{else} \end{cases}$ by excision.

We know $A - \text{unlink} \simeq S^2 \vee S^1 \sqcup S^2 \vee S^1$

and $\tilde{H}_n(S^2 \vee S^1) \approx \tilde{H}_n(S^2) \oplus \tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n=1,2 \\ 0 & \text{else} \end{cases}$

So $H_n(A - \text{unlink}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=0,1,2 \\ 0 & \text{else} \end{cases}$

Now we have

$$\cdots \to H_n(A-\text{unlink}) \to H_n(U) \to H_n(U, A-\text{unlink}) \to H_{n-1}(A-\text{unlink})$$

	$H_n(A-\text{unlink})$	$H_n(U)$	$H_n(U, A-\text{unlink})$	$H_{n-1}(A-\text{unlink})$
$n=0$	$\mathbb{Z} \oplus \mathbb{Z}$		0	0
$n=1$	$\mathbb{Z} \oplus \mathbb{Z}$		$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$

$n=2$	$\mathbb{Z}\oplus\mathbb{Z}$		$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$
$n=3$	0	0	0	$\mathbb{Z}\oplus\mathbb{Z}$

MV: $\quad H_n(A\cap B) \longrightarrow H_n(A)\oplus H_n(B) \longrightarrow H_n(X)$

Let U_ε be small tori neighborhoods of the unlink.
and let $U_{2\varepsilon}$ be a slightly larger one.
Then $U_{2\varepsilon}$ and $S^3 - U_\varepsilon$ cover S^3, and MV gives

$$\cdots \longrightarrow H_n(U_{2\varepsilon}\cap(S^3-U_\varepsilon)) \longrightarrow H_n(U_{2\varepsilon})\oplus H_n(S^3-U_\varepsilon) \longrightarrow H_n(S^3) \longrightarrow \cdots$$

$U_{2\varepsilon}\cap(S^3-U_\varepsilon) \simeq T^2 \sqcup T^2 \Rightarrow H_n(U_{2\varepsilon}\cap(S^3-U_\varepsilon)) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n=0,2 \\ \mathbb{Z}^4 & n=1 \\ 0 & \text{else} \end{cases}$

$U_{2\varepsilon} \simeq S^1 \sqcup S^1 \Rightarrow H_n(U_{2\varepsilon}) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n=0,1 \\ 0 & \text{else} \end{cases}$ and $H_n(S^3) = \begin{cases} \mathbb{Z} & n=0,3 \\ 0 & \text{else} \end{cases}$

So we get:

	$H_n(U_{2\varepsilon}\cap(S^3-U_\varepsilon))$	$\longrightarrow H_n(U_{2\varepsilon})\oplus H_n(S^3-U_\varepsilon)$	$\longrightarrow H_n(S^3)$
$n=4$	0	$0 \oplus 0$	0
$n=3$	0	$0 \oplus$	$\mathbb{Z}$
$n=2$	$\mathbb{Z}\oplus\mathbb{Z}$	$0 \oplus$	0
$n=1$	$\mathbb{Z}^4$	$\mathbb{Z}\oplus\mathbb{Z} \oplus \mathbb{Z}\oplus\mathbb{Z}$	0
$n=0$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z} \longrightarrow 0$

Since $H_2(S^3) = H_1(S^3) = 0$, $H_1(U_{2\varepsilon}\cap(S^3-U_\varepsilon)) \approx H_1(U_{2\varepsilon}) \oplus H_1(S^3-U_\varepsilon)$
So $H_1(S^3-U_\varepsilon) \approx \mathbb{Z}\oplus\mathbb{Z}$
We know $H_0(S^3-U_\varepsilon) = \mathbb{Z}$ since S^3-U_ε is connected.
We see that the image of $H_3(S^3) \longrightarrow H_2(U_{2\varepsilon}\cap(S^3-U_\varepsilon))$
cannot be $\mathbb{Z}\oplus\mathbb{Z}$, so the next map cannot be the 0 map.
Hence, $H_n(S^3-U_\varepsilon) \neq 0$.

If $\mathbb{Z}$ then $\mathbb{Z} \xrightarrow{\approx} \mathbb{Z}$ then $\mathbb{Z} \xrightarrow{\times 0} \mathbb{Z}\oplus\mathbb{Z}$ then $\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{inj} ?$

then $? = \mathbb{Z}\oplus\mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{\approx} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\approx} \mathbb{Z}\oplus\mathbb{Z} \to 0 \quad \text{fine.}$$

$\boxed{2B.1}$ (b) $h: S^k \to S^n$ embedding

then $\tilde{H}_i(S^n - h(S^k)) = \mathbb{Z}$ for $i = n-k-1$ and 0 otherwise.

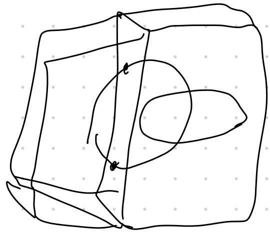
assume:

(a) $h: D^k \to S^n$, $\tilde{H}_i(S^n - h(D^k)) = 0 \ \forall i$.

$S^n - h(S^0) = S^n - \{2 \text{ points}\} \simeq D^{n-1} - \{1 \text{ point}\} \simeq S^{n-1}$

$\Rightarrow \tilde{H}_i(S^n - h(S^0)) \approx \tilde{H}_i(S^{n-1}) = \mathbb{Z}$ for $n-1$ and 0 otherwise. ✓

<u>Induction:</u>



$S^k = D^k_+ \cup D^k_-$ hemispheres.

$A := S^n - h(D^k_+)$ $\quad B = S^n - h(D^k_-)$

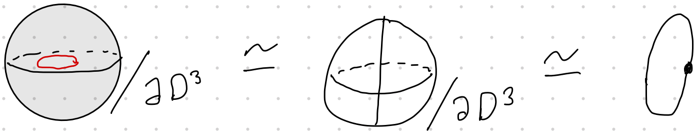
$$\to \underset{\tilde{H}_i(S^n - h(S^k))}{\underset{\|}{\tilde{H}_i(A \cap B)}} \to \underset{\substack{0\ \forall i \\ \text{by (a)}}}{\tilde{H}_i(A)\oplus\tilde{H}_i(B)} \to \tilde{H}_i(S^n - h(S^{k-1})) \to$$

$$\Rightarrow \tilde{H}_i(S^n - h(S^k)) \approx \tilde{H}_{i+1}(S^n - h(S^{k-1}))$$

$$\approx \tilde{H}_{i+k}(S^n - h(S^0)) = \mathbb{Z} \text{ for } \begin{array}{l} i+k = n-1 \\ i = n-k-1 \end{array}$$ ✓

First, let's compute $H_n(S^3 - S^1)$. View S^3 as $D^3/\partial D^3$

Then



$S^3 - S^1 \simeq S^1$. So $H_n(S^3 - S^1) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{else.} \end{cases}$

Now, for $S^3 -$ unlink, we use MV.

Let $A, B =$ hemispheres of $S^3 = D^3/\partial D^3$ each containing one circle.

Then $A \cap B = D^2/\partial D^2 = S^2 \Rightarrow H_n(A \cap B) = \begin{cases} \mathbb{Z} & n = 0,2 \\ 0 & \text{else.} \end{cases}$

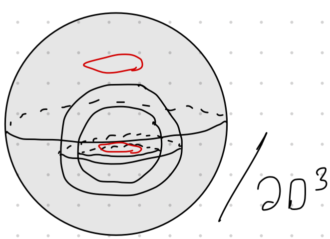
$A \simeq B^3 - S^1 \simeq S^2 \vee S^1 \Rightarrow \tilde{H}_n(A) = \begin{cases} \mathbb{Z} & n = 1,2 \\ 0 & \text{else} \end{cases} \Rightarrow H_n(A) = \begin{cases} \mathbb{Z} & n = 0,1,2 \\ 0 & \text{else} \end{cases}$

The same applies for B and we get a sequence:

$\cdots \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(S^3 - \text{unlink}) \longrightarrow H_{n-1}(A \cap B) \longrightarrow \cdots$

	$H_n(A) \oplus H_n(B)$		$H_n(S^3 - \text{unlink})$		$H_{n-1}(A \cap B)$
$n=4$	0		0		0
$n=3$	0		0	$\xhookrightarrow{\times 0}$	$\mathbb{Z}$
$n=2$	$\mathbb{Z} \oplus \mathbb{Z}$	$\twoheadrightarrow$	$\mathbb{Z}$	$\xrightarrow{\times 0}$	0
$n=1$	$\mathbb{Z} \oplus \mathbb{Z}$	$\hookrightarrow\!\!\twoheadrightarrow$	$\mathbb{Z} \oplus \mathbb{Z}$	$\xrightarrow{\times 0}$	$\mathbb{Z}$
$n=0$	$\mathbb{Z} \oplus \mathbb{Z}$		$\mathbb{Z}$		0

✓



$A =$ larger inner sphere

$B = S^3 -$ smaller inner sphere

$A \cap B \simeq S^2$

$A \simeq S^2 \vee S^1$

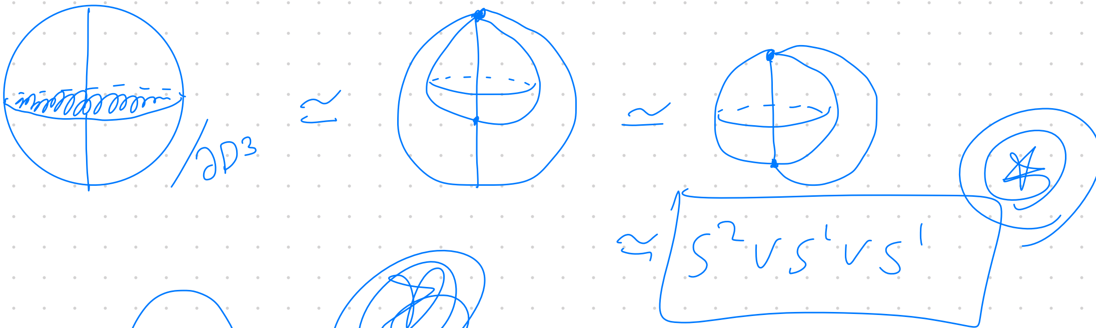
$B \simeq$  $\simeq S^2 \vee S^1$

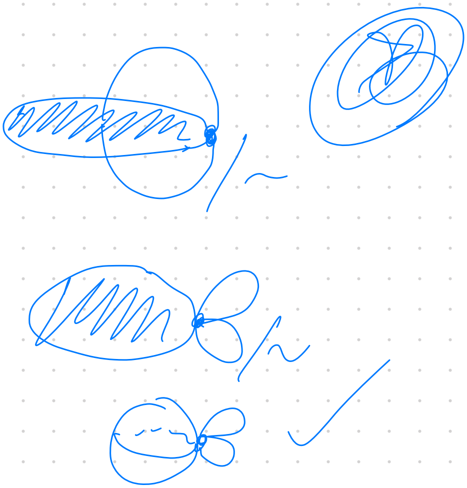
$(0,1) \longrightarrow 1$

$(1,0) \longrightarrow 1$

$(2,2) = 2(0,1) + 2(1,0) = 4$

$(x,-x) \longrightarrow 0$





Fall 2006 $\boxed{\#2}$

$T^2 = S^1 \times S^1$ torus. D_1, D_2 closed unit disks in $\mathbb{C}$.

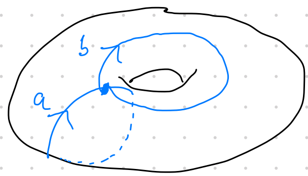
For integers p, q define $X_{pq} := T^2 \sqcup D_1 \sqcup D_2 / \sim$

where $\underset{\partial D_1}{e^{i\theta}} \sim \underset{T^2}{(e^{ip\theta}, 1)}$ and $\underset{\partial D_2}{e^{i\phi}} \sim \underset{T^2}{(1, e^{iq\phi})}$

Compute $\pi_1(X_{pq}; x_0)$

Let $x_0 = (1,1) \in T^2$. Let's form X_{pq} as a CW complex.

$e_0 = x_0$. $e_1^1 = S^1 \times \{1\}$, $e_1^2 = \{1\} \times S^1$.



e_1^1 determines a loop a with base x_0

and e_1^2 determines loop b with base x_0

To obtain X_{pq}, we attach 2-cells

e_2^0, e_2^1, e_2^2 along the following paths:

$aba^{-1}b^{-1}$, a^p, b^q

Hence we get:

$$\boxed{\pi_1(X_{pq}) = \langle a, b \mid aba^{-1}b^{-1} = a^p = b^q = 0 \rangle}$$

Fall 2006 $\boxed{\#3}$

X simply connected. $f, g : X \to S^1$

Claim: $f \simeq g$.

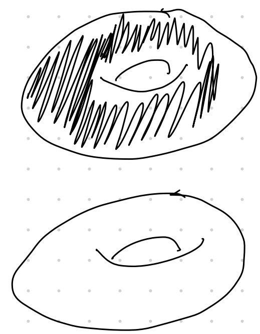
?

$$X \xrightarrow{f} S^1 \xrightarrow{\gamma} X$$

$$\pi_1(X) \underset{\times 0}{\xrightarrow{f_*}} \pi_1(S^1) \underset{\times 0}{\xrightarrow{\gamma_*}} \pi_1(X)$$

Fall 2006 $\boxed{\#4}$

Calculate $H_n(S^1 \times D^2, S^1 \times \partial D^2)$



$$H_n(S^1 \times D^2) \approx H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else} \end{cases}$$

$$H_n(S^1 \times \partial D^2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

$$\cdots \to H_n(\partial X) \to H_n(X) \to H_n(X, \partial X) \to H_{n-1}(\partial X) \to \cdots$$

For $n \geq 3$, we have

$$0 \to H_n(X, \partial X) \to H_{n-1}(\partial X) \to 0$$

implying $H_n(X, \partial X) \approx H_{n-1}(\partial X)$

So $H_n(X, \partial X) = \begin{cases} \mathbb{Z} & n = 3 \\ 0 & n \geq 4 \end{cases}$.

$$\to \underset{\mathbb{Z}}{H_2(\partial X)} \to \underset{0}{H_2(X)} \to H_2(X, \partial X) \to \underset{\mathbb{Z} \oplus \mathbb{Z}}{H_1(\partial X)} \to \underset{\mathbb{Z}}{H_1(X)} \xrightarrow{\times 0} \underset{0}{H_1(X, \partial X)} \xrightarrow{\times 0}$$

$$\xrightarrow{\times 0} \underset{\mathbb{Z}}{H_0(\partial X)} \xrightarrow{\approx} \underset{\mathbb{Z}}{H_0(X)} \xrightarrow{\times 0} \underset{0}{H_0(X, \partial X)} \to 0$$

$H_0(\partial X) \xrightarrow{i_*} H_0(X)$ is an isomorphism ?

So the maps before and after are 0-maps.

Since $H_0(X) \to H_0(X, \partial X)$ has image 0,
$H_0(X, \partial X) \to 0$ has kernel 0, which
means $\boxed{H_0(X, \partial X) = 0}$

Consider $\underset{\mathbb{Z}\oplus\mathbb{Z}}{H_1(\partial X)} \xrightarrow{i_*} \underset{\mathbb{Z}}{H_1(X)}$. Two generators for
$H_1(\partial X)$ are a, b, circles around the torus with one
going through the hole. i_* sends $(a,b) \mapsto b$ and
is surjective. Thus $\ker(H_1(X) \to H_1(X, \partial X)) = H_1(X)$
so it is the 0-map. Thus $H_1(X, \partial X) \xrightarrow{\times 0} H_0(\partial X)$ is
injective implying $\boxed{H_1(X, \partial X) = 0}$
The kernel of i_* is $\{(a,0)\}$, so $\{(a,0)\}$ is the
image of $H_2(X, \partial X) \to H_1(\partial X)$. And since $H_2(X) = 0$
we know this map is injective. Hence $\boxed{H_2(X, \partial X) = \mathbb{Z}}$

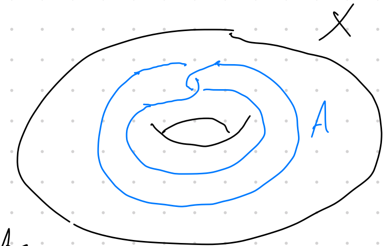
All together, we have $\boxed{H_n(X, \partial X) = \begin{cases} \mathbb{Z} & n = 2,3 \\ 0 & \text{else} \end{cases}}$

Spring 2007 $\boxed{\#5}$

Define def. retraction of X onto A.

Let A be the knot in the solid torus $X = S^1 \times D^2$ as below

Claim: no def. ret. from X to A.



Assume there is such a def. ret. which is a homotopy between the identity map on X and a retraction onto A, call it $r: X \to A$. with A fixed. Let the def. ret. be $F: X \times I \to X$ with $F(x,0) = x$ and $F(x,1) = r(x) \in A$. and $F(a,t) = a \ \forall a \in A, \forall t$.

Theres not even a retraction onto A.

For if so, then there is an induced injective homomorphism

$$\begin{array}{ccc} \pi_1(A; x_0) & \xrightarrow{i_*} & \pi_1(X; x_0) \\ \parallel & & \parallel \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

However, a loop going once around A is a generator for $\pi_1(A; x_0)$ but gets sent to the 0 element in $\pi_1(X; x_0)$ by i_* because A is nullhomotopic in X.

Thus i_* is the 0 map contradicting injectivity.

Hatcher 1.1.16 (c)

Fall 2007 $\boxed{\# 1}$

When is path connected used?

X path connected. $H_p(X;\mathbb{Z})=0$ $0<p\leq n$.

Compute $H_p(X\times S^n;\mathbb{Z})$ $0<p\leq n$.

$X\times\{x_0\}$ is a retract of $X\times S^n$, so we get a split SES:

$$0\longrightarrow H_p(X\times\{x_0\})\longrightarrow H_p(X\times S^n)\longrightarrow H_p(X\times S^n, X\times\{a_0\})\longrightarrow 0$$

$\forall$ p. Thus, $H_p(X\times S^n)\approx H_p(X)\oplus H_p(X\times S^n, X\times\{a_0\})$ $\forall$ p.

Let A and B be upper and lower hemispheres that cover S^n. Then $A\simeq B\simeq D^n\simeq\{\cdot\}$. and $A\cap B\simeq S^{n-1}$

Relative MV then gives

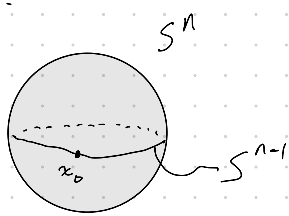
$$\cdots\longrightarrow H_p(X\times S^{n-1}, X\times\{a_0\})\longrightarrow H_p(X,X)\oplus H_p(X,X)\longrightarrow H_p(X\times S^n, X\times\{a_0\})\longrightarrow\cdots$$

The middle term is 0, so we get isomorphisms

$$H_p(X\times S^n, X\times\{a_0\})\approx H_{p-1}(X\times S^{n-1}, X\times\{a_0\})$$

By induction, $H_p(X\times S^n, X\times\{a_0\})\approx H_{p-n}(X)=0$ for $0<p\leq n$.

Hence $\boxed{H_p(X\times S^n)\approx H_p(X)=0 \quad \text{for } 0<p\leq n}$.



What about when $p=n$?

Do we get $H_n(X\times S^n, X\times\{a_0\})\approx H_0(X\sqcup X, X)$

$\cancel{\approx} H_0(X\sqcup\{a_0\})$

$\approx H_0(X)\oplus\mathbb{Z}$?

So perhaps $\mathbb{Z}$ for $p=n$.

For $p=n$ we get $H_0(X\sqcup X, X)\approx\tilde{H}_0(X\sqcup\{a_0\})\approx\tilde{H}_0(X)\oplus\mathbb{Z}\approx H_0(X)$ ✓

Fall 2007 $\boxed{\#2}$

why in $\mathbb{R}^3$? different if circles linked?

C_1, C_2 disjoint circles in $\mathbb{R}^3$.

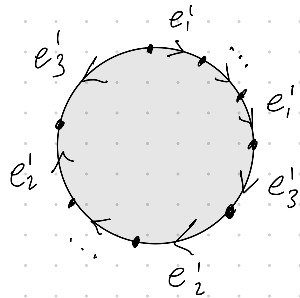
$A = S^1 \times [0,1]$ cylinder.

Attach $S^1 \times \{0\}$ to C_1 by a homeomorphism $f_1 : S^1 \to S^1$
and $S^1 \times \{1\}$ to C_2 by a homeomorphism $f_2 : S^1 \to S^1$

Compute $\pi_1(X)$ for this space X.

As a cell complex we have 2 0-cells: $e_1^0 = f_1(1) \in C_1$, $e_2^0 = f_2(1) \in C_2$; 3 1-cells $e_1^1 = C_1$, $e_2^1 = C_2$, e_3^1 which goes from e_1^0 to e_2^0. and one 2-cell e^2

The two cell is attached based on the degrees of f_1 and f_2.



$$\pi_1(X) = \langle a, b \mid a^{\deg(f_1)} b^{\deg(f_2)} \rangle$$

Spring 2008 $\boxed{\#3}$

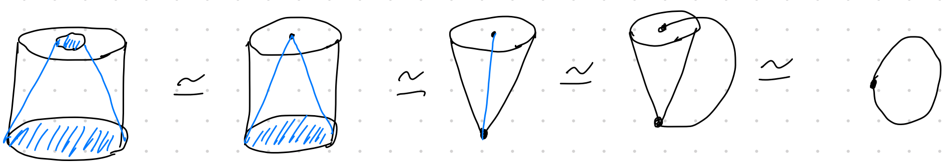
$f: X \to X$ null homotopic

$M_f = X \times [0,1]/\sim$ where $(x,0) \sim (f(x),1)$.

Compute $H_n(M_f)$.

We can view M_f as $X \times [0,1]$ where we attach $X \times \{0\}$ to $X \times \{1\}$ via f. Since f is homotopic to a constant map c_{x_0}, we can say M_f is homotopy equivalent to $M_f' = X \times [0,1]/\sim$ where $(x,0) \sim (x_0, 1)$. This is precisely $X \times [0,1] / X \times \{0\} \cup (x_0, 1)$. This is equivalent to the cone CX with a line going from a single point x_0 in the base to the tip.

Since CX is contractible we may collapse it and conclude $M_f \simeq S^1$.

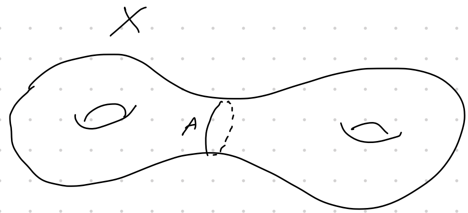


Hence $\boxed{H_n(M_f) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{else} \end{cases}}$

Spring 2009 $\boxed{\# 6}$

$X =$ surface of genus 2.

$A =$ simple closed curve



Compute $H_n(X, A)$ $n \geq 0$.

$H_n(X,A) \approx \tilde{H}_n(X/A)$ $\forall n$ since (X, A) is a good pair.

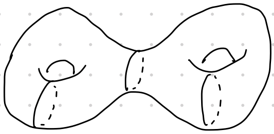
$X/A = T \vee T$ for the torus T.

$$\tilde{H}_n(T \vee T) \approx \tilde{H}_n(T) \oplus \tilde{H}_n(T)$$

$$\tilde{H}_n(T) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{else} \end{cases}$$

So, $$H_n(X, A) = \begin{cases} \mathbb{Z}^4 & n = 1 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ 0 & \text{else} \end{cases}$$

Fall 2009 $\boxed{\#6}$



$S =$ surface of genus 2 in $\mathbb{R}^3$.

$W =$ solid 2-torus. $H_n(W,S)$?

$$H_n(W) \underset{\text{homotopy invariance}}{\approx} H_n(S^1 \vee S^1) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$ because

$$\tilde{H}_n(S^1 \vee S^1) \approx \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$ because

$$\tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}.$$

$$H_n(S) = \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}^4 & n=1 \\ 0 & \text{else}. \end{cases}$$ Since S is the surface of genus 2.

Then we get a LES:

$$\cdots \longrightarrow H_n(S) \xrightarrow{i_*} H_n(W) \xrightarrow{j_*} H_n(W,S) \xrightarrow{\partial} H_{n-1}(S) \longrightarrow \cdots$$

For $n \geq 3$, we have

$$\longrightarrow \underset{0}{H_n(W)} \xrightarrow{j_*} H_n(W,S) \xrightarrow{\partial} H_{n-1}(S) \xrightarrow{i_*} \underset{0}{H_{n-1}(W)}$$

which implies $H_n(W,S) \approx H_{n-1}(S) \quad n \geq 3$

so $$H_n(W,S) = \begin{cases} \mathbb{Z} & n=3 \\ 0 & n \geq 4 \end{cases}$$

0	0
$H_0(W,S)$	0
$H_0(W)$	$\mathbb{Z}$
$H_0(S)$	$\mathbb{Z}$
$H_1(W,S)$	0
$H_1(W)$	$\mathbb{Z}\oplus\mathbb{Z}$
$H_1(S)$	$\mathbb{Z}^4$
$H_2(W,S)$	$\mathbb{Z}\oplus\mathbb{Z}$
$H_2(W)$	0
$H_2(S)$	$\mathbb{Z}$
$H_3(W,S)$	$\mathbb{Z}$

$\times 0$

$\times 0$

$\times 0$

$\operatorname{im} j_* = \ker 0 = H_0(W,S)$ $\quad \ker j_* = \operatorname{im} i_* = \mathbb{Z}$ $\Rightarrow j_* = \times 0 \Rightarrow \operatorname{im} j_* = 0$

$\cong \quad \ker i_* = 0 = \operatorname{im}\partial \Rightarrow \partial = \times 0$

$= \ker\partial = \operatorname{im} j_* = 0 \quad \operatorname{im}\partial = 0$

$j_* = \times 0$

c,d

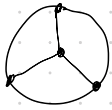
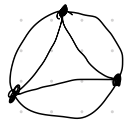
$\uparrow$

a,b,c,d surj. $\Rightarrow$ $\quad \ker i_* = \{a,b,0,0\} \cong \mathbb{Z}\oplus\mathbb{Z}$

$\ker\partial = 0 \quad \operatorname{im}\partial = \ker i_* =$ ~~at least $\mathbb{Z}\oplus\mathbb{Z}$~~ $\quad \operatorname{im}\partial = \mathbb{Z}\oplus\mathbb{Z}$

In sum, $\boxed{H_n(W,S) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n=2 \\ 0 & \text{else} \end{cases}}$

Fall 2010 $\boxed{\#1}$

$X_1 =$  $X_2 =$ 

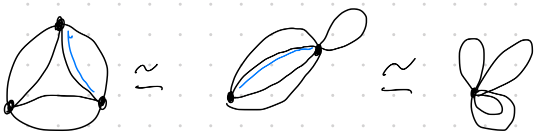
Compute their fundamental groups

 $\simeq S^1 \vee S^1 \vee S^1$

Thus $\pi_1(X_1) \approx \pi_1(S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

the free group on 3 elements.

And,

 $\simeq S^1 \vee S^1 \vee S^1 \vee S^1$

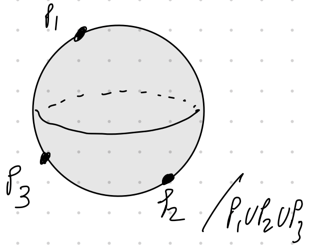
So $\pi_1(X_2) \approx \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

the free group on 4 elements.

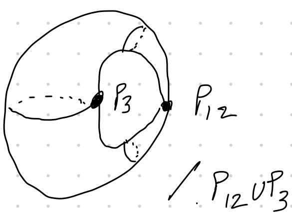
Fall 2010 $\boxed{\#2}$

P_1, P_2, P_3 distinct points in S^2. $X = S^2/\{P_1 \cup P_2 \cup P_3\}$

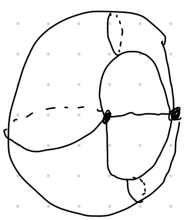
Compute $H_n(X;\mathbb{Z})$.



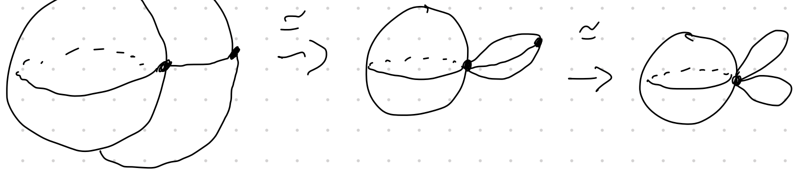
$\downarrow \simeq$



$\downarrow \simeq$



$\downarrow \simeq$



$X \simeq S^2 \vee S^1 \vee S^1$

$H_n(X;\mathbb{Z}) \approx H_n(S^2 \vee S^1 \vee S^1;\mathbb{Z})$

$\Rightarrow \tilde{H}_n(X;\mathbb{Z}) \approx \tilde{H}_n(S^2;\mathbb{Z}) \oplus \tilde{H}_n(S^1;\mathbb{Z}) \oplus \tilde{H}_n(S^1;\mathbb{Z})$

$\tilde{H}_n(S^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=2 \\ 0 & \text{else} \end{cases}$

$\tilde{H}_n(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$

$\Rightarrow \tilde{H}_n(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & \text{else} \end{cases}$

$\Rightarrow \boxed{H_n(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}}$

Spring 2011 $\boxed{\# 4}$

$f: X \to Y$ continuous. $C_f := \left((X \times [0,1]) \sqcup Y \right) / \sim$

where $(x,1) \sim f(x)$ $\forall x \in X$ and $(x,0) \sim (x',0)$ $\forall x, x' \in X$.

Claim: $\exists$ LES

$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \tilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots$$

$$\tilde{H}_{i+1}(C_f) \approx \tilde{H}_{i+1}(Y/f(X)) \approx H_{i+1}(Y, f(X))$$

We have a relative LES:

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_{i+1}(f(X)) \arrow[r, "i_*"] & H_{i+1}(Y) \arrow[r, "j_*"] & H_{i+1}(Y, f(X)) \arrow[r, "\partial"] \arrow[d, shift right, "g"', "\approx"] & H_i(f(X)) \arrow[r, "i_*"] & H_i(Y) \arrow[r, "j_*"] & \cdots \\
& H_{i+1}(X) \arrow[u, "f_*"] & & \tilde{H}_{i+1}(C_f) \arrow[u, shift right] & H_i(X) \arrow[u, "f_*"] & &
\end{tikzcd}$$

$$X \xrightarrow{f} f(X) \xrightarrow{i} Y$$

$$H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(f(X)) \xrightarrow{i_*} H_{i+1}(Y)$$

WEEK 2

Fall 1999 $\boxed{\#7}$

$\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$ closed. i.e. $d\omega = 0$.

$i^* : \Omega^n(\mathbb{R}^{n+1} - \{0\}) \to \Omega^n(S^n)$

homomorphism induced by inclusion $i : S^n \to \mathbb{R}^{n+1} - \{0\}$.

Claim: ω exact $\iff \int_{S^n} i^*(\omega) = 0$.

ω exact $\Rightarrow \exists\ \alpha \in \Omega^{n-1}(\mathbb{R}^{n+1} - \{0\})$ s.t. $d\alpha = \omega$.

$\Rightarrow \int_{S^n} i^*(\omega) = \int_{S^n} i^*(d\alpha) = \int_{S^n} d(i^*\alpha) \overset{\text{Stokes}}{=} \int_{\partial S^n} i^*\alpha = \int_{\emptyset} i^*\alpha = 0$.

Conversely, assume $\int_{S^n} i^*(\omega) = 0$.

$$H^n_{dR}(S^n) \xrightarrow[\sim]{i_*} H^n_{dR}(\mathbb{R}^{n+1} - \{0\})$$

$[\omega] \in H^n_{dR}(\mathbb{R}^{n+1} - \{0\})$ NTS $[\omega] = 0$.

Claim: $[i^*(\omega)] \in H^n_{dR}(S^n) \overset{i_*}{\longmapsto} [\omega]$

and $[i^*(\omega)] = 0$.

Fall 2000 $\boxed{\#1}$

$\omega \in \Omega^1(S^2)$. ω invariant under rotations.

i.e. $\phi^* \omega = \omega \quad \forall \phi \in SO(3)$.

Claim: $\omega = 0$.

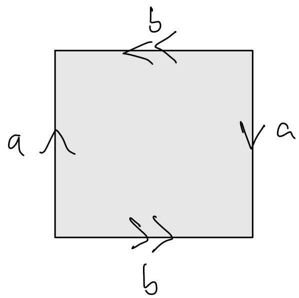
Fall 2000 $\boxed{\#6}$

$f : \mathbb{RP}^2 \to T^2$ continuous.

(a) Prove $f_* : \pi_1(\mathbb{RP}^2) \to \pi_1(T^2)$ trivial.

(b) Prove $f \simeq$ constant map.

$\mathbb{RP}^2$



$ab^{-1}ab^{-1} = 1$

$ab^{-1} = ba^{-1}$

$$\pi_1(\mathbb{RP}^2) = \langle a,b \mid ab^{-1}ab^{-1} = 1 \rangle \approx \langle a \mid a^2 = 1 \rangle$$

$$\pi_1(T^2) = \langle c,d \mid cdc^{-1}d^{-1} = 1 \rangle$$

$f_*(a) = c, \quad f_*(b) = d \Rightarrow f_*(ab^{-1}ab^{-1}) = cd^{-1}cd^{-1} = 1$

$f_*(a) = c, \quad f_*(b) = c \Rightarrow f_*(ab^{-1}ab^{-1}) = cc^{-1}cc^{-1} = 1$ ✓

$p : S^2 \to \mathbb{RP}^2$ covering map 2-sheeted.

$$p_* : \underset{\substack{\cong \\ 1}}{\pi_1(S^2)} \to \pi_1(\mathbb{RP}^2)$$

Since S^2 is a 2-sheeted cover, 1 is a subgp of $\pi_1(\mathbb{RP}^2)$ of index 2

$\Rightarrow |\pi_1(\mathbb{RP}^2)| = 2 \Rightarrow \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$.

$\pi_1(\mathbb{RP}^2) = \{0,1\}$.

$f_*(1) = c$ or $f_*(1) = d$ or $f_*(1) = 0$

$f_*(1+1) = c^2 \neq 0$ $\Rightarrow\Leftarrow$	$f_*(1+1) = d^2 \neq 0$ $\Rightarrow\Leftarrow$	✓

Hence, f_* must be trivial.

(b) $p : \mathbb{R}^2 \to T^2$ covering map.

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & T^2 \end{tikzcd}$$

$p_*(\pi_1(\mathbb{R}^2)) = p_*(0) = 0$

$f_*(\pi_1(\mathbb{RP}^2)) = 0 \subset 0$

$\Rightarrow$ lifting criterion

$\Rightarrow \exists\ \tilde{f} : \mathbb{RP}^2 \to \mathbb{R}^2$ s.t. $p \circ \tilde{f} = f$.

$p \circ \tilde{f} \simeq C_{p(0)}$, since $\tilde{f} \simeq C_0$.

and $p \circ C_0 = C_{p(0)}$.

Spring 2002 $\boxed{\#1}$

Spring 2002 $\boxed{\#2}$

$M(n) = \{n\times n \text{ matrices}\}$ identified with $\mathbb{R}^{n^2}$

$O(n) = \{\text{orthogonal matrices}\} \subset M(n)$

$= \{A \in M(n) : AA^T = Id\}$

Claim: $O(n)$ is a submfd of $M(n) = \mathbb{R}^{n^2}$

Claim: $T_{Id}O(n) = \{\text{antisymmetric matrices}\}$

$= \{A \in M(n) : A^T = -A\}$.

Let $f : M(n) \longrightarrow M(n)$ map $A \longmapsto AA^T$

Then $O(n) = f^{-1}(Id)$

$$df_A = \begin{pmatrix} \frac{\partial f^{11}}{\partial x^{11}} & \cdots & \frac{\partial f^{11}}{\partial x^{nn}} \\ \vdots & & \vdots \\ \frac{\partial f^{nn}}{\partial x^{11}} & \cdots & \frac{\partial f^{nn}}{\partial x^{nn}} \end{pmatrix}$$

$$f^{ij}_{(A)} = a^i \cdot a_j = \sum_K x^{iK} x^{jK}$$

$$\underset{A}{\begin{pmatrix} a^1 \\ \vdots \\ a^n \end{pmatrix}} \underset{A^T}{\begin{pmatrix} a^1 & \cdots & a^n \end{pmatrix}}$$

$$\frac{\partial f^{ij}}{\partial x^{pq}} = 0 \quad p \neq i, j \qquad \frac{\partial f^{ij}}{\partial x^{jq}} = x^{iq}$$

$$\frac{\partial f^{ij}}{\partial x^{iq}} = x^{jq}$$

Spring 2002 $\boxed{\#3}$

$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad f(x,y,z) = (\alpha x + \beta y,\ \gamma x + \delta y,\ \varepsilon z)$

$\alpha\delta - \beta\gamma = 1$. Find matrix of $f^*: \Lambda^2\mathbb{R}^3 \to \Lambda^2\mathbb{R}^3$ associated to basis $dy\wedge dz,\ dz\wedge dx,\ dx\wedge dy$.

$$f^*(\omega)(\underbrace{V_1, V_2}_{\in T_p\mathbb{R}^3}) = \omega(df_p(V_1), df_p(V_2))$$

$V_1 = x_1, y_1, z_1$

$V_2 = x_2, y_2, z_2$

$$df_p = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & \varepsilon \end{pmatrix}$$

$$df_p(V_1) = \begin{matrix} f^1(V_1) \\ f^2(V_1) \\ f^3(V_1) \end{matrix}$$

$$f^*(dy\wedge dx)(V_1, V_2) = dy\wedge dx\,(df_p(V_1), df_p(V_2))$$

$$= \det\begin{pmatrix} \gamma x_1 + \delta y_1 & \gamma x_2 + \delta y_2 \\ \alpha x_1 + \beta y_1 & \alpha x_2 + \beta y_2 \end{pmatrix}$$

Spring 2002 $\boxed{\text{II } 4}$

(a) Compute $\pi_1(\mathbb{R}P^2 - \{x\})$.

(b) $f: \mathbb{R}P^2 \to \mathbb{R}P^2$ not surjective. Claim: $f \simeq$ constant map.

(Hint: use a covering space).

(a) Claim: $\mathbb{R}P^2 - \{x\}$ def. retracts to $\mathbb{R}P^1$.

$$\mathbb{R}P^2 - \{x\} = (S^2/\sim) - \{x\} = (S^2 \setminus \underbrace{\{N,S\}}_{\text{north \& south poles,}})/\sim$$

Send a point in $(S^2 - \{N,S\})/\sim$ down the meridional line to a point on the equator. It's antipodal point will be sent to the antipodal destination on the equator.

Thus the image is $S^1/\sim$ or $\mathbb{R}P^1$.

Hence $\pi_1(\mathbb{R}P^2 - \{x\}) \approx \pi_1(\mathbb{R}P^1) = \boxed{\mathbb{Z}}$.

(b) Let $x \in \mathbb{R}P^2 \setminus f(\mathbb{R}P^2)$ be outside the image of f.

Then $f: \mathbb{R}P^2 \to \mathbb{R}P^2 \setminus \{x\}$.

$f_*: \pi_1(\mathbb{R}P^2) \to \pi_1(\mathbb{R}P^2 - \{x\})$

$$\begin{tikzcd} & \mathbb{R}P^1 \arrow[d] \\ \mathbb{R}P^2 \arrow[ur, dashed] \arrow[r] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} & S^1 \arrow[d, "q"] \\ \mathbb{R}P^2 \arrow[ur, dashed] \arrow[r, "r \circ f"'] & \mathbb{R}P^1 \end{tikzcd}$$

Spring 2002 $\boxed{\#5}$

$X = S^1 \times B^2 \quad \partial X = S^1 \times S^1.$

Compute $H_n(X, \partial X)$.

Relative Homology LES:

$$\cdots \longrightarrow H_n(\partial X) \longrightarrow H_n(X) \longrightarrow H_n(X, \partial X) \longrightarrow \cdots$$

X deformation retracts to S^1

$$\Rightarrow \quad H_n(X) \approx H_n(S^1) = \begin{cases} \mathbb{Z} & n = 0, 1 \\ 0 & \text{else.} \end{cases}$$

$$H_n(\partial X) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

Thus for $n \geq 3$ we have

$$0 \longrightarrow H_n(X, \partial X) \longrightarrow H_{n-1}(\partial X) \longrightarrow 0$$

$$\Rightarrow \quad H_n(X, \partial X) \approx H_{n-1}(\partial X) = \begin{cases} \mathbb{Z} & n = 3 \\ 0 & n > 3 \end{cases}$$

Then we have

$$\begin{array}{ccccccccc} H_2(\partial X) & \longrightarrow & H_2(X) & \longrightarrow & H_2(X, \partial X) & \xrightarrow{\partial} & H_1(\partial X) & \xrightarrow{i_*} & H_1(X) \\ \mathbb{Z} & & 0 & & \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \end{array}$$

$$\begin{array}{ccccccccc} \xrightarrow{j_*} H_1(X, \partial X) & \xrightarrow{\partial'} & H_0(\partial X) & \xrightarrow{i_*'} & H_0(X) & \xrightarrow{j_*'} & H_0(X, \partial X) & \longrightarrow & 0 \\ 0 & & \mathbb{Z} & & \mathbb{Z} & & & & \end{array}$$

$$\begin{aligned} i_*((1,0)) &= 0 \\ i_*((0,1)) &= 1 \end{aligned} \quad \Rightarrow \ker(i_*) = \{(a, 0)\} = \operatorname{im}(\partial) \Rightarrow H_2(X, \partial X) \approx \mathbb{Z}$$

since ∂ injective.

$\ker(j_*) = \operatorname{im}(i_*) = \mathbb{Z} \Rightarrow j_* = 0\text{-map}$

$\Rightarrow \partial'$ injective.

$H_0(X, \partial X) = \mathbb{Z}$ since path connected.

i_*' is an isomorphism $\Rightarrow \partial'$ and j_*' are 0-maps. Thus $H_1(X, \partial X) = 0$. And since $\operatorname{im}(j_*') = H_0(X, \partial X)$, we see $H_0(X, \partial X) = 0$.

Hence

$$\boxed{H_n(X, \partial X) = \begin{cases} \mathbb{Z} & n = 2, 3 \\ 0 & \text{else} \end{cases}}$$

Spring 2002 $\boxed{\# 6}$

$X = S^1 \vee S^1$. $p : \tilde{X} \to X$ covering space s.t.

$\tilde{X}$ connected and $p^{-1}(x)$ $\forall x \in X$ consists of 2 points.

Compute $\pi_1(\tilde{X})$.

Let v be the vertex in X.

Then $p^{-1}(v)$ is two points. That means $\tilde{X}$ has two vertices.

and since there are 2 edges in X, there will be 4 edges in $\tilde{X}$.

Thus $\tilde{X}$ is a graph with 2 vertices and 4 edges. Collapse one edge $\Rightarrow$ $\tilde{X} \simeq S^1 \vee S^1 \vee S^1$

$\Rightarrow$ $\pi_1(\tilde{X}) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

Spring 2002 $\boxed{\# 7}$

What are the compact connected surfaces S for which there exists an immersion $S \to S$ which is not a diffeomorphism? Hint: Euler characteristic.

Call the immersion $f: S \longrightarrow S$

$df_p : T_p S \longrightarrow T_p S$ injective. Same dimension $\Rightarrow$ surjective $\Rightarrow$ submersion $\Rightarrow$ local diffeomorphism.

f is a covering map.

$$\chi(M_g) = 2-2g$$
$$\chi(N_g) = 2-g$$

Spring 2003 $\boxed{\#1}$

M^{2n} compact orientable mfd (w/out bdry)
ω symplectic form on M: $\omega \in \Omega^2(M)$ s.t. $\overbrace{\omega \wedge \cdots \wedge \omega}^{n}$ does not vanish at any point.

Claim: $H^2_{dR}(M;\mathbb{R}) \neq 0$ by showing ω not exact.

$\omega \wedge \cdots \wedge \omega$ nonvanishing $\Rightarrow$ $\omega \wedge \cdots \wedge \omega$ orientation form

$\Rightarrow \int_M \omega \wedge \cdots \wedge \omega \neq 0$. if $\omega = d\alpha$ for covector α

then $\int_M d\alpha \wedge \cdots \wedge d\alpha \neq 0$.

But $d(\alpha \wedge \underbrace{d\alpha \wedge \cdots \wedge d\alpha}_{n-1}) = \underbrace{d\alpha \wedge \cdots \wedge d\alpha}_{n} - \cancelto{0}{\alpha \wedge 0 \wedge d\alpha \wedge \cdots \wedge d\alpha}$

$$= d\alpha \wedge \cdots \wedge d\alpha$$

So $\int_M d\alpha \wedge \cdots \wedge d\alpha = \int_M d(\alpha \wedge d\alpha \wedge \cdots \wedge d\alpha) = \int_\emptyset \alpha \wedge d\alpha \wedge \cdots \wedge d\alpha = 0$

by stokes, a contradiction.

Thus, ω not exact.

Hence $[\omega] \neq 0 \in H^2_{dR}(M;\mathbb{R})$

$\Rightarrow \quad H^2_{dR}(M;\mathbb{R}) \neq 0.$

Spring 2003 $\boxed{\#2}$

Show that $SL(n,\mathbb{R})$ of $n\times n$ matrices with $\det = 1$ is a mfd. What is its dimension?

$M(n,\mathbb{R})$, the set of all $n\times n$ matrices is a $n\times n$ dimensional mfd, since it is homeomorphic to $\mathbb{R}^{n\times n}$. We know that the determinant is a smooth map from $M(n,\mathbb{R}) \to \mathbb{R}$.

If 1 is a regular value of det, then $\det^{-1}(1) = SL(n,\mathbb{R})$ is a submanifold and therefore a manifold.

Let $f: M(n,\mathbb{R}) \to \mathbb{R}$ map $X \mapsto \det(X)$.

Then $df_X : T_X M(n,\mathbb{R}) \to T_{f(X)}\mathbb{R}$ for X in $SL(n,\mathbb{R})$

$$df_X = \left[\frac{\partial f}{\partial x_{11}}, \ldots, \frac{\partial f}{\partial x_{1n}}, \ldots, \ldots, \frac{\partial f}{\partial x_{nn}}\right] \quad \text{for } X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

$$f(X) = x_{11}\det(\mathrm{cof}(x_{11})) - x_{12}\det(\mathrm{cof}(x_{12})) + \cdots \pm x_{1n}\det(\mathrm{cof}(x_{1n}))$$

$$\frac{\partial f}{\partial x_{11}} = \det(\mathrm{cof}(x_{11})), \quad \frac{\partial f}{\partial x_{12}} = -\det(\mathrm{cof}(x_{12})), \quad \ldots, \quad \frac{\partial f}{\partial x_{1n}} = \pm\det(\mathrm{cof}(x_{1n})).$$

If $\frac{\partial f}{\partial x_{11}}, \ldots, \frac{\partial f}{\partial x_{1n}}$ all equal 0 then $f(X) = 0$, a contradiction.

Thus $df_X \neq 0 \Rightarrow 1$ regular value $\Rightarrow SL(n,\mathbb{R})$ submfd.

Spring 2003 $\boxed{\# 3}$

$\omega \in \Omega^2(\mathbb{R}^4)$, $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

f smooth functn on $\mathbb{R}^4$.

$$X = \frac{\partial f}{\partial y_1}\frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial f}{\partial y_2}\frac{\partial}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial}{\partial y_2} \quad \text{vect. field.}$$

Compute $\mathcal{L}_X \omega$.

Cartan's Magic Formula: $\mathcal{L}_X \omega = X \lrcorner (d\omega) + d(X \lrcorner \omega)$

$d\omega = 0 \implies X \lrcorner (d\omega) = 0$

$$(X \lrcorner \omega)(V) = \omega(X, V) = (dx_1 \wedge dy_1 + dx_2 \wedge dy_2)(X, V)$$

$$= dx_1 \wedge dy_1 (X, V) + dx_2 \wedge dy_2 (X, V)$$

$$= \det\begin{pmatrix} dx_1(X) & dx_1(V) \\ dy_1(X) & dy_1(V) \end{pmatrix} + \det\begin{pmatrix} dx_2(X) & dx_2(V) \\ dy_2(X) & dy_2(V) \end{pmatrix}$$

$$= f_{y_1} V_{y_1} - (-f_{x_1} V_{x_1}) + f_{y_2} V_{y_2} + f_{x_2} V_{x_2}$$

$$\implies X \lrcorner \omega = f_{x_1} dx_1 + f_{y_1} dy_1 + f_{x_2} dx_2 + f_{y_2} dy_2$$

$$d(X \lrcorner \omega) = \cancel{f_{y_1 x_1} dy_1 \wedge dx_1} + \cancel{f_{x_2 x_1} dx_2 \wedge dx_1} + \cancel{f_{y_2 x_1} dy_2 \wedge dx_1}$$
$$+ \cancel{f_{x_1 y_1} dx_1 \wedge dy_1} + \cancel{f_{x_2 y_1} dx_2 \wedge dy_1} + \cancel{f_{y_2 y_1} dy_2 \wedge dy_1}$$
$$+ \cancel{f_{x_1 x_2} dx_1 \wedge dx_2} + \cancel{f_{y_1 x_2} dy_1 \wedge dx_2} + \cancel{f_{y_2 x_2} dy_2 \wedge dx_2}$$
$$+ \cancel{f_{x_1 y_2} dx_1 \wedge dy_2} + \cancel{f_{y_1 y_2} dy_1 \wedge dy_2} + \cancel{f_{x_2 y_2} dx_2 \wedge dy_2}$$
$$= \boxed{0}$$

?

Spring 2003 $\boxed{\# 4}$

M compact oriented n-mfd (w/out bdry), $n > 1$.
Claim: $\exists$ differentiable map $f: M \longrightarrow S^n$ of degree 1.

$$\int_M f^* \omega = k \int_{S^n} \omega$$

Hatcher 3.3.7

?

Let U be a chart of M, an open nbd homeomorphic to an open nbd of $\mathbb{R}^n$. Let f map this chart to $S^n \setminus \{N\}$, the sphere minus the north pole. Then send $M \setminus U$ to $\{N\}$ the north pole.

Spring 2003 $\boxed{\#5}$

two coverings $p: \tilde{X} \to X$, $p': \tilde{X}' \to X$ are <u>equivalent</u> if

$\exists$ homeomorphism $\varphi: X \to \tilde{X}'$ s.t. $p' \circ \varphi = p$.

$X = S^1 \times S^1$ torus. Determine number of equiv. classes of all coverings $p: \tilde{X} \to X$ s.t. $p^{-1}(x_0)$ consists of 3 points (for arbitrary x_0).

---

2 generators of $\mathbb{Z} \times \mathbb{Z}$ and where can we send them in S_3?

?

Spring 2003 $\boxed{\#6}$

Compute $H_n(X;\mathbb{Z})$ where $X = \mathbb{R}^5 - A$, where $A = 4$ points.

$\mathbb{R}^5 - A \simeq D^5 - A \simeq S^4 \vee S^4 \vee S^4 \vee S^4$

where D^5 is a 5-dim'l ball.

Indeed, the first $\simeq$ is a def. retract.

To obtain the second $\simeq$, consider D^5 with one point of A in one hemisphere and the other 3 points in the other hemisphere. The 4-dim'l disk cutting the ball in half is contractible, so we may collapse it. Then this space $\simeq$ to $Y \vee Z$ where Y is a 5-dim'l ball minus a point and Z is a 5 dim'l ball minus 3 points. Then Y def. retracts to S^4. We do the same to the other three points and get $S^4 \vee S^4 \vee S^4 \vee S^4$.

Hence: $\tilde{H}_n(X;\mathbb{Z}) \approx \tilde{H}_n(S^4 \vee \cdots \vee S^4;\mathbb{Z}) \approx \tilde{H}_n(S^4;\mathbb{Z})^{\oplus 4}$

$$= \begin{cases} \mathbb{Z}^{\oplus 4} & n = 4 \\ 0 & \text{else} \end{cases}$$

Thus, $\boxed{H_n(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}^{\oplus 4} & n = 4 \\ \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}}$

Spring 2003 $\boxed{\#7}$

$f: B^n \longrightarrow \mathbb{R}^n$ continuous. $f(x) = x \quad \forall x \in S^{n-1}$.

Claim: $f(B^n)$ contains B^n

If not, then $\exists \; x \in B^n$ s.t. $x \notin f(B^n)$

i.e. $\forall y \in B^n, \; f(y) \neq x$.

Then $f: B^n \longrightarrow \mathbb{R}^n \setminus \{x\}$

and $\mathbb{R}^n - \{x\}$ deformation retracts to S^{n-1}

$g: B^n \longrightarrow S^{n-1}$ retraction $\Rightarrow\Leftarrow$.

$g_*: H_p(B^n) \longrightarrow H_p(S^{n-1})$ surj. $\Rightarrow\Leftarrow$.

$r \circ i: Id \Rightarrow r_* \circ i_* = Id$

$\Rightarrow r_*$ surj.

i_* inj.

Fall 2003 $\boxed{\#5}$

$X = S^1 \times S^2$. $\quad Y = S^1 \sqcup S^2 \sqcup S^3 / \{p_1, p_2, p_3\}$ for $p_n \in S^n$

a) Compute Homology

b) Compute fundamental groups

c) $X \cong Y$ homeo?

---

a) $Y = S^1 \vee S^2 \vee S^3 \Rightarrow \tilde{H}_n(Y) \approx \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^2) \oplus \tilde{H}_n(S^3)$

$$= \begin{cases} \mathbb{Z} & n = 1, 2, 3 \\ 0 & \text{else} \end{cases}$$

Lemma: $H_n(S^1 \times Z) \approx H_n(Z) \oplus H_{n-1}(Z)$

?

Spring 2004 $\boxed{\#2}$

Let $\omega = \dfrac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2(\mathbb{R}^3 - \{0\})$.

If $i : S^2 = \{x^2+y^2+z^2 = 1\} \hookrightarrow \mathbb{R}^3$ inclusion, compute $\int_{S^2} i^*\omega$.

Compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbb{R}^3$ maps $(x,y,z) \to (3x, 2y, 8z)$.

$$d\omega = \frac{(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{1/2}x^2}{(x^2+y^2+z^2)^3}\, dx\wedge dy\wedge dz$$

$$+ \frac{(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{1/2}y^2}{(x^2+y^2+z^2)^3}\, dy\wedge dz\wedge dx$$

$$+ \frac{(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{1/2}z^2}{(x^2+y^2+z^2)^3}\, dz\wedge dx\wedge dy$$

$$= \frac{3(x^2+y^2+z^2)^{3/2} - 3(x^2+y^2+z^2)^{3/2}}{(x^2+y^2+z^2)^3}\, dx\wedge dy\wedge dz = 0$$

Let $D^3 = \{x^2+y^2+z^2 \le 1\}$ so $\partial D^3 = S^2$. Stokes $\Rightarrow$

$$\int_{S^2} i^*\omega = \int_{\partial D^3} i^*\omega = \int_{D^3} d(i^*\omega) = \int_{D^3} i^*(d\omega) = \boxed{0}$$

? Why is this wrong?

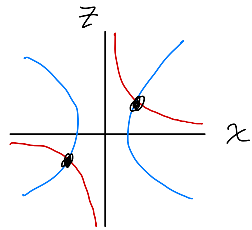
Similarly $\int_{S^2} j^*\omega = \boxed{0}$.

Spring 2004 $\boxed{\#3}$

Consider $X \subset \mathbb{R}^4$ defined by $x^2+y^2-z^2-w^2=1$ and $xz+yw=1$. Is X a smooth submanifold of $\mathbb{R}^4$?

---

x, z plane: $x^2-z^2=1$, $xz=1$



?

Spring 2004 $\boxed{\#5}$

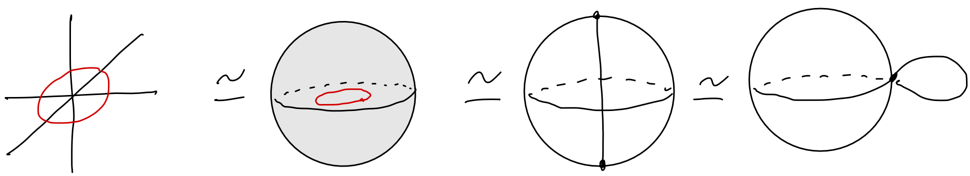
Let $S^1 = \{x^2+y^2=1, z=0\} \subset \mathbb{R}^3$

Calculate $\pi_1(\mathbb{R}^3 - S^1)$

$\mathbb{R}^3 - S^1$ deformation retracts to $D^3 - S^1$ where D^3 is a ball of radius 2.

D^3 deformation retracts to the sphere of radius 2, S^2, union the line segment going from north to south pole on the z-axis.

This is homotopy equivalent to $S^2 \vee S^1$ since the arc along S^2 connecting the poles is contractible. Visually,



Hence, $\pi_1(\mathbb{R}^3 - S^1) \approx \pi_1(S^2 \vee S^1) \approx \pi_1(S^2) * \pi_1(S^1) = \boxed{\mathbb{Z}}$

Spring 2004 $\boxed{\#7}$

Claim: $H_p(SX;\mathbb{Z}) \approx H_{p-1}(X;\mathbb{Z})$ for $p \geq 2$.

Compute H_0 and H_1

$SX = (X \times [0,1])/\sim$ where $(x,0) \sim (x',0)$ and $(x,1) \sim (x',1)$ $\forall\, x, x' \in X$.

Let $A = (X \times (\frac{1}{4}, 1])/\sim$, $B = (X \times [0, 3/4))/\sim$

Then $A \cap B \simeq X$, A contractible, B contractible

Thus, by Mayer-Vietoris we have a LES:

$$\cdots \longrightarrow H_p(A\cap B) \longrightarrow H_p(A) \oplus H_p(B) \longrightarrow H_p(SX) \longrightarrow \cdots$$

For $p \geq 2$ this gives

$$0 \longrightarrow H_p(SX) \longrightarrow H_{p-1}(A \cap B) \longrightarrow 0$$

since A, B contractible $\Rightarrow H_p(A), H_p(B) = 0$ for $p \geq 1$.

Hence, $H_p(SX) \approx H_{p-1}(A \cap B) \approx H_{p-1}(X)$ for $p \geq 2$.

Otherwise we have

$$0 \to H_1(SX) \to H_0(X) \to H_0(A) \oplus H_0(B) \to H_0(SX) \to 0$$

and for n path components in X (note SX is path connected):

$$0 \to H_1(SX) \xhookrightarrow{\partial} \mathbb{Z}^n \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$(a_1, \ldots, a_n) \longmapsto \left(\sum_1^n a_i, \sum_1^n a_i\right)$$

$\ker(i_*) = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum_1^n a_i = 0\} = \operatorname{im}(\partial)$

$\Rightarrow \quad \operatorname{im}(\partial) \approx \mathbb{Z}^{n-1}$

And since ∂ injective, we conclude $H_1(SX) \approx \mathbb{Z}^{n-1}$.

In summary, for X with n path components,

$$\boxed{H_p(SX, \mathbb{Z}) = \begin{cases} H_{p-1}(X, \mathbb{Z}) & p \geq 2 \\ \mathbb{Z}^{n-1} & p = 1 \\ \mathbb{Z} & p = 0 \end{cases}}$$

Fall 2007 $\boxed{\#1}$

X path connected. $H_p(X;\mathbb{Z}) = 0 \quad \forall \ 0<p\leq n.$

Compute $H_p(X\times S^n;\mathbb{Z}) \quad \forall \ 0<p\leq n.$

Let $A = X\times(S^n\setminus N)$ and $B = X\times(S^n\setminus S)$ for N,S the north and south poles of S^n.

Then $A\cap B = X\times(S^n\setminus\{N,S\})$ def. retracts to $X\times S^{n-1}$.

A, B def retract to $X\times\{S\}$, $X\times\{N\}$ both $\simeq X$.

$$\begin{array}{ccccccccc}
\cdots \to & H_p(A\cap B) & \to & H_p(A)\oplus H_p(B) & \to & H_p(X\times S^n) & \to & \cdots \\
& \wr\wr & & \wr\wr & & \wr\wr & & \\
\cdots \to & H_p(X\times S^{n-1}) & \to & H_p(X)\oplus H_p(X) & \to & H_p(X\times S^n) & \to & \cdots
\end{array}$$

So we get for $1<p\leq n$

$$\underbrace{H_p(X)\oplus H_p(X)}_{0} \to H_p(X\times S^n) \to H_{p-1}(X\times S^{n-1}) \to \underbrace{H_{p-1}(X)\oplus H_{p-1}(X)}_{0}$$

implying $H_p(X\times S^n) \approx H_{p-1}(X\times S^{n-1}) \quad \forall \ 1<p\leq n.$

Thus $H_p(X\times S^n) \approx H_1(X\times S^{n-p+1}) \quad \forall \ 1<p\leq n.$

We have

$$\underbrace{H_1(X)\oplus H_1(X)}_{0} \xrightarrow{i_*} H_1(X\times S^{n-p+1}) \xrightarrow{\partial} \underbrace{H_0(X\times S^{n-p})}_{\begin{cases}\mathbb{Z} & 0<p<n \\ \mathbb{Z}^2 & p=n\end{cases}} \xrightarrow{i_*} \underbrace{H_0(X)\oplus H_0(X)}_{\mathbb{Z}^2}$$

$$\begin{cases} a \longmapsto (a,a) \\ (a,b) \longmapsto (a+b, a+b) \end{cases}$$

$\ker \partial = \operatorname{im} j_* = 0 \Rightarrow \partial$ injective.

$$\operatorname{im} \partial = \ker i_* = \begin{cases} 0 & 0<p<n \\ \mathbb{Z} & p=n \end{cases}$$

$$\Rightarrow H_1(X \times S^{n-p+1}) = \begin{cases} 0 & 0<p<n \\ \mathbb{Z} & p=n \end{cases}$$

Hence, $$H_p(X \times S^n) \approx \begin{cases} 0 & 0<p<n \\ \mathbb{Z} & p=n \end{cases}$$

Fall 2007 $\boxed{\#2}$

C_0, C_1 disjoint circles in $\mathbb{R}^3$. $A = S^1 \times [0,1]$ cylinder.

$X = \mathbb{R}^3 \sqcup A$ gluing $S^1 \times \{0\}$ to C_0 and $S^1 \times \{1\}$ to C_1 by homeomorphisms f_0, f_1.

Compute $\pi_1(X)$.

Let $U = \mathbb{R}^3$, $V = C_0' \cup C_1' \cup A \cup \gamma'$ for γ a path connecting C_0 and C_1, and C_0', C_1', γ' small enough neighborhoods to maintain distinctness of C_0', C_1' and keep V open. Then $U, V, U \cap V$ path connected U, V open, and $U \cup V = X$. Hence SVK gives

$\pi_1(X) = \pi_1(\mathbb{R}^3) * \pi_1(C_0' \cup C_1' \cup A \cup \gamma') / \langle i(w) j(w)^{-1} \rangle$

for $w \in \pi_1(U \cap V)$, $i: \pi_1(U \cap V) \hookrightarrow \pi_1(U)$, $j: \pi_1(U \cap V) \hookrightarrow \pi_1(V)$ the induced inclusion maps.

$\pi_1(\mathbb{R}^3) = 1$. $C_0' \cup C_1' \cup A \cup \gamma' \simeq S^1 \vee S^1$ so $\pi_1(V) = \mathbb{Z} * \mathbb{Z}$. and $U \cap V \simeq S^1 \vee S^1$ so $\pi_1(U \cap V) = \mathbb{Z} * \mathbb{Z}$.

Let $a \in \pi_1(U \cap V)$ be a loop around C_0. and let $b \in \pi_1(U \cap V)$ be the loop $\gamma \cdot C_1 \cdot \gamma^{-1}$. Then in $\mathbb{R}^3$ these loops are contractible, but in V, a is sent to one of the generators c of $\pi_1(V)$. b is sent to dcd^{-1} for generators c, d of $\pi_1(V)$. Hence

$\pi_1(X) = \mathbb{Z} * \mathbb{Z} / \langle c, (dcd^{-1})^{-1} \rangle = \langle c, d \mid c = dcd^{-1} = 1 \rangle = \boxed{\mathbb{Z}}$.

Fall 2007 $\boxed{\#3}$

$M_n(\mathbb{R})$ vector space of $n \times n$ matrizes w/ coeffs in $\mathbb{R}$.

$\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$. Compute $d(\det)_{I_n}$ for $I_n \in M_n(\mathbb{R})$ identity.

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix} \longleftrightarrow (x_{11}, \dots, x_{1n}, \dots, x_{n1}, \dots, x_{nn}) \in \mathbb{R}^{n^2}$$

$$d(\det)_{I_n} = \left[\left.\frac{\partial(\det)}{\partial x_{11}}\right|_{I_n} \quad \cdots \quad \left.\frac{\partial(\det)}{\partial x_{nn}}\right|_{I_n} \right]$$

$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} x_{ij} \det(A_{\hat{ij}})$ for any $i \in \{1, \dots, n\}$, where $A_{\hat{ij}}$ is the matrix minor leaving out the ith row and jth column. $\det(I_{n\,\hat{ij}}) = 0$ unless $i = j$ since $I_{n\,\hat{ij}}$ has a row and column of 0's if $i \neq j$. And $\det(I_{n\,\hat{ii}}) = 1$ $\forall\, i \in \{1, \dots, n\}$.

Thus $$\left.\frac{\partial(\det)}{\partial x_{ij}}\right|_{I_n} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

Thus for any $A \in T_{I_n} M_n(\mathbb{R})$, with $A = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$

$d(\det)_{I_n}(A) = \sum_{i=1}^{n} x_{ii} = tr(A)$.

Hence
$$\begin{array}{ccc} d(\det)_{I_n} : T_{I_n} M_n(\mathbb{R}) & \longrightarrow & T_1 \mathbb{R} \\ \wr\wr & & \wr\wr \\ M_n(\mathbb{R}) & \longrightarrow & \mathbb{R} \\ A & \longmapsto & tr(A). \end{array}$$
is the trace map.

Fall 2007 $\boxed{\# 4}$

M compact, orientable n-dim'l mfd

$f : \partial M \xrightarrow{\cong} S^{n-1} \subset \mathbb{R}^n$. $F : M \to \mathbb{R}^n$ continuous

with $F|_{\partial M} = f$. Claim: $F(M)$ contains the center O of S^{n-1}.

Assume not. Then $F(M)$ deformation retracts onto S^{n-1}. Call this retraction r. Then, we have

$$S^{n-1} \xrightarrow{f^{-1}} \partial M \xhookrightarrow{i} M \xrightarrow{F} \mathbb{R}^n - \{0\} \xrightarrow{r} S^{n-1}$$

where i is the inclusion. This composition is the identity map since $r(F(i(f^{-1}(x)))) = r(F(f^{-1}(x))) = r(x) = x$.

Then the induced homomorphism $(r \circ F \circ i \circ f^{-1})_* : H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ is the identity map. But $H_{n-1}(\partial M) \xrightarrow{i_*} H_{n-1}(M)$ is the 0-map, so this is a contradiction.

Fall 2007 $\boxed{\#5}$

$\Omega = \{(x,y) \in \mathbb{R}^2 : 1 < x^2+y^2 < 10\}$ $\quad \omega = \dfrac{x\,dy - y\,dx}{4x^2+y^2}$

a) Show ω closed in Ω.

b) Show ω not exact in Ω.

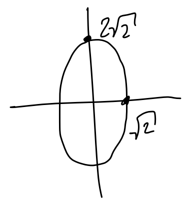
---

a) $d\omega = d\left(\dfrac{x}{4x^2+y^2}\right)\wedge dy - d\left(\dfrac{y}{4x^2+y^2}\right)\wedge dx$

$= \dfrac{4x^2+y^2-8x^2}{(4x^2+y^2)^2}\,dx\wedge dy - \dfrac{4x^2+y^2-2y^2}{(4x^2+y^2)^2}\,dy\wedge dx$

$= \dfrac{-4x^2+y^2+4x^2-y^2}{(4x^2+y^2)^2}\,dx\wedge dy = 0 \implies \omega$ closed in Ω.

b) Consider ellipse $E: 4x^2+y^2 = 8 \subset \Omega$ which is parametrized by



$(x,y) = (\sqrt{2}\cos\theta,\ 2\sqrt{2}\sin\theta)$

Then $dx = -\sqrt{2}\sin\theta\,d\theta$ and $dy = 2\sqrt{2}\cos\theta\,d\theta$

So $\omega = \dfrac{4\cos^2\theta\,d\theta + 4\sin^2\theta\,d\theta}{8} = \dfrac{1}{2}\,d\theta$

Thus, $\int_E \omega = \int_0^{2\pi} \frac{1}{2}\,d\theta = \pi \neq 0.$

If ω exact in E, then $\omega = d\alpha$ and Stokes gives

$\int_E \omega = \int_E d\alpha = \int_{\partial E} \alpha = \int_{\emptyset} \alpha = 0$, a contradiction.

So ω not exact in E. And since $E \subset \Omega$,

$\implies \omega$ not exact in Ω.

Fall 2007 $\boxed{\#6}$

$\varphi : \mathbb{R}^2 \to \mathbb{RP}^2$ $\quad (x,y) \longmapsto$ line passing through $(x,y,1)$.

$C = \{(x,y) \in \mathbb{R}^2 ; \ y^2 = x^3 - x\}$.

Claim: $\overline{\varphi(C)}$ diff. submfd of $\mathbb{RP}^2$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ send $(x,y) \longmapsto -x^3 + x + y^2$

Then $C = f^{-1}(0)$. Is 0 a regular value of f?

$$df_p = \left[\left.\frac{\partial f}{\partial x}\right|_p \ \left.\frac{\partial f}{\partial y}\right|_p \right] = \left[-3x^2+1 \quad 2y \right]_p \quad \text{for } p \in C.$$

$2y = 0 \Rightarrow y = 0 \Rightarrow -x^3 + x = 0 \Rightarrow x \in \{-1, 0, 1\}$

$\Rightarrow -3x^2 + 1 \in \{-2, 1\} \Rightarrow df_p \neq 0 \ \forall \ p \in C$.

Hence C is a submfd of $\mathbb{R}^2$ by regular val. Thm.

We see φ is a diffeomorphism onto its image $\varphi(\mathbb{R}^2)$ and $\overline{\varphi(\mathbb{R}^2)} = \mathbb{RP}^2$. Hence $\varphi(C)$ is a submanifold of $\varphi(\mathbb{R}^2)$ and $\overline{\varphi(C)}$ is a submanifold of $\overline{\varphi(\mathbb{R}^2)} = \mathbb{RP}^2$.

? Does the last part work?

Attempt #2

$\varphi : \mathbb{R}^2 \to \mathbb{RP}^2 \quad (x,y) \longmapsto$ line passing through $(x,y,1)$.

$C = \{(x,y) \in \mathbb{R}^2 ;\ y^2 = x^3 - x\}$.

$\frac{y^2}{x^2} = x - \frac{1}{x}$

$\frac{y}{x} = \sqrt{x - \frac{1}{x}} \to \pm\infty$ as $x \to \infty$.

Claim: $\overline{\varphi(C)}$ diff. submfd of $\mathbb{RP}^2$.

$$\varphi(C) = \{[x,y,1] \in \mathbb{RP}^2 : -x^3 + x + y^2 = 0\}$$

$$= \{[x,y,z] \in \mathbb{RP}^2 : -\left(\tfrac{x}{z}\right)^3 + \tfrac{x}{z} + \left(\tfrac{y}{z}\right)^2 = 0,\ z \neq 0\}$$

$$= \{[x,y,z] \in \mathbb{RP}^2 : -x^3 + z^2x + zy^2 = 0,\ z \neq 0\}$$

Consider $\{[x,y,z] \in \mathbb{RP}^2 : -x^3 + z^2x + zy^2 = 0\}$.

Then $z=0 \Rightarrow x=0 \Rightarrow [0,1,0]$ is the only element of $\mathbb{RP}^2$ with $z=0$ in this set. Hence this set $= \varphi(C) \cup \{[0,1,0]\}$. And it is $f^{-1}(0)$ for $f : \mathbb{RP}^2 \to \mathbb{R}$ defined by $f([x,y,z]) = -x^3 + z^2x + zy^2$, which is well defined since each term has degree 3, $\varphi(C) \cup \{[0,1,0]\}$ is closed since it is the preimage of a closed set $\{0\} \subset \mathbb{R}$. And since it is the smallest closed set containing $\varphi(C)$, it is $\overline{\varphi(C)}$.

Is 0 a regular value of f?

$$df_p = \begin{bmatrix} -3x^2 + z^2 & 2zy & 2zx + y^2 \end{bmatrix}_p \text{ for } p \in f^{-1}(0).$$

$df_p = 0 \Rightarrow 2zy = 0 \Rightarrow y = 0$ or $z = 0$.

$$\begin{cases} z = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0 \\ \quad \text{and} \\ z = 0 \Rightarrow -3x^2 = 0 \Rightarrow x = 0. \end{cases}$$ But $[0,0,0] \notin \mathbb{R}P^2 \Rightarrow\Leftarrow$.

$y = 0 \Rightarrow 2zx = 0 \Rightarrow x$ or $z = 0 \Rightarrow z^2$ or $-3x^2 = 0$ respectively $\Rightarrow z$ or $x = 0$ respectively. But again $[0,0,0] \notin \mathbb{R}P^2$.

Hence $df_p \neq 0 \ \forall p \in f^{-1}(0)$ and therefore $\overline{\varphi(C)}$ is a diff. submfd. of $\mathbb{R}P^2$.

Fall 2007 $\boxed{\#7}$

M, N compact, connected, n-dim'l mfds

$f: M \to N$ continuous. $H_n(f): H_n(M;\mathbb{Z}) \to H_n(N;\mathbb{Z})$ nonzero.

$f_*: \pi_1(M; x_0) \to \pi_1(N, f(x_0))$. Claim: $f_*(\pi_1(M; x_0))$ has finite index in $\pi_1(N; f(x_0))$. Hint: covering of N.

Hatcher Thm 3.26 $\Rightarrow$ $H_n(M;\mathbb{Z}), H_n(N;\mathbb{Z}) = \mathbb{Z}$ if M, N orientable or 0 if nonorientable. Since $H_n(f) \neq 0$, it must be that M, N orientable. Thus we can let $K = \deg(f) := \deg(H_n(f): \mathbb{Z} \to \mathbb{Z})$

By classification of covering spaces, $\exists$ covering $p: \tilde{N} \to N$ s.t. $p_*(\pi_1(\tilde{N})) = f_*(\pi_1(M)) \leq \pi_1(N)$.

Thus, we want to show that $p_*(\pi_1(\tilde{N}))$ has finite index, i.e, that $p: \tilde{N} \to N$ is a finite-sheeted covering.

Assume not, then $\tilde{N}$ not compact $\Rightarrow H_n(\tilde{N}) = 0$.

But then $H_n(f) = H_n(p \circ \tilde{f}) = H_n(p) \circ H_n(\tilde{f}): H_n(M) \to H_n(\tilde{N}) \to H_n(N)$ must be the zero-map, a contradiction.

?

M compact $\Rightarrow \tilde{f}: M \to \tilde{N}$ not surjective.

Then let $q \in \tilde{N} - \tilde{f}(M)$. $\exists$ nbd U of q that doesn't intersect $\tilde{f}(M)$. Let ω be supported on U.

Then $\int_M \tilde{f}^* \omega = 0 \Rightarrow \deg(\tilde{f}) = 0$.

Spring 2008 $\boxed{\#1}$

$p : \tilde{X} \to X$ covering. X path connected.

G automorphism group, consisting of homeomorphisms $\varphi : \tilde{X} \to \tilde{X}$ s.t. $p \circ \varphi = p$. $x_0 \in X$, $\tilde{x}_0 \in \tilde{X}$, $p(\tilde{x}_0) = x_0$.

For any two $\tilde{x}_0'$, $\tilde{x}_0'' \in p^{-1}(x_0)$, $\exists\ \varphi \in G$ s.t. $\varphi(\tilde{x}_0'') = \tilde{x}_0'$

Claim: $\exists$ exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}; \tilde{x}_0) \xrightarrow{p_*} \pi_1(X; x_0) \longrightarrow G \longrightarrow 1.$$

Assume $\tilde{X}$ path connected.

Let $[\gamma] \in \pi_1(X; x_0)$. γ lifts uniquely to a path $\tilde{\gamma}$ that starts at $\tilde{x}_0 \in \tilde{X}$. $\tilde{\gamma}$ ends at some $\tilde{x}_0' \in p^{-1}(x_0)$. By hypothesis, $\exists$ homeo $\varphi \in G$ s.t. $\varphi(\tilde{x}_0) = \tilde{x}_0'$. This is unique since deck trasformations are completely determined by where one point is sent.

Thus, we can define a map $f : \pi_1(X; x_0) \longrightarrow G$ that sends $[\gamma] \mapsto \varphi$ as above.

This is a homomorphism: $f([\gamma \cdot \gamma']) = \varphi' \circ \varphi$?

γ lifts to a path from $\tilde{x}_0$ to $\tilde{x}_0'$

γ' lifts to a path from $\tilde{x}_0'$ to $\tilde{x}_0''$

$\Rightarrow$ φ sends $\tilde{x}_0$ to $\tilde{x}_0'$, φ' sends $\tilde{x}_0'$ to $\tilde{x}_0''$.

$\gamma \cdot \gamma'$ lifts to a path from $\tilde{x}_0$ to $\tilde{x}_0''$ and $\varphi' \circ \varphi$ sends $\tilde{x}_0$ to $\tilde{x}_0''$ $\Rightarrow$ $f([\gamma \cdot \gamma']) = \varphi' \circ \varphi$.

What is $Im(f)$?

Let $\varphi \in G$. Then $\varphi(\tilde{x}_0) = \tilde{x}_0'$ for some $\tilde{x}_0' \in p^{-1}(x_0)$.

$\exists$ path from $\tilde{x}_0$ to $\tilde{x}_0'$ which is mapped to a loop γ at x_0 by p. $f([\gamma]) = \varphi$.

So $Im(f) = G = Ker(G \longrightarrow 1)$.

What is $Ker(f)$?

Let $[\gamma] \in \pi_1(X; x_0)$ s.t. $f([\gamma]) = Id$, which sends $\tilde{x}_0$ to $\tilde{x}_0$

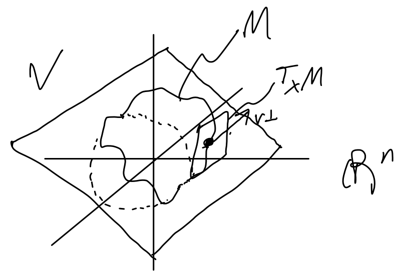
Then γ lifts to a loop $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_0$.

Hence $Ker(f) = p_*(\pi_1(\tilde{X}; x_0)) = im\ p_*$.

Thus the sequence is exact.

Spring 2008 $\boxed{\#2}$

$V \subset \mathbb{R}^n$ vector subspace, $\pi : \mathbb{R}^n \longrightarrow V$ orthogonal projection.
$M \subset \mathbb{R}^n$ submfd. Claim: $\pi|_M : M \to V$ immersion
$\iff T_x M \cap V^\perp = \{0\} \quad \forall\, x \in M.$



$$\ker(d\pi_x) = \{ v \in T_x M : d\pi_x(v) = 0 \}$$

WLOG take V to be $\mathbb{R}^m \times \{0\}^{n-m} \subset \mathbb{R}^n$.

Then $\pi((x_1, \dots, x_n)) = (x_1, \dots, x_m, 0, \dots, 0)$.

and $V^\perp = \{0\}^m \times \mathbb{R}^{n-m}$.

If $T_x M \cap V^\perp = \{0\}$ then $\forall\, v \in T_x M$, $\exists\, v_i \neq 0$ coordinate for $i \in \{1, \dots, m\}$. Then, $d\pi_x(v) \neq 0$.

Hence, $\ker(d\pi_x) = 0 \iff d\pi_x$ injective $\forall x \in M$
$\iff \pi|_M$ immersion.

And $\ker(d\pi_x) = 0 \;\; \forall\, x \in M \Rightarrow d\pi_x(v) \neq 0 \;\; \forall\, x \in M, \forall\, v \in T_x M$
$\Rightarrow v \notin V^\perp \Rightarrow T_x M \cap V^\perp = \{0\} \;\; \forall\, x \in M.$

Spring 2008 $\boxed{\#3}$

$f : X \to X$ $\quad f \simeq c_{x_0}$ constant map.

$M_f = X \times [0,1]/\sim \qquad (x,0) \sim (f(x),1)$.

Compute $H_n(M_f; \mathbb{Z})$.

By Hatcher example 2.48 $\exists$ LES:

$$\cdots \to H_n(X) \xrightarrow{\mathbb{1} - f_*} H_n(X) \to H_n(M_f) \to \cdots$$

and since $f \simeq c_{x_0}$, we get a LES:

$$\cdots \to H_n(X) \xrightarrow{\mathbb{1} - c_{x_0 *}} H_n(X) \to H_n(M_f) \to \cdots$$

For $n > 0$, $\mathbb{1} - c_{x_0 *} = \mathbb{1}$

and for $n = 0$ $\mathbb{1} - c_{x_0 *} = 0$-map.

Thus, for $n > 1$ we 0-maps going into and out of $H_n(M_f)$ by exactness of isomorphisms on either side.

$\Rightarrow H_n(M_f) = 0$ for $n > 1$ by exactness. we have

$$\cdots \to H_1(X) \xrightarrow{\cong} H_1(X) \xrightarrow{0} H_1(M_f) \to H_0(X) \xrightarrow{0} H_0(X) \to H_0(M_f) \xrightarrow{0} 0.$$

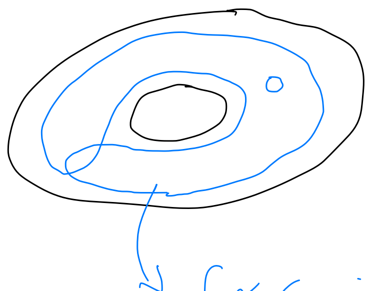
implying $H_1(M_f) \approx H_0(X)$ and $H_0(M_f) \approx H_0(X)$.

Summarizing: $\boxed{H_n(M_f) = \begin{cases} H_0(X) & n = 0,1 \\ 0 & \text{else.} \end{cases}}$

?

$f(X)$ contractible in X ?

Not necessarily:



$f \simeq c_{x_0}$ in X

$f(X)$ not contractible in X.

Spring 2008 $\boxed{\#4}$

differentiable $f: S^{2n-1} \to S^n$, $n \geq 2$. $\alpha \in \Omega^n(S^n)$.

$\int_{S^n} \alpha = 1$. $f^*(\alpha) \in \Omega^n(S^{2n-1})$.

(a) claim: $\exists\ \beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $f^*(\alpha) = d\beta$.

?

(b) claim: $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ independent of choice of β, α.

independent of β & α?

Recall $H^n(S^n) \xrightarrow{\approx} \mathbb{R}$, $\gamma \mapsto \int_{S^n} \gamma$.

(a)

$H^n_{dR}(S^{2n-1}) = 0 \Rightarrow$ all closed forms are exact.

$d\alpha \in \Omega^{n+1}(S^n) \Rightarrow d\alpha = 0 \Rightarrow d f^*(\alpha) = f^*(d\alpha) = 0$

Thus $f^*(\alpha)$ closed and therefore exact.

That is, $\exists\ \beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $f^*(\alpha) = d\beta$.

(b) $\beta \wedge d\beta \in \Omega^{2n-1}(S^{2n-1}) \Rightarrow \beta \wedge d\beta$ closed, $\in H^{2n-1}_{dR}(S^{2n-1})$

For the same α, pick another $\beta' \in \Omega^{n-1}(S^{2n-1})$

s.t. $f^*(\alpha) = d\beta'$.

Then $\beta \wedge d\beta - \beta' \wedge d\beta' = \beta \wedge d\beta - \beta' \wedge d\beta = (\beta - \beta') \wedge d\beta$.

And $\beta - \beta'$ is exact since $d(\beta - \beta') = d\beta - d\beta' = 0$

and $H^{n-1}_{dR}(S^{2n-1}) = 0$. Hence $\exists\ \gamma \in \Omega^{n-2}(S^{2n-1})$ s.t. $d\gamma = \beta - \beta'$

Then $d(\gamma \wedge d\beta) = d\gamma \wedge d\beta = (\beta - \beta') \wedge d\beta = \beta \wedge d\beta - \beta' \wedge d\beta'$

is exact. This means $[\beta \wedge d\beta] = [\beta' \wedge d\beta'] \in H^{2n-1}_{dR}(S^{2n-1})$

By the isomorphism this means

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} \beta' \wedge d\beta'.$$

Spring 2008 $\boxed{\#5}$

$$\omega = x\,dy\wedge dz + z\,dx\wedge dy + y\,dz\wedge dx \in \Omega^2(S^2)$$

Compute $\int_{S^2}\omega$.

$$d\omega = dx\wedge dy\wedge dz + dz\wedge dx\wedge dy + dy\wedge dz\wedge dx$$

$$= 3\,dx\wedge dy\wedge dz.$$

Stokes $\Rightarrow$ $\int_{S^2}\omega = \int_{\partial D^3}\omega = \int_{D^3} d\omega = \int_{D^3} 3\,dx\wedge dy\wedge dz$

$$= 3\,\mathrm{vol}(D^3) = 3\,\tfrac{4}{3}\pi = \boxed{4\pi}.$$

Fall 2008 $\boxed{\#6}$

$f : \mathbb{R} \to \mathbb{RP}^1$ sends $x \mapsto [x:1]$. $P(x)$ polynomial.

?

(a) claim: $\nexists$ diff. form ω on $\mathbb{RP}^1$ s.t. $f^*(\omega) = P(x)\,dx$.

no clue

(b) claim: $\exists$ vect field V on $\mathbb{RP}^1$ s.t. $f^*(V) = P(x)\frac{\partial}{\partial x}$

$\iff$ degree of $P(x) \le 2$.

(a) An atlas for $\mathbb{RP}^1$:

$U = \{[x:1] \in \mathbb{RP}^1\}$ $\quad \varphi : U \to \mathbb{R}$, $\varphi([x:1]) = x$.

$V = \{[1:y] \in \mathbb{RP}^1\}$ $\quad \psi : V \to \mathbb{R}$, $\psi([1:y]) = y$.

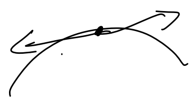
φ well defined: $\varphi([x:y]) = \varphi([\frac{x}{y}:1]) = \frac{x}{y}$

$\varphi([\lambda x:\lambda y]) = \varphi([\frac{\lambda x}{\lambda y}:1]) = \frac{x}{y}$. ✓ $\quad f$ is φ^{-1}

Let $\omega \in \Omega^1(\mathbb{RP}^1)$. In the chart U,

$\omega_x(v)$

$\omega_x(v)$



$\omega_x : T_x\mathbb{RP}^1 \to \mathbb{R}$

Fall 2008 $\boxed{\#4}$

Let T be obtained by revolving $\{(x,y,z) \mid z=0, (x-R)^2+y^2=r^2\}$ around the y-axis, $R>r$.

Compute $\int_T x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy$.

Call the integrand ω.

Then $d\omega = dx\wedge dy\wedge dz - dy\wedge dx\wedge dz + dz\wedge dx\wedge dy$

$= 3\,dx\wedge dy\wedge dz$

Let A be the solid torus obtained by revolving the disc $\{(x,y,z) \mid z=0, (x-R)^2+y^2\le r^2\}$ around the y-axis. $\partial A = T$.

Stokes $\Rightarrow \int_T \omega = \int_{\partial A}\omega = \int_A d\omega = \int_A 3\,dx\wedge dy\wedge dz = 3\,\mathrm{vol}(A)$.

$$\mathrm{Vol}(A) = \underbrace{\pi r^2}_{\substack{\text{vol of}\\ \text{disc.}}}\ \underbrace{2\pi R}_{\substack{\text{circumf.}\\ \text{of Rev.}}}$$

So $\boxed{\int_T \omega = 6\pi^2 r^2 R}$.

Spring 2009 $\boxed{\#2}$

$\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$, closed.

Claim: ω exact $\iff \int_{S^n} \omega = 0$

ω exact $\Rightarrow \exists\ \alpha \in \Omega^{n-1}(\mathbb{R}^{n+1} - \{0\})$ s.t. $\omega = d\alpha$

$\Rightarrow \int_{S^n} \omega = \int_{S^n} d\alpha = \int_{\partial S^n} \alpha = \int_{\emptyset} \alpha = 0$. (Stokes)

17.22

We know $H^n_{dR}(S^n) \approx \mathbb{R}$. Let ω be a smooth orientation form on S^n. 17.21 $\Rightarrow [\omega] \in H^n_{dR}(S^n)$ is a basis. Assume $[\eta] \neq 0 \in H^n_{dR}(S^n)$

then $[\eta] = c[\omega] \in H^n_{dR}(S^n) \approxeq \mathbb{R}$

i.e. $[\eta] = [c\omega]$

$\Rightarrow \quad \eta = c\omega + \alpha$ with α exact

$\Rightarrow \quad \int_{S^n} \eta = \int_{S^n} c\omega + \int_{S^n} \alpha = c\int_{S^n} \omega \neq 0$.

Finish

Fall 2009 $\boxed{\#5}$

$\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$, closed

$f, g : S^n \to \mathbb{R}^{n+1} - \{0\}$ differentiable.

Claim: $\int_{S^n} f^*(\omega) / \int_{S^n} g^*(\omega)$ rational when $\int_{S^n} g^*(\omega) \neq 0$.

Define $r : \mathbb{R}^{n+1} - \{0\} \to S^n$ by $r(x) = \frac{x}{|x|}$

Then $r \circ f, r \circ g : S^n \to S^n$ have degrees $k_f, k_g \in \mathbb{Z}$.

$\exists\ \eta \in \Omega^n(S^n)$ s.t. $r^*(\eta) = \omega$? Why r^* surjective.

Then $\int_{S^n} f^*(\omega) = \int_{S^n} f^*(r^*(\eta)) = \int_{S^n} (r \circ f)^*(\eta) = k_f \int_{S^n} \eta$

and $\int_{S^n} g^*(\omega) = k_g \int_{S^n} \eta \Rightarrow \int_{S^n} f^*(\omega) / \int_{S^n} g^*(\omega) = k_f / k_g \in \mathbb{Q}$.

Top degree cohomology isomorphism from homotop. equiv. def. retract.

$$\begin{array}{ccc} H^n_{dR}(S^n) & \xrightarrow[\cong]{r^*} & H^n_{dR}(S^n) \\ \mathbb{R} & & \mathbb{R} \\ \eta & & \omega \end{array}$$

Fall 2008 $\boxed{\#1}$

$d_f : \Omega^i(M) \to \Omega^{i+1}(M)$, $\omega \mapsto d\omega + df \wedge \omega$, f smooth.

(a) claim: d_f cochain map, i.e. $d_f \circ d_f = 0$.

(b) Let $H^i_f(M)$ be the ith cohomology gp of Cochain complex $(\Omega^i(M), d_f)$. Claim: $H^0_f(M) \cong \mathbb{R}$. when M is $\mathbb{R}$.

(a)
$$\begin{aligned}(d_f \circ d_f)(\omega) &= d_f(d\omega + df \wedge \omega) \\ &= d(d\omega + df\wedge\omega) + df \wedge (d\omega + df \wedge \omega) \\ &= \cancel{dd\omega}^{0} + d(df \wedge \omega) + df \wedge d\omega + \cancel{df \wedge df \wedge \omega}^{0} \\ &= \cancel{ddf \wedge \omega}^{0} - \cancel{df \wedge d\omega + df \wedge d\omega} \\ &= 0.\end{aligned}$$

(b) We have

$$0 \xrightarrow{d_f^0} \Omega^0(\mathbb{R}) \xrightarrow{d_f^1} \Omega^1(\mathbb{R})$$

$$H^0_f(\mathbb{R}) = \ker(d_f^1) = \{ g \in \Omega^0(\mathbb{R}) : d_f(g) = 0 \in \Omega^1(\mathbb{R}) \}$$

$$d_f(g) = dg + df \wedge g = dg + g\,df = 0$$

$$dg = -g\,df$$

$H^0_f(\mathbb{R}) = \{$

Fall 2008 $\boxed{\#2}$

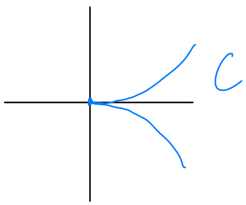
Fall 2008 $\boxed{\#3}$

$C = \{(x,y) : y^2 - x^3 = 0\}$, Claim: C not smooth submfld of $\mathbb{R}^2$.

Hint: space of tangent vectors in $T_{(0,0)}\mathbb{R}^2$ tangent to C?

$$C = \{(x,y) : y = \pm x^{3/2},\ x \geq 0\}$$

$$f(x) = x^{3/2} \Rightarrow f'(x) = \tfrac{3}{2}x^{1/2} \Rightarrow f'(0) = 0$$



$$i : C \hookrightarrow \mathbb{R}^2$$

$$di_{(x,y)} : T_{(x,y)}C \longrightarrow T_{(x,y)}\mathbb{R}^2$$

$$\gamma(t) = (t^{2/3}, t) \qquad t \in (-\infty, \infty)$$

$\gamma'(0) = (\infty, 1)$ undefined.

derivation v at $(0,0)$ defined as map from $C^\infty(\mathbb{R}^2)$ to $\mathbb{R}$ and satisfies product rule. $\quad v(fg) = f(vg) + (vf)g$

Let γ be a curve in C. $\gamma : (-1,1) \to C$

$\gamma'(0) =$

Assume it is a submfd. Then $i : C \hookrightarrow \mathbb{R}^2$ inclusion is an embedding $\Rightarrow$ $di_x : T_xC \to T_{i(x)}\mathbb{R}^2$ injective.

Let $\gamma : I \to C$ with $\gamma(0) = (0,0)$.

$$\gamma(t) = (x(t), y(t)).$$

$$\gamma'(t_0) = d\gamma_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right) = (x'(t), y'(t))\Big|_{t_0}$$

$$(i\circ\gamma)(t) = ((i\circ x)(t), (i\circ y)(t))$$

$$(i\circ\gamma)'(t) = di_{\gamma(t_0)}\, d\gamma_{t_0}\left(\frac{d}{dt}\Big|_{t_0}\right)$$

Let's assume $x'(t) \neq 0$, then

$$x^3 = y^2 \qquad 3x^2x' = 2yy'$$

Fall 2008 $\boxed{\#4}$

$T =$ revolving the circle $\{(x,y,z) \mid z=0, (x-R)^2+y^2=r^2\}$ around the y-axis, $R>r$. Compute
$$\int_T x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy.$$

Let ω be the integrand. Then
$$d\omega = dx\wedge dy\wedge dz - dy\wedge dx\wedge dz + dz\wedge dx\wedge dy$$
$$= 3\,dx\wedge dy\wedge dz$$
By Stokes:
$$\int_T \omega = \int_{\partial(D^2\times S^1)} \omega = \int_{D^2\times S^1} d\omega = 3\,\mathrm{vol}(D^2\times S^1)$$

where D^2 is a disk of radius r and S^1 is the circle of radius R.

Then
$$3\,\mathrm{vol}(D^2\times S^1) = 3\cdot(\pi r^2(2\pi R))$$
$$= \boxed{6\pi^2 r^2 R}$$

Fall 2008 $\boxed{\#5}$

B^3 closed 3-d ball. K closed, connected 1-dim'l submfd of B^3 with $\partial K = K \cap \partial B^3 = 2$ points. Compute homology of $B^3 - K$.

Let A be a tubular nbd of K, which exists since it is a 1-dim'l submfd and therefore does not intersect itself. Let $B = B^3 - K$.

Then $A \cap B = A \setminus K$ which deformation retracts to S^1.

And $A \cup B = B^3$ which is contractible.

M.V. gives:

$$\cdots \longrightarrow H_n(A \cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(A \cup B) \longrightarrow \cdots$$

	$H_n(S^1)$	$H_n(\{x_0\}) \oplus H_n(B^3-K)$	$H_n(\{x_0\})$
$n \geq 2$	0	$0 \oplus H_n(B^3-K)$ 0	0
$n = 1$	$\mathbb{Z}$	$0 \oplus H_1(B^3-K)$ $\mathbb{Z}$	0
$n = 0$	$\mathbb{Z}$	$\mathbb{Z} \oplus H_0(B^3-K)$ $\mathbb{Z}$	$\mathbb{Z}$

By exactness $H_2(B^3-K) = 0$. Since B^3-K is path connected, $H_0(B^3-K) = \mathbb{Z}$. Then we have left

$0 \longrightarrow \mathbb{Z} \longrightarrow H_1(B^3-K) \longrightarrow 0$. Exactness $\Rightarrow$

$H_1(B^3-K) = \mathbb{Z}$. In sum:

$$\boxed{H_n(B^3-K) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{else} \end{cases}}$$

Fall 2008 $\boxed{\#6}$

Recall covering spaces $p: \tilde{X} \to X$, $p': \tilde{X}' \to X$ isomorphic if $\exists$ homeo $\tilde{\phi}: \tilde{X} \to \tilde{X}'$ s.t. $p' \circ \tilde{\phi} = p$. Consider $p: \tilde{X} \to X$ of torus $X = S^1 \times S^1$ whose fiber $p^{-1}(x_0)$ at any point $x_0 \in X$ consists of 3 points. How many distinct isomorphism classes of such coverings are there?

" We know that n-sheeted covering spaces of X are classified by equivalence classes of homomorphisms $\pi_1(X; x_0) \to \Sigma_n$, the symmetric group on n elements, where the equivalence relation identifies a homomorphism ρ with each of its conjugates $h^{-1}\rho h$ by elements $h \in \Sigma_n$. " – Hatcher

Consider $\rho: \mathbb{Z}\langle a \rangle \times \mathbb{Z}\langle b \rangle \to \Sigma_3$ homomorphism, where $\Sigma_3 = \{(), (12), (13), (23), (123), (132)\}$. ρ is determined by $\rho(a), \rho(b)$.

Conjugates: $\underbrace{\{(12), (13), (23)\}}_{C_1}$ and $\underbrace{\{(123), (132)\}}_{C_2}$

- $\rho(a) = \rho(b) = ()$
- $\rho(a) = ()$, $\rho(b) \in C_1$
- $\rho(a) = ()$, $\rho(b) \in C_2$
- $\rho(a) \in C_1$, $\rho(b) = ()$
- $\rho(a) = \rho(b) \in C_1$
- $\rho(a) = \rho(b) \in C_2$
- $\rho(a) \neq \rho(b) \in C_2$
- $\rho(a) \neq \rho(b) \in C_1$ 2

- $\rho(a) \in C_2$, $\rho(b) = ()$
- $\rho(a) \in C_1$ $\rho(b) \in C_2$
- $\rho(a) \in C_2$ $\rho(b) \in C_1$

$$((12),(123)) \sim ((23),(132)) \sim ((13),(123))$$

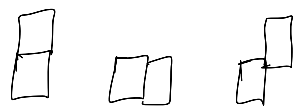
$$\wr \qquad\qquad \wr \qquad\qquad \wr$$

$$((12),(132)) \sim ((23),(123)) \sim ((13),(132))$$

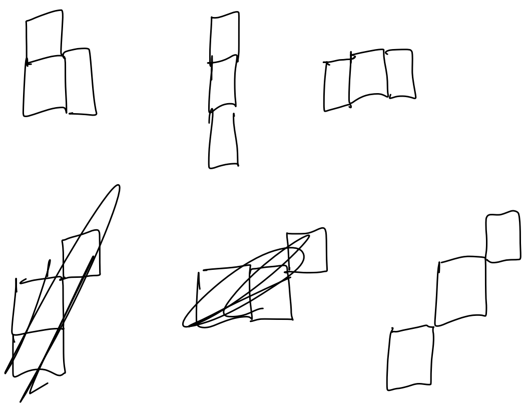
12

3 2-sheeted coverings

3 (2+1)-sheeted coverings



1 (1+1+1)-sheeted covering



Spring 2009 $\boxed{\#1}$

$S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ unit sphere in $\mathbb{R}^3$.

$f: S^2 \to \mathbb{R}^4$, $f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$

Claim: f immersion, $f(S^2)$ diffeomorphic to $\mathbb{RP}^2$.

$$df_p = \begin{pmatrix} 2x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \end{pmatrix} \quad \text{for } p = (x_1, x_2, x_3) \in S^2.$$

$\det(df_p^{\hat{4}}) = x_1(2x_1^2 + 2x_2^2)$

$\det(df_p^{\hat{3}}) = 2x_1(x_1x_2) + 2x_2(x_2^2) = 2x_2(x_1^2 + x_2^2)$

$\det(df_p^{\hat{2}}) = 2x_1(-x_1x_3) + 2x_2(x_2x_3) = 2x_3(x_2^2 - x_1^2)$

$\det(df_p^{\hat{1}}) = x_2(-x_1x_3) - x_1(x_2x_3) = -2x_1x_2x_3$.

where $df_p^{\hat{i}}$ is df_p with the ith row removed.

If $x_1 \neq 0$ then $\det(df_p^{\hat{4}}) \neq 0$.

If $x_2 \neq 0$ then $\det(df_p^{\hat{3}}) \neq 0$.

Otherwise p must be either $(0,0,1)$ or $(0,0,-1)$.

then df_p has rank 2. Thus $\operatorname{rank} df_p \geq 2$ $\forall\, p \in S^2$.

Since S^2 is 2-dim'l, this means f immersion.

Let $[x_1 : x_2 : x_3] \in \mathbb{RP}^2$ be represented by $(x_1, x_2, x_3) \in S^2$.

Then send $[x_1 : x_2 : x_3]$ to $f(x_1, x_2, x_3) \in f(S^2)$.

This is well-defined since $f(-x_1, -x_2, -x_3) = f(x_1, x_2, x_3)$.

We have

$$\begin{tikzcd} S^2 \arrow[rr, "\text{immersion}\ f"] \arrow[dr, "q"'] & & f(S^2) \subset \mathbb{R}^4 \\ & S^2/\sim = \mathbb{R}P^2 \arrow[ur, "g"'] & \end{tikzcd}$$

Let $(x_1^2 - x_2^2, x_1 x_2, x_1 x_3, x_2 x_3) = (y_1^2 - y_2^2, y_1 y_2, y_1 y_3, y_2 y_3)$

for $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S^2$ to be to points in $f(S^2)$.

If $x_1 = \pm y_1$ then $x_2 = \pm y_2$ and $x_3 = \pm y_3$. The same is true if $x_2 = \pm y_2$ or $x_3 = \pm y_3$.

Otherwise $x_1 \neq \pm y_1$ and $x_2 \neq \pm y_2$ and $x_3 \neq \pm y_3$:

We have $x_1^2 - x_2^2 = y_1^2 - y_2^2$

$$x_1^2 - y_1^2 = x_2^2 - y_2^2 \neq 0$$

Assume WLOG $x_1^2 - y_1^2 = x_2^2 - y_2^2 > 0$

then $\frac{x_1^2}{y_1^2} > 1$ and $\frac{y_2^2}{x_2^2} < 1$

then $(x_1 x_2)^2 \neq (y_1 y_2)^2 \Rightarrow x_1 x_2 \neq y_1 y_2$ a contradiction.

Hence $f(x) = f(y) \Rightarrow x = \pm y \Rightarrow g$ injective.

And g immersion b/c f immersion.

Hence, g diffeomorphism.

Spring 2009 $\boxed{\#2}$

ω closed $\in \Omega^n(\mathbb{R}^{n+1} - \{0\})$

Claim: ω exact $\iff \int_{S^n} \omega = 0$, for S^n unit sphere in $\mathbb{R}^n$.

ω exact $\Rightarrow$ $\omega = d\alpha$ for α in $\Omega^{n-1}(\mathbb{R}^{n+1} - \{0\})$

Then $\int_{S^n} \omega = \int_{S^n} d\alpha = \int_{\partial S^n} \alpha = \int_{\emptyset} \alpha = 0$.

Conversely, ω closed $\Rightarrow$ $\omega \in H^n_{dR}(\mathbb{R}^{n+1} - \{0\})$ and $\omega \in H^n_{dR}(S^n)$. There is an isomorphism

$$H^n_{dR}(S^n) \xrightarrow{\approx} \mathbb{R} \quad \text{sending} \quad \gamma \longmapsto \int_{S^n} \gamma$$

Thus if $\int_{S^n} \omega = 0$ then ω must be the 0 element in $H^n_{dR}(S^n)$ i.e. ω exact.

Spring 2009 $\boxed{\#3}$

$X = e^x \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$ vect. fields on all of $\mathbb{R}^2$

What Z satisfy $[X, Z] = 0$ and $[Y, Z] = 0$.

$$Z = g(x,y)\frac{\partial}{\partial x} + h(x,y)\frac{\partial}{\partial y}$$

$$[X,Z]f = XZf - ZXf$$

$$= X(gf_x + hf_y) - Z e^x f_x$$

$$= e^x(g_x f_x + \cancel{g f_{xx}} + h_x f_y + \cancel{h f_{xy}}) - g(e^x f_x + \cancel{e^x f_{xx}})$$

$$- \cancel{h e^x f_{xy}}$$

$$= e^x(g_x f_x + h_x f_y - g f_x) = 0$$

$$\Rightarrow h_x = 0, \quad g_x = g$$

$$[Y,Z]f = YZf - ZYf$$

$$= Y(gf_x + hf_y) - Zf_y$$

$$= g_y f_x + \cancel{g f_{xy}} + h_y f_y + \cancel{h f_{yy}} - \cancel{g f_{xy}} - \cancel{h f_{yy}}$$

$$= g_y f_x + h_y f_y = 0$$

$$\Rightarrow h_y = 0 \quad g_y = 0.$$

$$\boxed{Z = c_1 e^x \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial y}} \quad \text{for } c_1, c_2 \in \mathbb{R}.$$

$$[X,Z]f = X(c_1 e^x f_x + c_2 f_y) - Z e^x f_x$$

$$= e^x\left(\cancel{c_1 e^x} f_x + \cancel{c_1 e^x f_{xx}} + \cancel{c_2 f_{xy}}\right)$$

$$- c_1 e^x\left(\cancel{e^x} f_x + \cancel{e^x f_{xx}}\right) - \cancel{c_2 e^x} f_{xy}$$

$$= 0 \quad \checkmark$$

$[Y,Z]f = 0$ similarly.

Spring 2009 $\boxed{\#4}$

north pole

$$\pi_n(X; x_0) = \{f : S^n \to X \mid f(N) = x_0\}/\simeq$$

where $\simeq$ is basepoint-preserving homotopies.

Compute $\pi_n(T^P)$ for $T^P = \underbrace{S^1 \times \cdots \times S^1}_{P}$.

Hatcher prop 4.1:

A covering space projection $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces isomorphisms $P_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ $\forall\, n \geq 2$.

Proof: Let $[f] \in \pi_n(X, x_0)$, $n \geq 2$. Then we have

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ S^n \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd}$$

This satisfies the lifting criterion because $\pi_1(S^n) = 0 \subset P_*(\pi_1(\tilde{X}, \tilde{x}_0))$

Thus, $\exists\ \tilde{f} : S^n \to \tilde{X}$ a lift $\forall\ [f] \in \pi_n(X, x_0)$

That is, $\forall [f] \in \pi_n(X, x_0)$, $\exists [\tilde{f}] \in \pi_n(\tilde{X}, \tilde{x}_0)$ s.t. $P_*([\tilde{f}]) = [f]$.

So P_* is surjective, and we know it is injective already.

Claim: $\mathbb{R}^P$ is a covering space of T^P.

Define $p : \mathbb{R}^P \to T^P$ by $p(x_1, \ldots, x_p) = (e^{ix_1}, \ldots, e^{ix_p})$.

This is a covering space because for arbitrary $(e^{i\theta_1}, \ldots, e^{i\theta_p}) \in T^P$, Let $U = \{(e^{i\phi_1}, \ldots, e^{i\phi_p}) \mid \phi_i \in (\theta_i - \varepsilon, \theta_i + \varepsilon)\}$ for small ε. $p^{-1}(U) = \{(x_1, \ldots, x_p) \mid x_i \in (2\pi k + \theta_i - \varepsilon, 2\pi k + \theta_i + \varepsilon), k \in \mathbb{Z}\}$

which is a disjoint collection of sets homeomorphic to U.

Hence. $\pi_n(T^p) \approx \pi_n(\mathbb{R}^p) = 0 \quad \forall \ n \geq 2 \quad \forall \ p.$

And for $n=1$, we know $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$ if X, Y path connected. Thus, $\pi_1(T^p) = \mathbb{Z}^p$.

In sum,

$$\boxed{\pi_n(T^p) = \begin{cases} \mathbb{Z}^p & \text{if } n=1 \\ 0 & n \geq 2 \end{cases}}$$

Just use Hatcher 4.2

Spring 2009 $\boxed{\#5}$

Compute $\pi_1(\mathbb{R}^3 - K)$. $K =$ union of vertical axis $\{x=0, y=0\}$ and unit circle $\{x^2+y^2=1, z=0\}$.

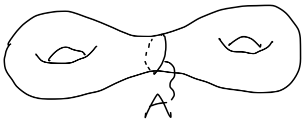
$\mathbb{R}^3 - K$ deformation retracts to $B-K$ where B is the ball of radius 2 centered at the origin. This is homotopy equivalent to the torus, by hollowing out the holes created by K.

Hence $\pi_1(\mathbb{R}^3 - K) \approx \pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$.

To see the second step, note that the half disc minus a point  is deformation retracts to the circle . Do this homotopy equivalence all around the circle to get $B-K \simeq T^2$.

Spring 2009 $\boxed{\#6}$

$X = M_2$ surface of genus 2. A as below.



Compute $H_n(X, A)$ $\forall\, n \geq 0$.

(X, A) is a good pair since A is a def. retract of a nbd in X.

Then, $H_n(X, A) \approx \tilde{H}_n(X/A)$ and $X/A = T^2 \vee T^2$.

And $\tilde{H}_n(T^2 \vee T^2) = \tilde{H}_n(T^2) \oplus \tilde{H}_n(T^2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ \mathbb{Z}^4 & n = 1 \\ 0 & \text{else} \end{cases}$.

Hence,

$$\boxed{H_n(X, A) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ \mathbb{Z}^4 & n = 1 \\ 0 & \text{else} \end{cases}}$$

Fall 2009 $\boxed{\#1}$

$f: M \to N$. M, N compact oriented mfds of same dimension n.

$f^*(\pi_1(M)) \subset \pi_1(N)$ has finite index as a subgr.

(a) Prove $[\pi_1(N) : f^*(\pi_1(M))]$ divides the degree of f.

(b) Give ex. where $[\pi_1(N) : f^*(\pi_1(M))] \neq$ degree of f.

(a) We know $H_n(M), H_n(N) = \mathbb{Z}$, so $f_*: H_n(M) \longrightarrow H_n(N)$ is multiplication by an integer k which is the degree of x.

$$\mathbb{Z} \to \mathbb{Z}$$
$$x \mapsto kx$$

Denote $l := [\pi_1(N) : f^*(\pi_1(M))]$. Let $p: \tilde{N} \to N$ be a covering with $p^*(\pi_1(\tilde{N})) = f^*(\pi_1(M))$, guaranteed to exist by classification of covering spaces. Then the lifting criterion is satisfied and $\exists\ \tilde{f}: M \to \tilde{N}$. Also, since $p^*(\pi_1(\tilde{N}))$ has index l, it is an l-sheeted covering.

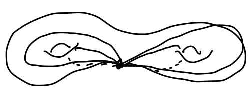
$\tilde{N}$ is n-dim'l as well, so we get induced homomorphism on homology $f_* = p_* \circ \tilde{f}_* : H_n(M) \to H_n(N)$.

Hence $\deg(f) = \deg(p)\deg(\tilde{f})$. In particular $\deg(p)$ divides $\deg(f)$, and $\deg(p)$ is l.

(b) $T^2 \to T^2$

$\pi_1(M_2) \longrightarrow \pi_1(T^2)$

$\mathbb{Z}^4 \qquad \mathbb{Z} \times \mathbb{Z}$



$$\begin{matrix} a \\ b \\ c \\ d \end{matrix} \longmapsto \begin{matrix} a \\ b \\ 0 \\ 0 \end{matrix}$$

$$\begin{tikzcd} & \tilde{N} \arrow[d, "p"] & \text{l-sheeted} \\ M \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"] & N \end{tikzcd}$$

$$\deg(f) = \deg(p \circ \tilde{f}) = \deg(p)\deg(\tilde{f})$$

$$\begin{tikzcd} & H_n(\tilde{N}) \arrow[d, "p^*"] & \mathbb{Z} \\ \mathbb{Z} \quad H_n(M) \arrow[ur, "\tilde{f}^*"] \arrow[r, "f^*"'] & H_n(N) & \mathbb{Z} \end{tikzcd}$$

?

vlad

Fall 2009 $\boxed{\#2}$

Is there a diff. map $\mathbb{R}^2 \to \mathbb{R}^2$ that sends vect. field $\frac{\partial}{\partial x}$ to v.f. $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and sends v.f. $\frac{\partial}{\partial y}$ to v.f. $Y = -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$?

$df_p : T_p\mathbb{R}^2 \longrightarrow T_p\mathbb{R}^2$

$(1,0) \longmapsto (x,1)$

$(0,1) \longmapsto (-1,x)$

Hence, $df_p = \begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix}$ in matrix form.

Then $\frac{\partial f^1}{\partial x} = x$, $\frac{\partial f^1}{\partial y} = -1$, $\frac{\partial f^2}{\partial x} = 1$, $\frac{\partial f^2}{\partial y} = x$

for $f(x,y) = (f^1(x,y), f^2(x,y))$.

Claim: $\nexists$ $f^2 : \mathbb{R}^2 \to \mathbb{R}$ s.t. $\frac{\partial f^2}{\partial x} = 1$ and $\frac{\partial f^2}{\partial y} = x$

The only f^2 satisfying $\frac{\partial f^2}{\partial y} = x$ are of the form $xy + g(x)$. Then $\frac{\partial f^2}{\partial x} = y + g'(x) \neq 1$.

Hence the answer to the original question is $\boxed{\text{no}}$.

Fall 2009 $\boxed{\#3}$

Let $f: S^n \to S^n$ degree 5.

(a) Show $\exists\ x_1 \in S^n$ s.t. $f(x_1) = -x_1$.

(b) Show $\exists\ x_2 \in S^n$ s.t. $f(x_2) = x_2$.

If $\nexists\ x_1 \in S^n$ s.t. $f(x_1) = -x_1$, then we can draw a line segment from each x to $f(x)$ that doesn't pass through the origin. Let this path be parameterized by $l_x(t)$. Since the line doesn't pass through 0, the path $\frac{l_x(t)}{|l_x(t)|}$ is well defined and lies in S^n. These paths give a homotopy from the identity map to f. Then $\deg(f) = \deg(Id) = 1 \implies \Leftarrow$. This proves (a).

Similarly, if (b) is false then we get a homotopy from f to the antipodal map which has degree ± 1, a contradiction. In this case we draw line segments from $f(x)$ to $-x$.

Fall 2009 $\boxed{\#4}$

M compact submfd of $\mathbb{R}^n$ w/ dim $\le n-3$.

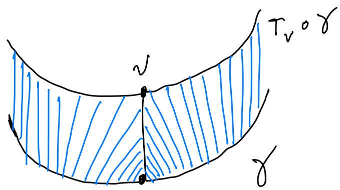
$f: B^2 \to \mathbb{R}^n$ differentiable. $T_v : \mathbb{R}^n \to \mathbb{R}^n$ translation along $v \in \mathbb{R}^n$.

(a) Show $\exists$ arbitrarily small vectors $v \in \mathbb{R}^n$ s.t. the image of $T_v \circ f$ is disjoint from M.

(b) Conclude that $\mathbb{R}^n - M$ simply connected.

First we show (a) $\Rightarrow$ (b).

Let γ be a loop based at x_0 in $\mathbb{R}^n - M$, which is open. $\forall\, x \in \gamma$, there is an open ball nbd containing x in $\mathbb{R}^n - M$, call it $B_{\varepsilon_x}(x)$ where ε_x is the radius. This is an open cover of γ which is compact since $\gamma : [0,1] \to \mathbb{R}^n - M$. Thus, $\exists$ $x_1, \dots, x_k$ finite points s.t. $\gamma \subset \bigcup_{i=1}^k B_{\varepsilon_{x_i}}(x_i)$. Let $\varepsilon = \min\{\varepsilon_{x_i}\}_{i=1}^k$. Then there is a vector v s.t. $|v| < \varepsilon$ and $(T_v \circ f)(B^2)$ disjoint from M. Now, we can build a homotopy from γ to the constant map C_{x_0}. First there is a homotopy from γ to $\tilde{\gamma} := \nu \cdot T_v \circ \gamma \cdot \nu^{-1}$ where ν is the path from x_0 to $x_0 + v$. We can see that in this digram:



Then $\tilde{\gamma}$ is homotopic to C_{x_0} since $T_v \circ \gamma \simeq C_{x_0+v}$ with $T_v \circ f$ serving as a homotopy.

Since γ was arbitrary, $\pi_1(\mathbb{R}^n - M)$ trivial.

Now we prove (a).

Let $g: M \times B^2 \longrightarrow \mathbb{R}^n$ be smooth. $\dim(M \times B^2) < n$ so by Sard's thm, $g(M \times B^2)$ has measure 0 in $\mathbb{R}^n$.

if we can have $g(M \times B^2) = \{v \in \mathbb{R}^n : (T_v \circ f)(B^2) \cap M \neq \emptyset\}$ then we are done because if (a) is false then $\{v \in \mathbb{R}^n : (T_v \circ f)(B^2) \cap M \neq \emptyset\}$ would have positive measure.

Define $g(x, y) = x - f(y) \in \mathbb{R}^n$ for $x \in M \subset \mathbb{R}^n$, $y \in B^2$.

Then
$$\begin{aligned} g(M \times B^2) &= \{v \in \mathbb{R}^n : v = x - f(y),\ x \in M,\ y \in B^2\} \\ &= \{v \in \mathbb{R}^n : x = v + f(y),\ x \in M,\ y \in B^2\} \\ &= \{v \in \mathbb{R}^n : x = (T_v \circ f)(y),\ x \in M,\ y \in B^2\} \\ &= \{v \in \mathbb{R}^n : (T_v \circ f)(B^2) \cap M \neq \emptyset\}. \end{aligned}$$

Fall 2009 $\boxed{\#5}$

$\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$ closed. $f, g : S^n \longrightarrow \mathbb{R}^{n+1} - \{0\}$ diff.

Claim: the ratio $\dfrac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)} \in \mathbb{Q}$ when denominator $\neq 0$.

Define $r : \mathbb{R}^{n+1} - \{0\} \longrightarrow S^n$ by $r(x) = \frac{x}{|x|}$, a retraction.

Then $r \circ f, r \circ g : S^n \longrightarrow S^n$ have degrees K_f and $K_g \in \mathbb{Z}$.

That is, $\int_{S^n} (r \circ f)^* \alpha = K_f \int_{S^n} \alpha$ for α any diff. form. on S^n.

Is there an $\alpha \in \Omega^n(S^n)$ s.t. $r^*(\alpha) = \omega$?

$i : S^n \hookrightarrow \mathbb{R}^{n+1} - \{0\}$ is the inclusion map.

then $r \circ i = Id \Rightarrow i^* \circ r^* = (r \circ i)^* = Id^* = Id$

$\Rightarrow r^*$ surjective, so yes, $\exists\ \alpha$ s.t. $r^*(\alpha) = \omega$.

Then $(r \circ f)^* \alpha = f^*(r^*(\alpha)) = f^*(\omega)$ and $(r \circ g)^* = g^*(\omega)$.

Then

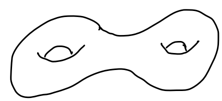
$\int_{S^n} f^*(\omega) = K_f \int_{S^n} \alpha$ and $\int_{S^n} g^*(\omega) = K_g \int_{S^n} \alpha$

Hence, $\dfrac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)} = \dfrac{K_f}{K_g} \in \mathbb{Q}$ when $K_g \neq 0$.

Fall 2009 $\boxed{\#6}$

$S =$ surface of genus $2 \in \mathbb{R}^3$. W is the closure of the bounded component of $\mathbb{R}^3 - S$, i.e. the solid with boundary S, $W \simeq S^1 \vee S^1$. Compute $H_n(W, S)$.

We get the relative long exact sequence of homology:

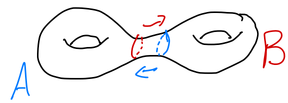


$$\cdots \longrightarrow H_n(S) \longrightarrow H_n(W) \longrightarrow H_n(W,S) \longrightarrow \cdots$$

$$\tilde{H}_n(W) \approx \tilde{H}_n(S^1 \vee S^1) \approx \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

So $H_n(W) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$

Let's do M.V. for S. Let A = left torus and B = right torus as such:



Then $A \cap B \simeq S^1$, $A, B = T^2 - \{*\} \simeq S^1 \vee S^1$

So we get:

$$\cdots \longrightarrow \underbrace{H_n(S^1)}_{\begin{cases} \mathbb{Z} & n=0,1 \\ 0 & \text{else} \end{cases}} \longrightarrow \underbrace{H_n(S^1 \vee S^1) \oplus H_n(S^1 \vee S^1)}_{\begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=0 \\ \mathbb{Z}^4 & n=1 \\ 0 & \text{else} \end{cases}} \longrightarrow H_n(S) \longrightarrow \cdots$$

For $n \geq 3$ we have $0 \to H_n(S) \to 0 \Rightarrow H_n(S) = 0$.

Otherwise, we have (since $H_0(S) = \mathbb{Z}$ by path connectedness)

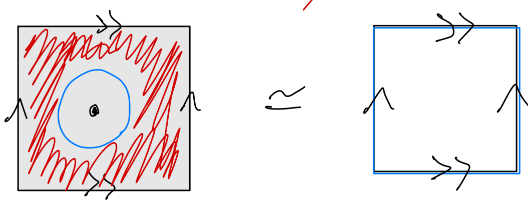
$$0 \to H_2(S) \to \mathbb{Z} \to \mathbb{Z}^4 \to H_1(S) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$$

Consider $H_1(S^1) \xrightarrow{i_*, j_*} H_1(S^1 \vee S^1) \oplus H_1(S^1 \vee S^1)$

$\mathbb{Z} \longrightarrow \mathbb{Z}^4$

If we have a loop in $A \cap B$, then we have a loop around the puncture in A. ~~As we can see in the square representation of the torus, the loop gets sent to $aba^{-1}b^{-1}$ as $T - \{*\}$ gets sent to $S^1 \vee S^1$. And $aba^{-1}b^{-1} = 0$ in homology~~

this is a boundry in the torus of red cell So 0 in homology.



Hence $\mathbb{Z} \to \mathbb{Z}^4$ is the 0-map, and we have

$$0 \xrightarrow{\times 0} H_2(S) \longrightarrow \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z}^4$$

implying $H_2(S) \approx \mathbb{Z}$ by exactness.

The map $H_0(S^1) \longrightarrow H_0(A) \oplus H_0(B)$ has Kernel 0

$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$

$x \longmapsto (a, b)$ So $\operatorname{im}(H_1(S) \to \mathbb{Z}) = 0$.

Then we have $\xrightarrow{\times 0} \mathbb{Z}^4 \longrightarrow H_1(S) \xrightarrow{\times 0}$ implying $H_1(S) \approx \mathbb{Z}^4$ by exactness. Hence
$$H_n(S) \simeq \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^4 & n = 1 \\ 0 & \text{else} \end{cases}$$

Then our original sequence is

$$\cdots \longrightarrow H_n(S) \longrightarrow H_n(W) \longrightarrow H_n(W, S) \longrightarrow \cdots$$

For $n \geq 4$ we have $0 \longrightarrow H_n(W,S) \longrightarrow 0 \implies \boxed{H_n(W,S) = 0.}$

For $n = 3$ we have $0 \longrightarrow H_3(W,S) \longrightarrow \mathbb{Z} \longrightarrow 0 \implies \boxed{H_3(W,S) = \mathbb{Z}}.$

otherwise we have

$$0 \to H_2(W,S) \longrightarrow \mathbb{Z}^4 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(W,S) \longrightarrow \mathbb{Z} \to \mathbb{Z} \longrightarrow H_0(W,S) \longrightarrow 0$$

The final $\mathbb{Z} \to \mathbb{Z}$ is an isomorphism, so $\boxed{H_0(W,S) = 0}$ since $H_0(W,S) \hookrightarrow 0$. This also means $\mathrm{im}(\mathbb{Z} \oplus \mathbb{Z} \rightsquigarrow H_1(W,S))$

$= \mathrm{Ker}(H_1(W,S) \xrightarrow{\times 0} \mathbb{Z}) = H_1(W,S)$.

Consider $H_1(S) \longrightarrow H_1(W)$ which has $\mathrm{Ker} = \mathbb{Z} \oplus \mathbb{Z}$

$\mathbb{Z}^4 \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$ and image $= \mathbb{Z} \oplus \mathbb{Z}$

$$\left.\begin{matrix} a \\ b \\ c \\ d \end{matrix}\right\} \longmapsto \left\{\begin{matrix} x \\ 0 \\ y \\ 0 \end{matrix}\right.$$

Then $\mathrm{im}(H_2(W,S) \hookrightarrow \mathbb{Z}^4) = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \boxed{H_2(W,S) = \mathbb{Z} \oplus \mathbb{Z}}$

and $\mathrm{Ker}(\mathbb{Z} \oplus \mathbb{Z} \twoheadrightarrow H_1(W,S)) = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow$ image $= \boxed{0 = H_1(W,S)}$

In Sum:
$$\boxed{H_n(W,S) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ \mathbb{Z} & n = 3 \\ 0 & \text{else} \end{cases}}$$

Fall 2009 $\boxed{\#7}$

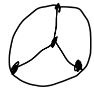
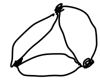
M compact, connected submfd of oriented mfd N of dim n, $\dim M = \dim N - 1$. Claim: M orientable $\Longleftrightarrow$ it admits arbitrarily small connected nbds U s.t. $U - M$ is disconnected.

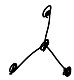
I.e. $\Leftrightarrow$ $\forall$ open $V \subset N$ containing M, $\exists$ open connected $U \subset V$ s.t. $U - M$ not connected. $M \subset U$?

Let η be an orientation form on N. i.e. η nonvanishing n-form. Let M be orientable with ω orientation form. Since M is compact, it is covered by finitely many, positively oriented coordinate charts $U^1, \ldots, U^k$ with positively oriented coordinate frames $(E_i^1), \ldots, (E_i^k)$, Then we get a positive direction at each point based on the orientation of N, which is consistent.

? Ask

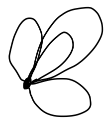
Fall 2010 $\boxed{\#1}$

Compute $\pi_1(X_1)$, $\pi_1(X_2)$ for $X_1 =$  and $X_2 =$ 

These are CW complexes. Let $A =$  which is contractible, and since (X_1, A) good pair we get

$\pi_1(X_1) \approx \pi_1(X_1/A) = \pi_1(S^1 \vee S^1 \vee S^1) = \boxed{\mathbb{Z}*\mathbb{Z}*\mathbb{Z}}$

$B =$  is contractible as well and (X_2, B) good pair.

Then collapsing it gives  $= S^1 \vee S^1 \vee S^1 \vee S^1$

Thus $\pi_1(X_2) \approx \pi_1(X_2/B) = \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1) = \boxed{\mathbb{Z}*\mathbb{Z}*\mathbb{Z}*\mathbb{Z}}$

Fall 2010 $\boxed{\#2}$

Let p_1, p_2, p_3 be three distinct points in S^2

$X = S^2/\{p_1, p_2, p_3\}$. Compute $H_n(X;\mathbb{Z})$.

$S^2/\{p_1,p_2,p_3\}$ is homotopy equivalent to S^2 union two 1-cells glued from p_1 to p_2 and from p_2 to p_3. This is because these two 1-cells are contractible and form a good pair with S^2. Then this is homotopy equivalent with $S^2 \vee S^1 \vee S^1$ by collapsing the paths in S^2 connecting p_1 to p_2 and p_2 to p_3.

Hence $\tilde{H}_n(X) \approx \tilde{H}_n(S^2 \vee S^1 \vee S^1) = \tilde{H}_n(S^2) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1)$

$$= \begin{cases} \mathbb{Z} & n=2 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

Thus $\boxed{H_n(X) = \begin{cases} \mathbb{Z} & n=0,2 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}}$

Fall 2010 $\boxed{\#3}$

Gauss Bonet

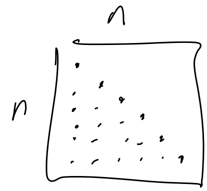
Fall 2010 $\boxed{\#4}$ $\boxed{\text{Daniel}}$

$M_n(\mathbb{R}) = \{n\times n \text{ matrices}\}$. $O(n) = \{A \in M_n(\mathbb{R}) : AA^T = id\}$

Claim: $O(n)$ smooth mfd. what is it's dimension?

Define $f : M_n(\mathbb{R}) \longrightarrow Sym_n(\mathbb{R})$

$$A \longmapsto AA^T$$



$$df_A : T_A M_n(\mathbb{R}) \longrightarrow T_{AA^T} Sym_n(\mathbb{R})$$

$$\mathbb{R}^{n^2} \longrightarrow \mathbb{R}^{n(n+1)/2}$$

$$n + n-1 + n-2 + n-3 + \cdots + n-(n-1)$$

$$= n^2 - \sum_{i=1}^{n-1} i$$

$$= n^2 - \frac{n(n-1)}{2}$$

$$= \frac{2n^2 - n^2 + n}{2} = \frac{n^2+n}{2}$$

$$= \frac{n(n+1)}{2}$$

$$df_A(B) = \left.\frac{d}{dt}\right|_{t=0} f(A+tB)$$

$$= \lim_{h\to 0} \frac{f(A+hB) - f(A)}{h}$$

$$= \lim_{h\to 0} \frac{(A+hB)(A+hB)^T - AA^T}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{AA^T} + AhB^T + hBA^T + hBhB^T - \cancel{AA^T}}{h}$$

$$= AB^T + BA^T + \cancel{hBB^T}^{\,0}$$

So for $C \in Sym_n(\mathbb{R})$ if $\exists\, B$ s.t. $C = AB^T + BA^T$ then df_A surjective and Id reg. value $\Rightarrow O(n)$ submfd.

Indeed, let $B = \frac{1}{2}CA$. Then $AB^T + BA^T = \frac{1}{2}AA^TC + \frac{1}{2}CAA^T = C$ since $f(A) = Id$. Dimension is then $n^2 - n(n+1)/2 = n(n-1)/2$.

Fall 2010 $\boxed{\#5}$

$\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ $\quad d\omega = dx_1 \wedge \cdots \wedge dx_n$

Claim: $\forall p \in \mathbb{R}$, $\quad \alpha = \dfrac{1}{(x_1^2 + \cdots + x_n^2)^p}\, \omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$

is not exact. Hint: S^{n-1}

If α exact, the $\exists\, \eta \in \Omega^{n-2}(\mathbb{R}^n - \{0\})$ s.t. $\alpha = d\eta$.

Then by stokes, $\int_{S^{n-1}} \alpha = \int_{S^{n-1}} d\eta = \int_{\partial S^{n-1}} \eta = 0$

But $\alpha = \omega$ on S^{n-1} $\forall p \in \mathbb{R}$, so by Stokes again

$$\int_{S^{n-1}} \alpha = \int_{S^{n-1}} \omega = \int_{B^n - \{0\}} d\omega = \mathrm{vol}(B^n - \{0\}) \neq 0.$$

Fall 2010 $\boxed{\#6}$

$\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ w/ coords $x_1, y_1, \dots, x_n, y_n$.

$f : \mathbb{R}^{2n} \to \mathbb{R}$ smooth, Find vect. field X s.t.

$i_X \omega = df$ where i_X denotes the interior product. Then

compute Lie derivative $\mathcal{L}_X \omega$.

Recall: $(i_X \omega)_p (V) = \omega_p (X_p, V)$ for $p \in \mathbb{R}^{2n}$, $V \in T_p \mathbb{R}^{2n}$

$$dx_i \wedge dy_i (X_p, V) = \det \begin{pmatrix} x_i(X_p) & x_i(V) \\ y_i(X_p) & y_i(V) \end{pmatrix} = x_i(X_p) y_i(V) - x_i(V) y_i(X_p)$$

$$\omega_p (X_p, V) = \sum_{i=1}^n x_i(X_p) y_i(V) - x_i(V) y_i(X_p)$$

$$df_p(V) = \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_p x_i(V) + \left.\frac{\partial f}{\partial y_i}\right|_p y_i(V)$$

Hence $x_i(X) = \dfrac{\partial f}{\partial y_i}$ and $y_i(X) = \dfrac{\partial f}{\partial x_i}$

By Cartan's magic formula,

$$\mathcal{L}_X \omega = X \lrcorner\, d\omega + d(X \lrcorner\, \omega) = X \lrcorner\, d\omega + \cancel{d(df)} = X \lrcorner\, d\omega$$

$d\omega = 0$ so $X \lrcorner\, 0 = 0 \Rightarrow \mathcal{L}_X \omega = 0$.

Fall 2010 $\boxed{\# 7}$

X top. space. $H_p(X;\mathbb{Z})$ finite. $H^{p+1}(X;\mathbb{Q}) = 0$.

$u \in C^{p+1}(X;\mathbb{Z}) = \mathrm{Hom}(C_{p+1}(X;\mathbb{Z}),\mathbb{Z})$, $\delta u = 0$.

(a) Show $\forall\ \alpha \in C_p(X;\mathbb{Z})$ with $\partial\alpha = 0$, $\exists\ k \in \mathbb{Z}-\{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$.

$[\alpha] \in H_p(X;\mathbb{Z})$, a finite group $\Rightarrow k[\alpha] = 0 \in H_p(X;\mathbb{Z})$ for some $k \in \mathbb{Z}-\{0\}$. That means $[k\alpha] = 0$

$\Rightarrow k\alpha = \partial\beta$ for some $\beta \in C_{p+1}(X;\mathbb{Z})$. $\square$

(b) Show $\exists$ hom $L_u : H_p(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ s.t. $L_u([\alpha]) = \frac{1}{k}u(\beta)$ for every $k \in \mathbb{Z}-\{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$.

Namely show that $L_u([\alpha])$ independent of k, β and or representative α of $[\alpha]$.

$\delta u = 0 \Rightarrow [u] \in H^{p+1}(X;\mathbb{Q}) = 0 \Rightarrow \exists\ v \in C^p(X;\mathbb{Z})$ s.t. $\delta v = u$. Thus for β' with $k\alpha = \partial\beta'$ as well, we have

$u(\beta') = \delta v(\beta') = v(\partial\beta') = v(k\alpha) = v(\partial\beta) = \delta v(\beta) = u(\beta)$.

k and k' are the same by order.

Spring 2011 $\boxed{\#1}$

$$\omega = x_1 \, dx_2 \wedge dx_3 \wedge dx_4 + x_2 \, dx_1 \wedge dx_3 \wedge dx_4$$
$$+ x_3 \, dx_1 \wedge dx_2 \wedge dx_4 \qquad \text{Compute } \int_{S^3} \omega.$$

$$d\omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_4$$
$$+ dx_3 \wedge dx_1 \wedge dx_2 \wedge dx_4$$
$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

By stokes,

$$\int_{S^3} \omega = \int_{\partial B^4} \omega = \int_{B^4} d\omega = \int_{B^4} dx_1 dx_2 dx_3 dx_4 = \boxed{\mathrm{vol}(B^4)}.$$

Spring 2011 $\boxed{\#2}$

$M = \{(x,y) : x, y \in \mathbb{R}^3,\ \|x\| = 1,\ \|y\| = 1,\ \langle x, y\rangle = 0\}$.

Claim: M smooth cpct embedded submfd of $\mathbb{R}^6$ and can be identified with unit tangent bundle of S^2.

Define $M_1 = \{(x,y) : x, y \in \mathbb{R}^3,\ \|x\| = 1\}$ and $f_1 : \mathbb{R}^6 \to \mathbb{R}$ by $f_1(x,y) = \|x\|^2 = x_1^2 + x_2^2 + x_3^2$.

Then $M_1 = f_1^{-1}(1)$ and $(df_1)_p = [2x_1\ 2x_2\ 2x_3\ 0\ 0\ 0] \neq 0$ $\forall p \in M_1$. Thus M_1 submfd of $\mathbb{R}^6$.

Define $M_2 = \{(x,y) \in M_1 : \|y\| = 1\}$ and $f_2 : M_1 \to \mathbb{R}$ by $f_2(x,y) = \|y\|^2 = y_1^2 + y_2^2 + y_3^2$. Then $M_2 = f_2^{-1}(1)$ and $(df_2)_p = [0\ 0\ 0\ 2y_1\ 2y_2\ 2y_3] \neq 0$ $\forall p \in M_2$.

Thus, M_2 submfd of M_1.

Finally define $f : M_2 \to \mathbb{R}$ by $f(x,y) = \langle x, y\rangle$

Then $f^{-1}(0) = M$ and $df_p = [y_1\ y_2\ y_3\ x_1\ x_2\ x_3] \neq 0$ $\forall p \in M$ $\Rightarrow$ M submfd of M_2 and therefore of $\mathbb{R}^6$.

And it must be cpct since $\|(x,y)\|^2 = \|x\|^2 + \|y\|^2 = 2$.

We can identify M with the unit tangent bundle by identifying x with a point on the sphere S^2 and y with a unit tangent vector at x, since the normal vector at x is just x and all tangent vects char. by $\langle x, y\rangle = 0$.

Spring 2011 $\boxed{\#3}$

$\mathbb{RP}^n = S^n/\sim$, $x \sim -x$.

(a) Use covering spaces to compute $\pi_1(\mathbb{RP}^n)$

(b) CW decomp. of $\mathbb{RP}^n$ $n \geq 1$

(c) Use CW decomp to compute $H_k(\mathbb{RP}^n)$, $k \geq 0$.

(d) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable? Explain.

(a) $\mathbb{RP}^1 \cong S^1$ so $\pi_1(\mathbb{RP}^1) \approx \pi_1(S^1) = \mathbb{Z}$.

For $n > 1$, S^n is a 2-sheeted cover of $\mathbb{RP}^n$ and S^n is simply connected $\Rightarrow$ it is a universal cover. Thus the trivial subgroup has index 2 in $\pi_1(\mathbb{RP}^n)$. That means $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$.

In sum, $\boxed{\pi_1(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & n=1 \\ \mathbb{Z}_2 & n>1 \end{cases}}$

(b) For $n \geq 2$, $\mathbb{RP}^n = \mathbb{RP}^{n-1} \sqcup D^n$ where $\partial D^n = S^{n-1}$ is glued via the quotient map $q: S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$ identifying antipodal points. $\mathbb{RP}^1$ is composed of a 0-cell and a 1-cell, so $\mathbb{RP}^n$ is composed of an i-cell for each $i \in \{0, \dots, n\}$, glued as above.

(c) $H_k(\mathbb{RP}^n) = 0$ for $k > n$. For $k \leq n$ we have $H_k(\mathbb{RP}^n) = H_k(X^k) = H_k(\mathbb{RP}^k)$ where X^k is the k-dim'l subcomplex of $\mathbb{RP}^n$.

$H_1(\mathbb{RP}^1) \approx H_1(S^1) = \mathbb{Z}$.

For $K > 1$, we have relative homology LES:

$$\cdots \longrightarrow H_K(X^{K-1}) \longrightarrow H_K(\mathbb{RP}^K) \longrightarrow H_K(\mathbb{RP}^K, X^{K-1}) \longrightarrow H_{K-1}(X^{K-1}) \longrightarrow \cdots$$

for X^{K-1} the $(K-1)$-dim'l subcomplex of $\mathbb{RP}^K$.

$$H_K(\mathbb{RP}^K, X^{K-1}) \approx \tilde{H}_K(\mathbb{RP}^K / X^{K-1}) \approx \tilde{H}_K(S^K) = \mathbb{Z}$$

And $H_{K-1}(\mathbb{RP}^K) \approx H_{K-1}(\mathbb{RP}^{K-1})$, so we get

$$0 \longrightarrow H_K(\mathbb{RP}^K) \longrightarrow \mathbb{Z} \longrightarrow H_{K-1}(\mathbb{RP}^{K-1}) \longrightarrow H_{K-1}(\mathbb{RP}^K)$$

The map $H_{K-1}(\mathbb{RP}^{K-1}) \xrightarrow{i_*} H_{K-1}(\mathbb{RP}^K)$ is the 0-map b/c a $(K-1)$ chain in $\mathbb{RP}^{K-1}$ included in $\mathbb{RP}^K$ is the boundary of a K-dim'l cell in the CW decomp.

Thus, $$H_K(\mathbb{RP}^K) \hookrightarrow \mathbb{Z} \twoheadrightarrow H_{K-1}(\mathbb{RP}^{K-1}).$$

Hence, $$H_K(\mathbb{RP}^K) = \begin{cases} \mathbb{Z} & \text{if } H_{K-1}(\mathbb{RP}^{K-1}) = 0 \\ 0 & \text{if } H_{K-1}(\mathbb{RP}^{K-1}) = \mathbb{Z} \end{cases}$$

In sum, $$H_K(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & K = 0 \\ 0 & K > n \\ \mathbb{Z} & K \text{ odd, } 0 < K \leq n \\ 0 & K \text{ even, } 0 < K \leq n \end{cases}$$

for $n \geq 1$

(d) Hence, since $H_n(M^n) = \mathbb{Z} \Leftrightarrow M^n$ orientable, for M^n an n dim'l closed, connected mfd, then $\mathbb{RP}^n$ orientable when n odd.

Spring 2011 $\boxed{\#4}$

$f: X \to Y$ continuous. $C_f = \left((X \times [0,1]) \sqcup Y \right) / \sim$

$(x,1) \sim f(x)$ $\forall x \in X$. $(x,0) \sim (x',0)$ $\forall x, x' \in X$.

Claim: $\exists$ LES:

$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \tilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots$$

Let $A = X \times [0, 3/4] / \sim \;\simeq \{*\}$ $\quad H_i(A) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{else} \end{cases}$

$\quad B = X \times [1/4, 1] / \sim \;\simeq Y$

Then $A \cap B \simeq X$, and by M.V. we get L.E.S.

$$\cdots \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(C_f) \to \cdots$$
$$\wr\wr$$
$$\cdots \to H_i(X) \to H_i(\{*\}) \oplus H_i(Y) \to H_i(C_f) \to \cdots$$

For $i > 0$ this is

$$\cdots \to H_i(X) \to H_i(Y) \to H_i(C_f) \to \cdots$$

and for $i = 0$ this is

$$\cdots \to H_0(X) \to \mathbb{Z} \oplus H_0(Y) \to H_0(C_f) \to 0$$

Thus, we can switch out this section for

$$\cdots \to H_0(X) \to H_0(Y) \to \tilde{H}_0(C_f) \to 0$$

and the whole LES is

$$\cdots \longrightarrow H_i(X) \longrightarrow H_i(Y) \longrightarrow \tilde{H}_i(C_f) \longrightarrow \cdots$$

since $\tilde{H}_i(C_f) = H_i(C_f)$ for $i > 0$.

We can make the switch because

$H_0(X) \longrightarrow \mathbb{Z} \oplus H_0(Y)$ and $H_0(X) \longrightarrow H_0(Y)$ both have kernel 0. and $\operatorname{im}(\mathbb{Z} \oplus H_0(Y) \longrightarrow H_0(C_f))$

$= H_0(C_f) \implies \operatorname{im}(H_0(Y) \longrightarrow H_0(C_f)) = \tilde{H}_0(C_f)$.

The last thing we check is that

$H_i(X) \longrightarrow H_i(Y)$ is the map f_*

We have:

$$\begin{tikzcd} H_i(A \cap B) \arrow[r, "j_*, j'_*"] & H_i(A) \oplus H_i(B) \arrow[d, "r_*"] \\ H_i(X) \arrow[u, "i_*"] \arrow[r, dashed] & H_i(Y) \end{tikzcd}$$

where i, j, j' are inclusions and r is a retraction. from $B \simeq Y$.

Then $(r \circ j' \circ i)x = r(x) = f(x)$ since the retraction just follows f into Y. Hence the map is indeed f_*.

Spring 2011 $\boxed{\#5}$

Recall a Lie Group G is a smooth mfd which is also a group, and whose group operations multiplication and inverse are smooth maps.

Claim: G connected lie group $\Rightarrow \pi_1(G)$ abelian.

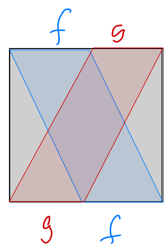
We can choose the basepoint to be the identity element in G, e. Let $[f], [g] \in \pi_1(G, e)$.

Then $f, g : I \longrightarrow G$ with $f(0) = f(1) = g(0) = g(1) = e$.

Define $H : I \times I \longrightarrow G$ by $H(s,t) = f_t(s)\, g_t(s)$

where $f_t(s) = \begin{cases} Ce(s) & s \in [0, t/2] \\ f(2s-t) & s \in [t/2, (t+1)/2] \\ Ce(s) & s \in [(t+1)/2, 1] \end{cases}$ $\qquad f(2(\frac{t}{2}) - t) = f(0) = e$, $\quad f(2(\frac{t+1}{2}) - t) = f(1) = e$

and $g_t(s) = \begin{cases} Ce(s) & s \in [0, (1-t)/2] \\ g(2s+t-1) & s \in [(1-t)/2, (2-t)/2] \\ Ce(s) & s \in [(2-t)/2, 1] \end{cases}$ $\qquad g(2(\frac{1-t}{2}) + t - 1) = g(0) = e$, $\quad g(2(\frac{2-t}{2}) + t - 1) = g(1) = e$



$H(s,0) = f_0(s)\, g_0(s) = \left\{ \begin{matrix} f(2s) & s \in [0, 1/2] \\ g(2s-1) & s \in [1/2, 1] \end{matrix} \right\} = (f \cdot g)(s)$

$H(s,1) = f_1(s)\, g_1(s) = \left\{ \begin{matrix} f(2s-1) & s \in [1/2, 1] \\ g(2s) & s \in [0, 1/2] \end{matrix} \right\} = (g \cdot f)(s)$

$H(0,t) = H(1,t) = e$. Hence $f \cdot g \simeq g \cdot f$ and $\pi_1(G)$ abelian.

Spring 2011 $\boxed{\#6}$

$M \subset \mathbb{R}^3$ embedded cpct oriented surface w/out bdry of genus $g \geq 1$. Show Gaussian curvature K of M must vanish somewhere.

Gauss Bonet gives

$$\int_M K \, dA = 2\pi \chi(M) = 2\pi(2-2g) \leq 0$$

So K must be negative somewhere on M.

All mfds with genus $g \geq 1$ also must have K positive somewhere on M. Thus it must vanish somewhere by continuity.

WEEK 4

Spring 2012 $\boxed{\#1}$

M compact n-dim'l mfd. Claim: M cannot be immersed in $\mathbb{R}^n$.

Assume $f: M \to \mathbb{R}^n$ is an immersion

then $df_p : T_pM \to T_{f(p)}\mathbb{R}^n$ is injective $\forall\, p \in M$

$\mathbb{R}^n \to \mathbb{R}^n$

which mens it is surjective $\forall\, p \in M$ by dimension.

Thus f is also a submersion and therefore a local diffeomorphism.

Since f is smooth, $f(M)$ compact in $\mathbb{R}^n$.

Let $\{U_i\}_{i=1}^{k}$ be an open cover of M s.t.

$F|_{U_i} : U_i \to F(U_i)$ is a diffeomorphism $\forall\, i$.

Then it is also a homeomorphism $\forall\, i$, and therefore $F(U_i)$ open and $\bigcup_{i=1}^{k} F(U_i) = M$ is open in $\mathbb{R}^n$.

Then $\mathbb{R}^n = \mathbb{R}^n - M \sqcup M$ is the disjoint union of two open sets $\Rightarrow$ $\mathbb{R}^n$ disconnected $\Rightarrow\Leftarrow$.

Spring 2012 $\boxed{\#2}$

$\Sigma_{1,1} = T^2 -$ small disc.

(a) compute $H_n(\Sigma_{1,1})$

(b) $\Sigma_2 =$ closed, oriented surface of genus 2. Use (a) to compute $H_n(\Sigma_2)$

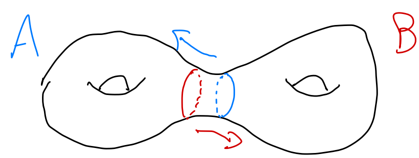
(a) $T^2 -$ small disc def. retracts to $S^1 \vee S^1$

So, $\tilde{H}_n(\Sigma_{1,1}) \approx \tilde{H}_n(S^1 \vee S^1) \approx \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1)$

$$= \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \quad H_n(\Sigma_{1,1}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

(b) Let $A, B = 2$ overlapping punctured tori as in the diagram:



Then $A \cup B = \Sigma_2$,

$A, B \simeq \Sigma_{1,1}$ and $A \cap B \simeq S^1$.

Hence, M.V. gives $\cdots \to H_n(S^1) \to H_n(\Sigma_{1,1}) \oplus H_n(\Sigma_{1,1}) \to H_n(\Sigma_2) \to \cdots$

For $n \geq 3$ we have

$$0 \to H_n(\Sigma_2) \to 0 \quad \text{implying} \quad H_n(\Sigma_2) = 0.$$

Otherwise, we have

Since Σ_2 path connected

$$0 \to H_2(\Sigma_2) \to \mathbb{Z} \to \mathbb{Z}^4 \to H_1(\Sigma_2) \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$$

Consider the map $H_1(A\cap B) \xrightarrow{i_*, j_*} H_1(A)\oplus H_1(B)$.

A 1-chain in $A\cap B$ is the boundary of a 2-chain in A or B $\Rightarrow$ i_*, j_* are 0-maps.

Hence, $\mathbb{Z}^4 \hookrightarrow H_1(\Sigma_2)$. Consider the map $H_0(A\cap B) \to H_0(A)\oplus H_0(B)$ is injective $\Rightarrow$ $H_1(\Sigma_2) \to \mathbb{Z}$ is the 0-map $\Rightarrow$ $\mathbb{Z}^4 \twoheadrightarrow H_1(\Sigma_2)$. Hence $H_1(\Sigma_2) = \mathbb{Z}^4$

$\mathbb{Z} \xrightarrow{\times 0} \mathbb{Z}^4$ also $\Rightarrow$ $H_2(\Sigma_2) \twoheadrightarrow \mathbb{Z}$ and the $0 \to H_2(\Sigma_2)$ $\Rightarrow$ $H_2(\Sigma_2) \hookrightarrow \mathbb{Z}$. Hence $H_2(\Sigma_2) = \mathbb{Z}$

In sum,

$$\boxed{H_n(\Sigma_2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^4 & n = 1 \\ 0 & \text{else} \end{cases}}$$

Spring 2012 $\boxed{\#3}$

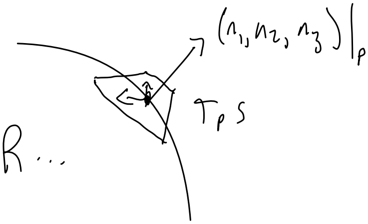
S oriented embedded surface in $\mathbb{R}^3$ and ω area form on S

$\omega_p(e_1, e_2) = 1 \quad \forall p \in S$ and any orthonormal basis (e_1, e_2) of T_pS

If (n_1, n_2, n_3) is the unit normal vect. field of S,

then prove that $\omega = n_1\, dy \wedge dz - n_2\, dx \wedge dz + n_3\, dx \wedge dy$.

$$\omega = P\, dy \wedge dz + Q\, dx \wedge dz + R\, dx \wedge dy \in \Omega^2(S)$$



$$\omega_p(e_1, e_2) = P(p)\,(dy \wedge dz)(e_1, e_2) + Q(p)\,(dx \wedge dz)(e_1, e_2) + R \ldots$$

$$= P(p)\left(e_{1y} e_{2z} - e_{1z} e_{2y}\right) + Q(p)\left(e_{1x} e_{2z} - e_{1z} e_{2x}\right)$$
$$+ R(p)\left(e_{1x} e_{2y} - e_{1y} e_{2x}\right) = 1$$

$$= P n_1 + Q n_2 + R n_3 = 1$$

$$e_1 \times e_2 = \det \begin{pmatrix} e_{1x} & e_{1y} & e_{1z} \\ e_{2x} & e_{2y} & e_{2z} \\ i & j & k \end{pmatrix}$$

$$(\underbrace{e_{1y} e_{2z} - e_{1z} e_{2y}}_{n_1}, \ldots$$

$$P n_1 + Q n_2 + R n_3 = n_1^2 + n_2^2 + n_3^2$$
$$n_1(P - n_1) + n_2(Q - n_2) + n_3(R - n_3) = 0$$

$\forall\, p \in S.$

$$dx \wedge \omega = dx \wedge P\, dy \wedge dz = \frac{\partial P}{\partial x}\, dx \wedge dy \wedge dz = 0$$

$$d\omega = \frac{\partial P}{\partial x} dx \wedge dy \wedge dz + \frac{\partial Q}{\partial y} dy \wedge dx \wedge dz + \frac{\partial R}{\partial z} dz \wedge dx \wedge dy$$

Spring 2012 $\boxed{\#4}$

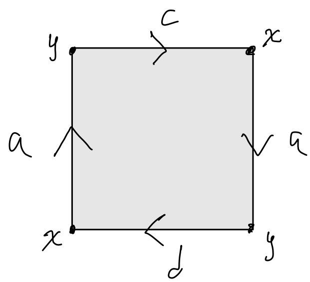
$X = M_1 \cup M_2$ M_1 and M_2 mobius bands and

$M_1 \cap M_2 = \partial M_1 = \partial M_2$. Mobius band $= ([-1,1]\times[-1,1])/\sim$

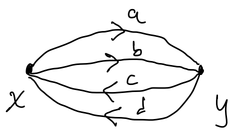
$(1,y) \sim (-1,-y)$.

(a) compute $\pi_1(X)$ (b) no b/c $\simeq$ Klein bottle

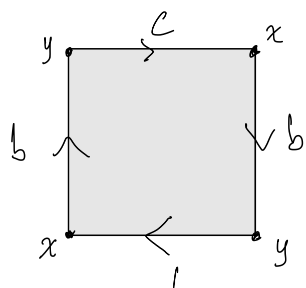
(b) $X \simeq$ compact orientable surface of genus g for some g?



As a CW complex, X has two 0-cells, x, y. It has four 1-cells a, b, c, d. So before attaching 2-cells it looks like



which is homotopy equivalent to $S^1 \vee S^1 \vee S^1$ and has fundamental gp $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \langle ab^{-1}, ac, ad \rangle$



We attach two 2-cells along the paths $acad$ and $bcbd = ba^{-1}acba^{-1}ad$

$$= (ab^{-1})^{-1}ac(ab^{-1})^{-1}ad$$

Then $\pi_1(X) = \langle ab^{-1}, ac, ad \mid acad = (ab^{-1})^{-1}ac(ab^{-1})^{-1}ad = 1 \rangle$

$$= \langle u, v, w \mid vw = u^{-1}vu^{-1}w = 1 \rangle$$

$$= \langle u, v \mid u^{-1}vu^{-1}v^{-1} = 1 \rangle = \boxed{\langle u, v \mid v = uvu \rangle}$$

Spring 2012 $\boxed{\#5}$

Determine connected covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$

$\pi_1(\mathbb{RP}^{14} \vee \mathbb{RP}^{15}) = \pi_1(\mathbb{RP}^{14}) * \pi_1(\mathbb{RP}^{15})$.

For $n \geq 2$, S^n is a 2-sheeted universal cover of $\mathbb{RP}^n$ (since S^n simply connected). Thus, it's induced (trivial) image in $\pi_1(\mathbb{RP}^n)$ has index 2 $\Rightarrow \pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$

So $\pi_1(\mathbb{RP}^{14} \vee \mathbb{RP}^{15}) = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1 \rangle$

The connected covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$ thus correspond to the subgroups of $\mathbb{Z}_2 * \mathbb{Z}_2$.

$a,\ b,\ ab,\ ba,\ aba,\ abab,$

$$\left\{ 1 \begin{matrix} b, & ba, & bab, & baba, & \dots \\ a, & ab, & aba, & abab, & \dots \end{matrix} \right\}$$

$\langle ab \cdots ba \rangle = \{1, ab\cdots ba\}$ $\mathbb{Z}_2$

$\langle ba \cdots ab \rangle =$

$\mathbb{Z}$

$\vdots$
ababa
aba
a
$\cdots$ baba ba 1 ab abab $\cdots$
b
bab
babab
$\vdots$

$\langle ab \rangle = \left\{ \begin{matrix} 1, ab, abab, ababab, \ldots \\ ba, baba, baba, \ldots \end{matrix} \right\}$

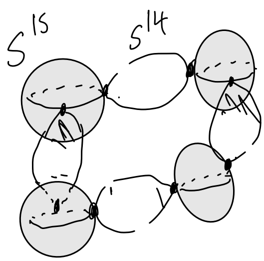
which has index 2

indeed a or b multiplied by $\langle ab \rangle$ gives you the rest of the group.

$\langle \underbrace{ab \cdots ab}_{n(ab)'s} \rangle = \left\{ \begin{matrix} 1, n(ab), 2n(ab), 3n(ab), \ldots \\ n(ba), 2n(ba), 3n(ba), \ldots \end{matrix} \right\}$

which has index $2n$

The "rings" of alternating S^{14}, S^{15} are the connected covers:



even-sheeted covers.

Spring 2012 $\boxed{\#6}$

$f : M \to N$ smooth, X, Y vect. fields on M, N.

$f_* X = Y$, i.e. $f_*(X(x)) = Y(f(x))$ $\forall x \in M$.

Claim: $f^* \mathcal{L}_Y \omega = \mathcal{L}_X (f^* \omega)$, $\omega \in \Omega^1(N)$.

$$\mathcal{L}_Y \omega = d(Y \lrcorner \omega) + Y \lrcorner d\omega$$

$$\begin{aligned}
(f^* \mathcal{L}_Y \omega)_x (v) &= (f^* d(Y \lrcorner \omega))_x (v) + (f^*(Y \lrcorner d\omega))_x (v) \\
&= d(Y \lrcorner \omega)_{f(x)} (df_x(v)) + (Y \lrcorner d\omega)_{f(x)} (df_x(v)) \\
&= \frac{d}{dt}\Big|_{t=0} \left((Y \lrcorner \omega)((f \circ \gamma)(t)) \right) + d\omega_{f(x)} \left(Y(f(x)), df_x(v) \right) \\
&= \frac{d}{dt}\Big|_{t=0} \left(\omega (Y((f \circ \gamma)(t))) \right) + d\omega_{f(x)} \left(f_*(X(x)), df_x(v) \right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_X (f^* \omega)_x (v) &= (d(X \lrcorner f^* \omega))_x (v) + (X \lrcorner d f^* \omega)_x (v) \\
&= \frac{d}{dt}\Big|_{t=0} \left((X \lrcorner f^* \omega)(\gamma(t)) \right) + d f^* \omega_x (X(x), v) \\
&= \frac{d}{dt}\Big|_{t=0} \left(f^* \omega (X(\gamma(t))) \right) + f^* d\omega_x (X(x), v) \\
&= \frac{d}{dt}\Big|_{t=0} \left(\omega (f_* X(\gamma(t))) \right) + d\omega_{f(x)} \left(f_*(X(x)), df_x(v) \right) \\
&= \frac{d}{dt}\Big|_{t=0} \left(\omega (X(f \circ \gamma(t))) \right) + d\omega_{f(x)} \left(f_*(X(x)), df_x(v) \right)
\end{aligned}$$

where $\gamma : [-1,1] \to M$, $\gamma(0) = x$, $\gamma'(0) = v$

$\Rightarrow$ $f \circ \gamma : [-1,1] \to N$, $(f \circ \gamma)(0) = f(x)$, $(f \circ \gamma)'(0) = df_x v$

Spring 2012 $\boxed{\#7}$

X, Y vect. fields on $\mathbb{R}^4 - \{0\}$

$$X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}$$

$$Y = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}.$$

Is the rank 2 distribution orthogonal to these two vect. fields integrable?

See Chapter 19 Lee Smooth mfds.

$$[X, Y]f = XYf - YXf$$

$$XYf = X\left(-x_2 \frac{\partial f}{\partial x_1} + x_1 \frac{\partial f}{\partial x_2} - x_4 \frac{\partial f}{\partial x_3} + x_3 \frac{\partial f}{\partial x_4}\right)$$

$$= -x_1 x_2 f_{11} + x_1 f_2 + x_1^2 f_{12} - x_1 x_4 f_{13} + x_1 x_3 f_{14}$$

$$- x_2 f_1 - x_2^2 f_{12} + x_1 x_2 f_{22} - x_2 x_4 f_{23} + x_2 x_3 f_{24} \ldots$$

$$Yf = Y\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}\right)$$

$$= -x_2 f_1 - x_1 x_2 f_{11} - x_2 x_2 f_{12} - x_2 x_3$$

	$-x_2 \frac{\partial f}{\partial x_1}$	$x_1 \frac{\partial f}{\partial x_2}$	$-x_4 \frac{\partial f}{\partial x_3}$	$x_3 \frac{\partial f}{\partial x_4}$
$x_1 \frac{\partial}{\partial x_1}$	1	B $+x_1 f_2$	2	3
$x_2 \frac{\partial}{\partial x_2}$	A $-x_2 f_1$	4	5	6
$x_3 \frac{\partial}{\partial x_3}$	7	8	9	D $+x_3 f_4$
$x_4 \frac{\partial}{\partial x_4}$	10	11	C $-x_4 f_3$	12

	$x_1 \frac{\partial f}{\partial x_1}$	$x_2 \frac{\partial f}{\partial x_2}$	$x_3 \frac{\partial f}{\partial x_3}$	$x_4 \frac{\partial f}{\partial x_4}$
$-x_2 \frac{\partial}{\partial x_1}$	1 $-x_2 f_1$	B	7	10
$x_1 \frac{\partial}{\partial x_2}$	A	4 $+x_1 f_2$	8	11
$-x_4 \frac{\partial}{\partial x_3}$	2	5	9 $-x_4 f_3$	D
$x_3 \frac{\partial}{\partial x_4}$	3	6	C	12 $+x_3 f_4$

Hence all the terms are the same and

$[X, Y] = 0.$

Fall 2012 $\boxed{\#1}$

$S^1 \subset \mathbb{R}^2$, $T^2 = S^1 \times S^1$. $A \subset T^2$

$A = \{(x,y,z,w) \in T^2 : (x,y) = (0,1) \text{ or } (z,w) = (0,1)\}$.

Compute $H^*(T^2, A)$.

Fall 2012 $\boxed{\#2}$

$X \wedge Y = X \times Y/\sim \quad (x, y_0) \sim (x_0, y)$

Claim: $H_n(S^1 \times S^1) \approx H_n(S^1 \wedge S^1 \wedge S^2)$

but U.C.'s don't have $\approx$ Homology.

$S^1 \wedge S^1 = S^1 \times S^1/\sim \quad (x, y_0) \sim (x_0, y)$

$$H_n(S^1 \times S^1) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases}$$

$(S^1 \wedge S^1) \wedge S^2 = (S^1 \wedge S^1) \times S^2/\sim \quad ((x_0, y_0), z) \sim ((x, y), z_0)$

$= S^1 \times S^1 \times S^2/\sim \quad (x_0, y, z) \sim (x, y_0, z) \sim (x, y, z_0)$

$= S^1 \times S^1 \times S^2 / \left\{ \{x_0\} \times S^1 \times S^2 \cup S^1 \times \{y_0\} \times S^2 \cup S^1 \times S^1 \times \{z_0\} \right\} =: A$

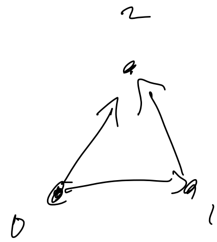
good pair? ✓ $\quad H_n(S^1 \wedge S^1 \wedge S^2) \approx H_n(S^1 \times S^1 \times S^2, A)$

$S^1 \times S^1 \times S^2 = S(S S^2)$ where SX is suspension X.

Fall 2012 $\boxed{\#3}$

1-skel.

2 skel



(a) Compute $H_n(X)$

(b) $\pi_1(X)$ Claim: nonabelian.

(a) $H_n(X) = 0 \quad \forall\ n \geq 3$

$$C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\times 0} C_0(X) \xrightarrow{\times 0} 0$$

$\partial(e_1) = a - a = 0$

$\partial(e_2) = 3b$

$\partial(e_3) = 2a + 2b$

$$\frac{\langle a, b\rangle}{\langle 2a+2b, 3b\rangle} = \mathbb{Z}$$

(b) $\pi_1(X) = \langle a, b \mid aa^{-1} = b^3 = abab = 1 \rangle$

$= \{0, a, b, b^2, ab, aba, ab^2$

$ba \neq ab$?

If abelian $\Rightarrow$ $abab = aabb = b^2 = 1 \Rightarrow\!\Leftarrow$

Fall 2012 $\boxed{\#4}$

? $\exists$ smooth embedding of $\mathbb{RP}^2$ into $\mathbb{R}^2$?

Assume so. $f: \mathbb{RP}^2 \longrightarrow \mathbb{R}^2$ smooth embedding

Then $df_p : T_p\mathbb{RP}^2 \longrightarrow T_{f(p)}\mathbb{R}^2 \approx \mathbb{R}^2$ injective

and by dimension surjective.

$f: \mathbb{RP}^2 \longrightarrow f(\mathbb{RP}^2) \subset \mathbb{R}^2$ bijective, so

$f^{-1}: \underset{\mathbb{R}^2}{\underset{\cap}{f(\mathbb{RP}^2)}} \longrightarrow \mathbb{RP}^2$ well defined.

$$\begin{tikzcd}
 & S^2 \arrow[d, "p"] \\
f(\mathbb{RP}^2) \arrow[r, "f^{-1}"] \arrow[ur, dashed, "\tilde{f}^{-1}"] & \mathbb{RP}^2
\end{tikzcd}$$

$$S^2 \xrightarrow{p} \mathbb{RP}^2 \xrightarrow{f} \mathbb{R}^2 \overset{\cong}{\xrightarrow{\phi^{-1}}} S^2$$

degree 2

Fall 2012 $\boxed{\#5}$

M mfd, $C^\infty(M)$ = algebra of C^∞ functions $M \to \mathbb{R}$. Explain relationship b/w vect. fields on M and $C^\infty(M)$. Consider v.f. X, Y on M as maps $C^\infty(M) \to C^\infty(M)$. Is XY a v.f.? How about $[X,Y] = XY - YX$?

A derivation $v_p : C^\infty(M) \to \mathbb{R}$ satisfies

$v_p(fg) = v(f)g(p) + f(p)v(g)$, linear

Is $(XY)_p$ a derivation $\forall\, p \in M$?

$$(XY)_p(fg) = X_p(Y(fg)) = X_p\big(Y(f)g + fY(g)\big)$$

$$= (XY)_p(f)g(p) + Y(f)(p)X(g)(p)$$

$$+ X(f)(p)Y(g)(p) + f(p)(XY)_p(g)$$

So if $Y(f)(p)X(g)(p) + X(f)(p)Y(g)(p) \neq 0$ for any $p \in M$, then XY is not a vector field.

$$(XY)_p(fg) - (YX)_p(fg) = XYfg + \cancel{YfXg} + \cancel{XfYg}$$

$$+ fXYg$$

$$- YXfg - \cancel{XfYg} - \cancel{YfXg}$$

$$- fYXg$$

$$= [X,Y](f)\, g + f[X,Y](g) \quad \checkmark$$

Fall 2012 $\boxed{\#6}$

$S =$ unit sphere in $\mathbb{R}^4$ $\quad x^2+y^2+z^2+w^2=1.$

Compute $\int_S \omega$, $\quad \omega = (w+w^2)\, dx\wedge dy\wedge dz.$

$d\omega = (1+2w)\, dw\wedge dx\wedge dy\wedge dz$. By Stokes

$$\int_S \omega = \int_{B^4} d\omega = \int_{B^4} (1+2w)\, dw\,dx\,dy\,dz$$

$$= -\mathrm{Vol}(B^4) + \int_{B^4} 2w\, dw\,dx\,dy\,dz$$

$$= -\mathrm{Vol}(B^4) + \int_{B^3}\left(\int_{-\sqrt{1-z^2-y^2-x^2}}^{\sqrt{1-z^2-y^2-x^2}} 2w\, dw\right) dx\,dy\,dz$$

$$= -\mathrm{Vol}(B^4) + \int_{B^3} 0\, dx\,dy\,dz$$

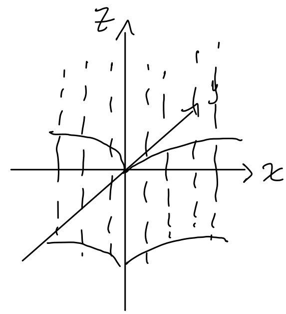
$$= \boxed{-\mathrm{Vol}(B^4)}$$

where $B^4 = \{(w,x,y,z)\in\mathbb{R}^4 : x^2+y^2+z^2+w^2 \le 1\}$

Fall 2012 $\boxed{\#7}$

$\{x^2 = y^3\} \subset \mathbb{R}^3$ submfd? Call it M.

Assume so. Then



$i : M \hookrightarrow \mathbb{R}^3$ is an immersion.

$(x,y,z) \mapsto (x,y,z)$

Consider the chart $\phi : M \to \mathbb{R}^2$

$(x,y,z) \mapsto (x,z)$

Then $\phi^{-1} : \mathbb{R}^2 \to M$

$(x,z) \mapsto (x, x^{2/3}, z)$

So $i \circ \phi^{-1} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ call it f

$(x,z) \longmapsto (x, x^{2/3}, z)$

Then $df_p = \begin{pmatrix} 1 & 0 \\ \frac{2}{3}x^{-1/3} & 0 \\ 0 & 1 \end{pmatrix}$ is undefined when $x=0$.

a contradiction.

Spring 2013 $\boxed{\#1}$

$$w = (x^2+x+y)\,dy\wedge dz \qquad \text{in } \mathbb{R}^3$$

(a) Calculate $\int_{S^2} w$

(b) closed form α on $\mathbb{R}^3$ s.t. $i^*\alpha = i^*w$?

(a) $dw = (2x+1)\,dx\wedge dy\wedge dz$. Stokes gives

$$\int_{S^2} w = \int_{B^3} dw = \int_{B^3} 2x\,dxdydz + \mathrm{vol}(B^3)$$

$$= \boxed{\mathrm{vol}(B^3)}, \text{ since } B^3 \text{ is symetric across the } x\text{-axis}$$

(b) Assume $\exists\ \alpha$ s.t. $d\alpha = 0$ and $i^*\alpha = i^*w$.

Then $\int_{S^2} i^*\alpha = \int_{S^2} i^*w = \int_{S^2} w \neq 0$

But $\int_{S^2} i^*\alpha = \int_{B^3} d(i^*\alpha) = \int_{B^3} i^*(d\alpha) = 0$

$\Rightarrow\Leftarrow$.

Spring 2013 $\boxed{\#2}$

Find all points in $\mathbb{R}^3$ s.t. $\exists$ nbd of the point s.t. $(x, x^2+y^2+z^2-1, z)$ can serve as a local coordinate system.

Consider points of the form $(x,0,z) \in \mathbb{R}^3$.

Let U be a nbd of $(x_0, 0, z_0) =: P$. U contains a ball centered at $(x_0, 0, z_0)$, call it $B_\varepsilon(P)$, radius $\varepsilon > 0$.

Then $(x_0, -\frac{\varepsilon}{2}, z_0)$ and $(x_0, \frac{\varepsilon}{2}, z_0) \in B_\varepsilon(P)$.

But both are mapped to $(x_0, x_0^2 + \frac{\varepsilon^2}{4} + z_0^2 - 1, z_0)$

So this cannot serve as a local coordinate system.

For all other points $(x_0, y_0, z_0) \in \mathbb{R}^3$, $y_0 \neq 0$, we can find small ball s.t. y's are all the same sign inside the ball

$\phi: U \to \mathbb{R}^3$, $\phi(x,y,z) = (x, x^2+y^2+z^2-1, z)$

$\phi^{-1}: \phi(U) \to U$, $\phi^{-1}(u,v,w) = (u, \sqrt{v+1-u^2-w^2}, w)$

ϕ injective? if $x_0^2+y_0^2+z_0^2-1 = x_1^2+y_1^2+z_1^2-1$ and $x_0=x_1$, $z_0=z_1$ then $y_0^2=y_1^2$ and if $\mathrm{sgn}(y_0)=\mathrm{sgn}(y_1)$ then $y_0=y_1$, so yes.

Thus ϕ is a homeomorphism onto $\phi(U)$ and we are done.

Spring 2013 $\boxed{\#3}$

Prove $\mathbb{RP}^n$ smooth mfd of dim n.

Let $q : S^n \longrightarrow \mathbb{RP}^n$ be the quotient map $q(x) = [x]$. Then let $U_i = q(V_i^{+/-})$ where $V_i^{+/-} = \{(x_1, \dots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1} : x_i >/< 0\}$. Note $q(V_i^-) = q(V_i^+)$

$\{U_i\}_1^n$ cover $\mathbb{RP}^n$. Let $\phi_i : U_i \longrightarrow \mathbb{R}^n$

$$\phi_i([x]) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right)$$

well defined b/c $\left(\frac{-x_1}{-x_i}, \dots, \frac{-x_{i-1}}{-x_i}, \frac{-x_{i+1}}{-x_i}, \dots, \frac{-x_{n+1}}{-x_i}\right) =$

and continuous, and inverse

$$\phi_i^{-1}(u_1, \dots, u_n) = [u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n]$$

which is also continuous. So homeo with $\mathbb{R}^n$

Hence $\mathbb{RP}^n$ is a smooth mfd of dim n.

Spring 2013 $\boxed{\#4}$

(a) Every closed 1-form on S^n, $n>1$ exact

(b) (a) $\Rightarrow$ every closed 1-form on $\mathbb{RP}^n$, $n>1$ exact.

(a) $H^1_{dR}(S^n) \approx H_1(S^n) = 0 \quad \forall\, n>1$.

$\Rightarrow$ closed 1-forms are exact.

(b) Let $f: \mathbb{RP}^n \longrightarrow S^n$ be a section of the quotient map $q: S^n \longrightarrow \mathbb{RP}^n$

Then $Id = q \circ f : \mathbb{RP}^n \xrightarrow{f} S^n \xrightarrow{q} \mathbb{RP}^n$ induces

$$f_* \circ q_* : H^1_{dR}(\mathbb{RP}^n) \xrightarrow{q_*} \underset{0}{H^1_{dR}(S^n)} \xrightarrow{f_*} H^1_{dR}(\mathbb{RP}^n)$$

the identity. Then f_* must be surjective and $H^1_{dR}(\mathbb{RP}^n) = 0$.

i.e. closed 1-forms exact.

Spring 2013 $\boxed{\#5}$

$X = \mathbb{R}^3 - \text{axes}$, Compute $\pi_1(X)$, $H_*(X)$.

$X \simeq B^3 - \text{axes} \simeq S^2 - \text{axes} = S^2 - \{6 \text{ points}\}$

$\simeq D^2 - \{5 \text{ points}\}$

$\simeq (S^1)^{\vee 5}$

$\Rightarrow \boxed{\pi_1(X) = F_5}$, free group on 5 elts.

$$\boxed{H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^{\oplus 5} & n = 1 \\ 0 & \text{else} \end{cases}}$$

$r: X \longrightarrow S^2 - \text{axes}$ deformation retract.

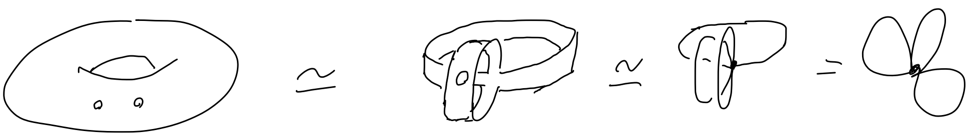
$r(x) = \dfrac{x}{|x|}$

Spring 2013 $\boxed{\#6}$

$$X = T^2 - \{p, q\}, \quad p \neq q$$

Compute $H_*(X; \mathbb{Z})$, $\pi_1(X)$.

$$T^2 - \{p,q\} \simeq S^1 \vee S^1 \vee S^1$$



So
$$\boxed{\begin{aligned} &\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = F_3 \\ &H_n(X) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases} \end{aligned}}$$

Spring 2013 $\boxed{\#7}$

(a) Find all 2-sheeted covering spaces of $S^1 \times S^1$.

(b) X path connected, locally path connected, $\pi_1(X)$ finite, $f : X \to S^1 \Rightarrow f$ null homotopic.

~~(a) Let $\tilde{T}$ be a 2-sheeted covering space of $S^1 \times S^1$.~~

~~$p : \tilde{T} \to T := S^1 \times S^1$.~~

~~$p_* : \pi_1(\tilde{T}) \to \pi_1(T) = \mathbb{Z} \times \mathbb{Z}$~~

~~$p_*(\pi_1(\tilde{T}))$ has index 2 in $\mathbb{Z} \times \mathbb{Z}$~~

$$\mathrm{Hom}\left(\pi_1(S^1 \times S^1), \Sigma_2\right) = \{\text{2-sheeted covering spaces}\}$$

$\Sigma_2 = \{(), (12)\} \approx \mathbb{Z}_2$.

Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2$ be a homomorphism.

f determined by $f((0,1))$ and $f((1,0))$.

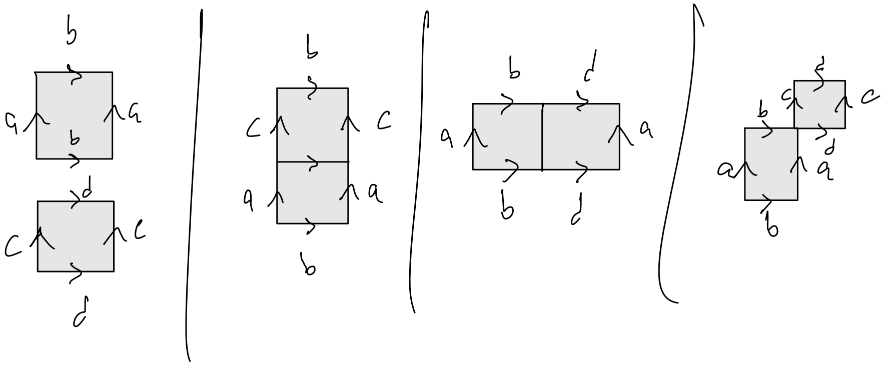
- $f((0,1)) = 0$ and $f((1,0)) = 0$

$\cdot \qquad = 1 \qquad\qquad = 0$

$\cdot \qquad = 0 \qquad\qquad = 1$

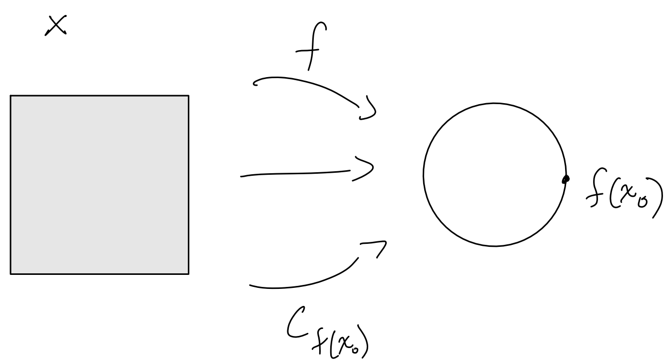
$\cdot \qquad = 1 \qquad\qquad = 1$

Each of these four are different 2-sheeted coverings, corresponding to



(b) X path connected, locally path connected, $\pi_1(X)$ finite, $f: X \to S^1 \Rightarrow f$ null homotopic.

$\overset{?}{\exists} H: \;\; I$



$H(x,t) =$

$$f_* : \pi_1(X) \longrightarrow \pi_1(S^1) \text{ hom}$$

$\Rightarrow$ $f_* = 0$ - map b/c any element in $\pi_1(X)$ has finite order and the only elt with finite order in $\pi_1(S^1) = \mathbb{Z}$ is 0.

$\Rightarrow$ f lifts to some $\tilde{f} : X \longrightarrow \mathbb{R}$ since $f_*(\pi_1(X)) = 0 \subset P_*(\pi_1(\mathbb{R})) = 0$ for universal covering map $p : \mathbb{R} \to S^1$.

Then $\tilde{f} \simeq C_0$ constant map by

$H(x,t) = \tilde{f}(x)(1-t)$. $\Rightarrow p \circ \tilde{f} \simeq C_{p(0)}$

Then $f = p \circ \tilde{f} \simeq C_{p(0)}$.

I.e. f nullhomotopic.

Spring 2013 $\boxed{\#8}$

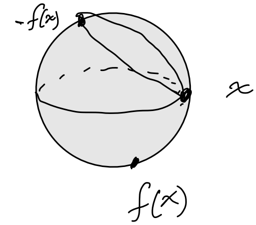
(a) $f: S^n \to S^n$ no fixed point $\Rightarrow \deg(f) = (-1)^{n+1}$.

(b) X has S^{2n} as a universal covering space

$\Rightarrow \pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

(a) $f: S^n \to S^n$ no fixed point.

Then $\exists$ straight line path γ_x from x to $-f(x)$ that doesn't pass through the origin. Then $\frac{\gamma_x}{|\gamma_x|}$ is a path from x to $-f(x)$ on S^n.



This defines a homotopy

$H: S^n \times I \to S^n$

$H(x,t) = \frac{\gamma_x(t)}{|\gamma_x(t)|}$ between Id

and $\alpha \circ f$, w/ α the antipodal map

Hence $\deg(\alpha)\deg(f) = \deg(\alpha \circ f) = \deg(Id) = 1$.

We know $\deg(\alpha) = (-1)^{n+1}$, so

$\deg(f)$ also $= (-1)^{n+1}$.

(b) X has S^{2n} as a universal covering space

$\Rightarrow$ $\pi_1(X)$ must be $\{1\}$ or $\mathbb{Z}_2$?

Since S^{2n} the universal cover of X,

$\pi_1(X) = G(S^{2n})$ the group of deck transformations on S^{2n}. This group acts freely on S^{2n}. That means

$\forall f \in G(S^{2n})$ nontrivial, $f: S^{2n} \to S^{2n}$ has no fixed points. $\overset{(a)}{\Rightarrow}$ $\deg(f) = (-1)^{2n+1} = -1$ $\forall f \in G(S^{2n})$ non trivial. Thus we define a map $d: G(S^{2n}) \longrightarrow \{\pm 1\}$ sending f to $\deg(f)$. d is a homomorphism because $\deg(fg) = \deg(f)\deg(g)$. And $\ker(d)$ $=$ trivial elt in $G(S^{2n})$ by what we said above. Thus d injective homomorphism

$\Rightarrow$ $\pi_1(X) = G(S^{2n}) \subset \mathbb{Z}_2$.

Fall 2013 $\boxed{\#1}$

$X = S^2/\{N,S\}$, N,S north, south poles.

(a) Cell decomp of X, compute $H_i(X)$ $\forall i \geq 0$.

(b) $\pi_1(X)$

(c) draw pic of universal cover of X & all other connected covering spaces.

(a) Let e_0 be $N=S$, 0-cell.

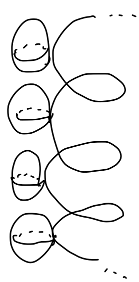
Let e_1 be a meridional line from S to N, 1-cell

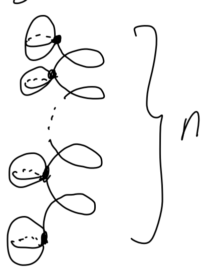
Let e_2 be a 2-cell glued along e_1 then back down e_1 to obtain $S^2/\{N,S\}$

$H_2(X) = H_2(X_2, X_1)$

(b) $S^2/\{N,S\} \simeq S^2 \vee S^1 \Rightarrow \pi_1(X) = \mathbb{Z}$.

(c)





Fall 2013 $\boxed{\#2}$

M compact N connected, $f: M \to N$ submersion. Claim: f surjective.

$N = f(M) \sqcup (N - f(M))$. Suppose $N - f(M) \neq \emptyset$.

M compact $\Rightarrow$ $f(M)$ compact $\Rightarrow$ $N - f(M)$ open.

f submersion $\Rightarrow$ $\dim M \geq \dim N$.

Fall 2013 $\boxed{\#3}$

$O(n) = \{A \in M_n(\mathbb{R}) \mid AA^T = id\}$ smooth mfd.

Let $f : M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$A \longmapsto AA^T$

Then $O(n) = f^{-1}(id)$.

If id is a regular value of f

then $O(n)$ is an embedded smooth mfd.

$df_A(B) = (f \circ \gamma)'(0)$ where

$\gamma(t) = A + tB$

$(f\circ\gamma)(t) = (A+tB)(A+tB)^T = (A+tB)(A^T+tB^T)$

$= AA^T + tBA^T + tAB^T + t^2 BB^T$

$(f\circ\gamma)'(t) = BA^T + AB^T + 2tBB^T$

$(f\circ\gamma)'(0) = BA^T + AB^T$

For any $C \in M_n(\mathbb{R})$, let $B = \frac{1}{2}CA$

Then $df_A(B) = \frac{1}{2}CAA^T + \frac{1}{2}AA^TC = C \Rightarrow df_A$ surjective $\forall A \in O(n)$.

Fall 2013 $\boxed{\#4}$

Compute de Rham cohomology of $S^1 = \mathbb{R}/\mathbb{Z}$ from the definition.

$H^n_{dR}(X) = $ closed n-forms / exact n-forms

All 1-forms on S^1 are closed because 2-forms are zero on 1-dim'l S^1.

Let ω be a closed 1-form on S^1.

Assume $\omega = d\alpha$ exact as well.

Then $\int_{S^1} \omega = \int_{S^1} d\alpha = \int_{\partial S^1} \alpha = \int_{\emptyset} \alpha = 0$

Define a map

$$I : H^1_{dR}(X) \longrightarrow \mathbb{R}$$
$$[\omega] \longmapsto \int_{S^1} \omega$$

Then $\ker(I) = 0 \Rightarrow I$ injective

I surjective b/c $\int_{S^1} \lambda\omega = \lambda \int_{S^1} \omega$.

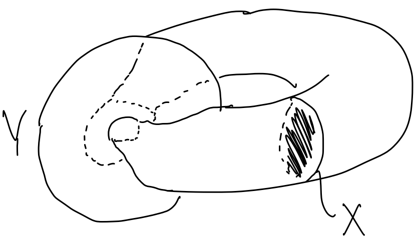
Fall 2013 $\boxed{\#5}$

$f, g : X \to Y$ continuous. $Z = (X \times [0,1]) \sqcup Y / \sim$

$(x,0) \sim f(x)$ and $(x,1) \sim g(x)$ $\forall\, x \in X$.

Claim: $\exists$ LES

$$\cdots \to H_n(X) \to H_n(Y) \to H_n(Z) \to H_{n-1}(X) \to \cdots$$



$((X \times \{0\} \cup \{1\}) \sqcup Y,\ Z)$

good pair

Fall 2013 $\boxed{\#6}$

$L(p,q) :=$ quotient of $S^3 \subset \mathbb{C}^2$ by $\mathbb{Z}/p$ action generated by $(z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$, p,q coprime.

(a) Compute $\pi_1(L(p,q))$

(b) any continuous map $L(p,q) \to T^2$ null homotopic

Let $f: S^3 \longrightarrow L(p,q)$ be the quotient map.

Let $z = (z_1, z_2) \in S^3$. Let $U = \{y \in S^3 : |y - z| < \varepsilon\}$ for ε small enough so that $\{gU : g \in \mathbb{Z}/p\}$ are distinct.

Then $f(U) = f(gU)$ $\forall g \in \mathbb{Z}/p$. And $gU \cong f(U)$ $\forall g \in \mathbb{Z}/p$ because of the subspace topology.

Thus, we have a universal p-sheeted covering space $f: S^3 \longrightarrow L(p,q)$. Each element $g \in \mathbb{Z}/p$ is a deck transformation since it is just a rotation.

Conversely, let h be a deck transformation, $z \in S^3$.

$\exists$ $g \in \mathbb{Z}/p$ s.t. $g(z) = h(z)$ and deck transforming of S^3 are determined by where a single point is sent b/c S^3 path connected $\Rightarrow$ $g = h$

So $\mathbb{Z}/p = G(S^3)$ group of deck transformations.

Define $\varphi : \pi_1(L(p,q)) \longrightarrow G(S^3) = \mathbb{Z}/p$ by sending
$[\gamma] \longmapsto g$ where

γ is a loop lifting to a path in S^3 from z_0 to z_1 and g is the deck transformation sending z_0 to z_1.

φ is a surjective homomorphism with kernel the loops lifting to loops in S^3, i.e. elts of $f_*(\pi_1(S_3)) =$ trivial subgp.

Hence φ is an iso.

$\pi_1(L(p,q)) = \mathbb{Z}/p$

(b)

2014 Spring $\boxed{\#1}$

2014 Spring $\boxed{\#2}$

2014 spring $\boxed{\#3}$

2014 Spring $\boxed{\#4}$

Consider the following vect. Fields in $\mathbb{R}^2$:

?

$$X = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial y}.$$

Determine whether or not $\exists$ (locally defined) coordinate system (s,t) in a nbd of $(x,y)=(0,1)$ s.t.

$$X = \frac{\partial}{\partial s}, \quad Y = \frac{\partial}{\partial t}$$

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\phi(x,y) = (s(x,y), t(x,y))$

Then $\frac{\partial}{\partial s} = d\phi_p\left(\frac{\partial}{\partial x}\right)$ and $\frac{\partial}{\partial t} = d\phi_p\left(\frac{\partial}{\partial y}\right)$

$$= \frac{\partial s}{\partial x}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x}\frac{\partial}{\partial y} \qquad = \frac{\partial s}{\partial y}\frac{\partial}{\partial x} + \frac{\partial t}{\partial y}\frac{\partial}{\partial y}$$

Then we want

$$\frac{\partial s}{\partial x} = 2, \quad \frac{\partial t}{\partial x} = x, \quad \frac{\partial s}{\partial y} = 0, \quad \frac{\partial t}{\partial y} = 1$$

This is satisfied by $s = 2x$, $t = \frac{1}{2}x^2 + y$ e.g.

Then at $(0,1)$, $d\phi_{(0,1)} = \begin{pmatrix} 2 & 0 \\ x & 1 \end{pmatrix}\Big|_{(0,1)} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ invertible.

$\Rightarrow$ by inverse function thm. $\exists$ nbd of $(0,1)$ s.t. ϕ is a diffeomorphism.

2014 Spring $\boxed{\#5}$

M diff. mfd. $T^*M = \{(x,u) ; x \in M, u : T_xM \to \mathbb{R} \text{ linear}\}$

Claim: T^*M orientable mfd.

Let (V, ϕ) be a chart in M, $\phi : V \xrightarrow{\cong} \phi(V) \subset \mathbb{R}^n$.

Let $\tilde{V} = \{(x,u) \in T^*M : x \in V\}$ and $\tilde{\phi} : \tilde{V} \to \mathbb{R}^n \times \mathbb{R}^n$

be defined by $\tilde{\phi}(x, u) = \left(\phi(x), u\left(\frac{\partial}{\partial x^1}\right), \ldots, u\left(\frac{\partial}{\partial x^n}\right)\right)$.

Which is a homeomorphism since ϕ is a homeomorphism and u is linear. Thus we see that T^*M is a mfd.

An orientation form for T^*M would be a $2n$-form w, nonvanishing.

Define $w_{(x,u)} : \left(T_{(x,u)}(T^*M)\right)^{2n} \longrightarrow \mathbb{R}$

2014 Spring $\boxed{\#6}$

2014 Spring $\boxed{\#7}$

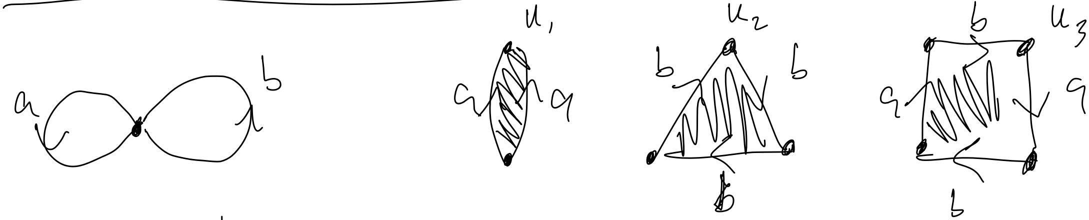
M compact m-dim'l submfd of $\mathbb{R}^m \times \mathbb{R}^n$.
Let M_x for $x \in \mathbb{R}^m$ be the translation of M by x in $\mathbb{R}^m \times \mathbb{R}^n$. Then the space of points $x \in \mathbb{R}^m$ s.t. $M_x \cap \mathbb{R}^n$ infinite has measure 0 in $\mathbb{R}^m$.

Define $f: M \subset \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$
$$(x,y) \longmapsto x$$
Then f sends a point in M to the $x \in \mathbb{R}^m$ s.t. $M_x \cap \mathbb{R}^n$ contains that point. Assume f sends an infinite amount of points in M to some x_0. (i.e. $M_{x_0} \cap \mathbb{R}^n$ infinite). We claim x_0 is a critical value. If not, then by regular level set Thm, $f^{-1}(x_0)$ is a submfd of M of dimension $m-m=0$. I.e. $f^{-1}(x_0)$ is a collection of distinct points in M. Since f continuous and $\{x_0\}$ closed then $f^{-1}(x_0)$ closed and therefore compact b/c M compact. By the subspace topology the points in $f^{-1}(x_0)$ are open, so there can only be finitely many, otherwise $\{\{y\} : y \in f^{-1}(x_0)\}$ is an infinite open cover with no finite subcover. Hence x_0 critical value and by Sard's they have measure 0 in $\mathbb{R}^m$.

Fall 2012 $\boxed{\#7}$

Does $x^2 = y^3$ define a smooth submfd in $\mathbb{R}^3$?

Impl. Func. Thm.



1-skel

2-skel

$$0 \longrightarrow \underset{\underset{\mathbb{Z}^3 \langle u_i \rangle}{\text{SS}}}{C_2} \xrightarrow{\partial_2} \underset{\underset{\mathbb{Z}^2 \langle a,b \rangle}{\text{SS}}}{C_1} \xrightarrow{\partial_1} \underset{\underset{\mathbb{Z}}{\text{SS}}}{C_0} \longrightarrow 0$$

$\partial_2(u_1) = a - a = 0$ $\qquad$ $\partial_1(a) = v - v = 0$

$\partial_2(u_2) = b+b+b = 3b$ $\qquad$ $\partial_1(b) = v - v = 0$

$\partial_2(u_3) = a+b+a+b = 2a+2b$

$\ker(\partial_2) = \langle u_1 \rangle \approx \mathbb{Z}$ $\qquad$ $\ker(\partial_1) = C_1$

$\text{im}(\partial_2) = \langle 3b, 2a+2b \rangle$

$$H_1(X) = \ker(\partial_1)/\operatorname{im}(\partial_2) = \langle a,b \mid aba^{-1}b^{-1} = b^3 = a^2b^2 = 1\rangle$$

$$= \langle a,b \mid 3b = 2(a+b) = 0\rangle$$

$$\overset{\text{abeln}}{=} \langle c,d \mid 3d = 2c = 0\rangle$$

$$= \mathbb{Z}_2 \oplus \mathbb{Z}_3$$

$$H_2(X) = \mathbb{Z}$$

✓

$H_0(X) = \mathbb{Z}$ b/c path connected.

$$\pi_1(X) = \mathbb{Z} * \mathbb{Z} / \langle aa^{-1},\ b^3,\ abab\rangle$$

$$= \langle a,b \mid b^3 = abab = 1\rangle$$

$$= \langle ab, b \mid b^3 = (ab)^2 = 1\rangle$$

$$\cong \langle c,d \mid c^2 = d^3 = 1\rangle$$

$$= \mathbb{Z}_2 * \mathbb{Z}_3$$

in which $cd \neq dc$. nonabelian.

$$\omega_p = a_1\, dy\wedge dz - a_2\, dx\wedge dz + a_3\, dx\wedge dy$$

for $a_i \in \mathbb{R}$. write $e_1 = (u_x, u_y, u_z) \in \mathbb{R}^3$

$e_2 = (v_x, v_y, v_z) \in \mathbb{R}^3$

Then $(n_1, n_2, n_3) = e_1 \times e_2 = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ i & j & k \end{vmatrix}$

$\Rightarrow \quad n_1 = \det\begin{pmatrix} u_y & u_z \\ v_y & v_z \end{pmatrix},\ n_2 = -\det\begin{pmatrix} u_x & u_z \\ v_x & v_z \end{pmatrix},\ n_3 = \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$

And $\omega_p(e_1, e_2) = a_1\, dy\wedge dz(e_1, e_2) - a_2\, dx\wedge dz(e_1, e_2)$

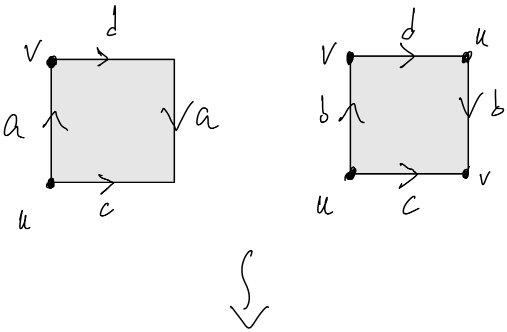
$+ a_3\, dx\wedge dy(e_1, e_2)$

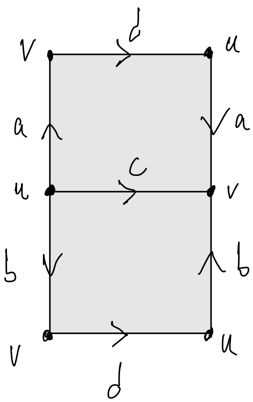
$$= a_1 \det\begin{pmatrix} u_y & u_z \\ v_y & v_z \end{pmatrix} - a_2 \det\begin{pmatrix} u_x & u_z \\ v_x & v_z \end{pmatrix} + a_3 \det\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

$$= a_1 n_1 + a_2 n_2 + a_3 n_3 = 1 = n_1^2 + n_2^2 + n_3^2$$

$\Rightarrow \quad a_1 = n_1, \quad a_2 = n_2, \quad a_3 = n_3.$

b/c this is true $\forall\, p \in S$.

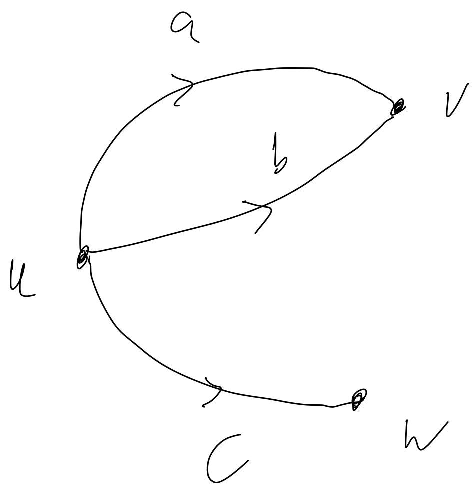


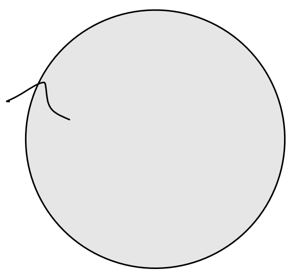


$= \text{Klein bottle} =: K$

Hence $\quad \pi_1(X) = \pi_1(K) = \langle x, y \mid xyxy^{-1} = 1 \rangle$

(a)





Fall 2014 $\boxed{\#1}$

Universal cover of (X,x) compact.

Claim: $\pi_1(X,x)$ finite.

Let $p: \tilde{X} \longrightarrow X$ be the universal cover.

$p_*(\pi_1(\tilde{X}))$ = trivial subgroup has index equal to the number of sheets in the covering.

Let U be some open set in X.

and let $K \subset U$ be some closed subset.

Then $p^{-1}(K)$ is a closed, and since $\tilde{X}$ compact

$\Rightarrow$ $p^{-1}(K)$ compact. And $p^{-1}(U)$ is an open cover of $p^{-1}(K)$ made of disjoint sets homeomorphic to U. Hence any subcover is the whole cover $\Rightarrow$ finitely sheeted

$\Rightarrow$ index of trivial subgroup of $\pi_1(X)$ is finite $\Rightarrow$ $|\pi_1(X)|$ finite.

Fall 2014 $\boxed{\#2}$

Claim: T^2 and $S^1 \vee S^1 \vee S^2$ have isomorphic homology gps but are not homeomorphic.

$$\tilde{H}_n(S^1 \vee S^1 \vee S^2) \approx \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^2)$$

$$\Rightarrow \quad H_n(S^1 \vee S^1 \vee S^2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases} = H_n(T^2).$$

But they are not homeomorphic because

$\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(S^1 \vee S^1 \vee S^2) = \mathbb{Z} * \mathbb{Z}$

are not equal.

Fall 2014 $\boxed{\#3}$

$f: S^n \to S^n$ continuous.

i) if f has no fixed points $\Rightarrow f \simeq$ antipodal map.

ii) if $n = 2m$, $\Rightarrow \exists\, x \in S^{2m}$ s.t. $f(x) = x$ or $f(x) = -x$.

(i) Let $x \in S^n$. Since $f(x) \neq x$, $\exists$ straight line from $-x$ to $f(x)$ that doesn't pass through the origin in $\mathbb{R}^{n+1}$. We then map this straight line to a path γ_x in S^n via the retraction $r: \mathbb{R}^{n+1} \setminus \{0\} \to S^n$ sending $x \longmapsto \frac{x}{|x|}$.

Then $H: S^n \times I \to S^n$ defined by $H(x,t) = \gamma_x(t)$ is a homotopy with $H(x,0) = \gamma_x(0) = -x$ the antipodal map and $H(x,1) = \gamma_x(1) = f(x)$. H is continuous because f is continuous, the straight line homotopy through $\mathbb{R}^{n+1} \setminus \{0\}$ is continuous and r continuous.

(ii) If $\exists$ fixed point, done. If not, (a) $\Rightarrow$ $f \simeq$ antipodal map, α. Then $\deg(f) = \deg(\alpha) = (-1)^{2m+1} = -1$

Then $\deg(\alpha \circ f) = (-1)(-1) = 1 \Rightarrow \alpha \circ f \not\simeq \alpha$

$\overset{(a)}{\Rightarrow}$ $\alpha \circ f$ has a fixed point $\Rightarrow -f(x) = x \Rightarrow f(x) = -x$ for some $x \in S^{2m}$.

2014 Fall $\boxed{\#4}$

See sol'ns.

2014 Fall $\boxed{\#5}$

$X \subset \mathbb{R}^3$ closed submfd homeomorphic to sphere with $g > 1$ handles,

$\Rightarrow \exists$ nonempty open subset on which Gaussian curvature K is negative.

$$\int_X K \, dA = 2\pi \chi(X) = 2\pi(2-2g) < 0$$

by Gauss-Bonnet.

Hence we must have a nonempty open set where $K < 0$.

2014 Fall $\boxed{\# 6}$

M nonempty closed oriented d-dim'l mfd.

ω d-form, X smooth vect. field on M.

Claim: $\mathcal{L}_X \omega$ vanishes at some point on M.

Cartan $\Rightarrow$ $\mathcal{L}_X \omega = X \lrcorner (d\omega) + d(X \lrcorner \omega)$

$= d(X \lrcorner \omega)$, since $d\omega$ is

a $(d+1)$-form on d-dim'l M and is therefore 0.

$$\int_M \mathcal{L}_X \omega = \int_M d(X \lrcorner \omega) = \int_{\partial M = \emptyset} X \lrcorner \omega = 0$$

Hence $\mathcal{L}_X \omega$ must vanish at some point on M.

2014 Fall $\boxed{\#7}$

Show $\mathbb{CP}^1$ smooth oriented mfd by atlas construction.

Let an element of $\mathbb{CP}^1$ be represented by $[z]$ where $z \in \mathbb{C}^2$. $[z] = [z']$ iff $z = \lambda z'$ for some $\lambda \in \mathbb{C}$. Then Define $U_1 = \{[z] \in \mathbb{CP}^1 : z_1 \neq 0\}$

$U_2 = \{[z] \in \mathbb{CP}^2 : z_2 \neq 0\}$.

U_1, U_2 cover $\mathbb{CP}^1$ and are open.

Define $\phi_1 : U_1 \longrightarrow \mathbb{C}$ by $\phi_1([z]) = \frac{z_2}{z_1}$

$\phi_2 : U_2 \longrightarrow \mathbb{C}$ by $\phi_2([z]) = \frac{z_1}{z_2}$

Well defined b/c $\phi_1([\lambda z]) = \frac{\lambda z_2}{\lambda z_1} = \frac{z_2}{z_1} = \phi_1([z])$

Then $\phi_1^{-1} : \phi_1(U_1) \longrightarrow \mathbb{CP}^1$ is defined by

$\phi_1^{-1}(w) = [(1, w)]$. Similarly $\phi_2^{-1}(w) = [(w, 1)]$.

Then $\phi_1^{-1}\left(\frac{z_2}{z_1}\right) = \left[\left(1, \frac{z_2}{z_1}\right)\right] = \left[z_1\left(1, \frac{z_2}{z_1}\right)\right] = [z]$

and $\phi_1([(1, w)]) = \frac{w}{1} = w$. So these are inverses and are continuous, so ϕ_1, ϕ_2 are homeomorphisms onto their images.

We just need to check compatibility

That is,

$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \longrightarrow \phi_2(U_1 \cap U_2)$ must be smooth.

$\phi_2([(1,w)]) = \frac{1}{w}$, $w \neq 0$ b/c we're considering $U_1 \cap U_2$.

This is smooth, so we are done.

Fall 2015 $\boxed{\#1}$

(a) Let $f, g : X \to Y$. We say $H : X \times I \to Y$ is a homotopy between f and g if H continuous, $H(x,0) = f(x)$, $H(x,1) = g(x)$. $(f \simeq g)$

We say $f : X \to Y$ is a homotopy equivalence between X and Y if $\exists\, g : Y \to X$ s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. $(X \simeq Y)$

(b) $\mathbb{R} \simeq \{0\} \subset \mathbb{R}$ by def. retract but no bijective map between spaces, so $\mathbb{R} \not\cong \{0\}$.

(c) S^2 and S^3 have trivial fundamental group but they are not homeomorphic because $H_2(S^2) = \mathbb{Z}$, $H_2(S^3) = 0$.

(d) $S^1 \vee S^1$ and $T^2 = S^1 \times S^1$.

$H_1(S^1 \vee S^1) = H_1(S^1) \oplus H_1(S^1) = \mathbb{Z} \oplus \mathbb{Z} = H_1(T^2)$

but $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$, while $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$.

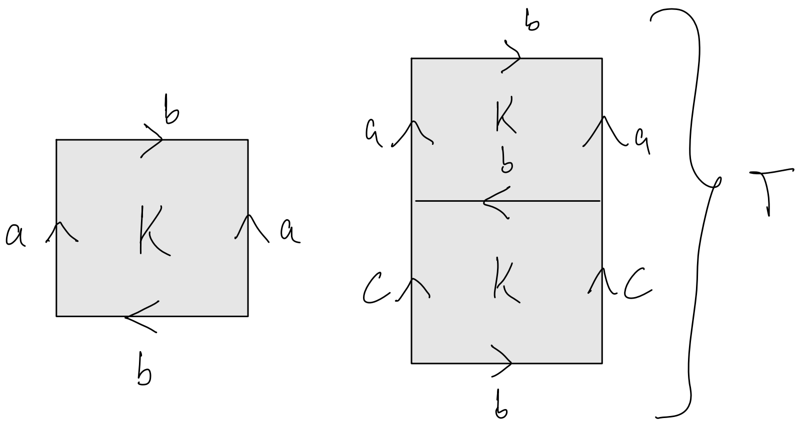
Fall 2015 $\boxed{\#2}$

$T = S^1 \times S^1$, $K =$ Klein Bottle.

(a) describe 2-sheeted covering $p : T \longrightarrow K$.

(b) $x_0 \in T$, $y_0 = p(x_0) \in K$. Give generators for $\pi_1(T; x_0)$, $\pi_1(K; y_0)$. For each generator of $\pi_1(T; x_0)$ express its image under $p_* : \pi_1(T; x_0) \to \pi_1(K; y_0)$.

(a)



map $a \mapsto a$
$b \mapsto b$
$c \mapsto a$

(b) We see $\pi_1(T; x_0) = \mathbb{Z} \times \mathbb{Z} \langle ca, b \rangle$

$\pi_1(K; p(x_0)) = \langle a, b \mid aba^{-1}b = 1 \rangle$

and $p_*(ca) = a^2$, $p_*(b) = b$

Fall 2015 $\boxed{\#3}$

$\Sigma_g, \Sigma_{g'}$ closed orientable surfaces of genus $g, g' > 0$.

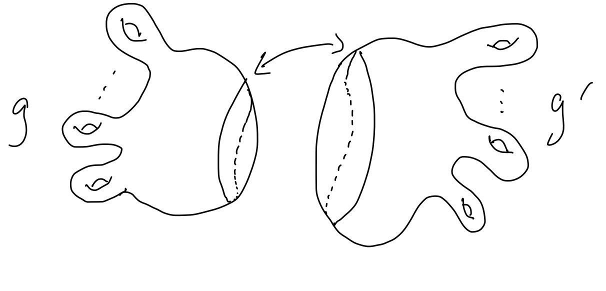
$f: B^2 \to \Sigma_g$ embedding. Simple closed curve $\gamma = f(S^1) \subset \Sigma_g$.

$\gamma' = f'(S^1) \subset \Sigma_{g'}$ with $f': B^2 \to \Sigma_{g'}$.

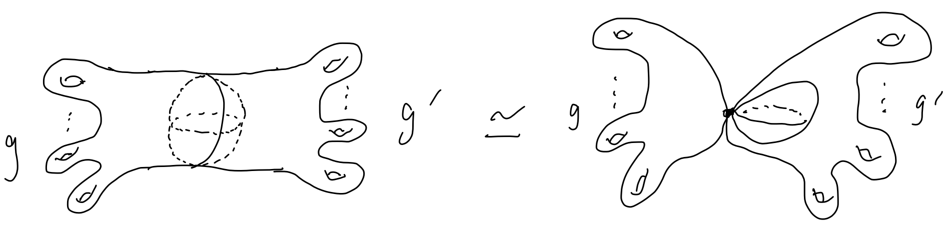
X = top. space obtained by gluing Σ_g and $\Sigma_{g'}$ along γ and γ'. I.e. $X = \Sigma_g \sqcup \Sigma_{g'}$ by gluing $f(x)$ to $f'(x)$ $\forall x \in S^1$.

(a) compute $\pi_1(X)$.

f embedding $\Rightarrow$ $\exists$ homotopy from γ to the constant map at $\gamma(0)$. Same for γ'.



$\Sigma_g \vee \Sigma_{g'} \vee S^2$



Hence $\pi_1(X) \approx \pi_1(\Sigma_g) * \pi_1(\Sigma_{g'}) * \pi_1(S^2)$

$$= \boxed{(\mathbb{Z} \times \mathbb{Z})^{*g+g'}}$$

(b) $\tilde{H}_n(X) \approx \tilde{H}_n(\Sigma_g) \oplus \tilde{H}_n(\Sigma_{g'}) \oplus \tilde{H}_n(S^2)$

$$= \begin{cases} 0 & n=0 \\ \mathbb{Z}^{\oplus 2(g+g')} & n=1 \\ \mathbb{Z}^{\oplus 3} & n=2 \\ 0 & n>2 \end{cases}$$

(c) No because $H_2(X) \neq H_2(\Sigma_g \times \Sigma_{g'})$.

2015 Fall $\boxed{\#4}$

M n-dim'l mfd. $\omega \in \Omega^{n-1}(M)$. $\int_N \omega = 0$ $\forall$ $(n-1)$-dim'l oriented closed submfd N of M.

Claim: $d\omega = 0$.

Let $x \in M$. There is a chart centered at x, call it (U, ϕ). So $\phi : U \longrightarrow \mathbb{R}^n$ with $\phi(x) = 0$.

$\exists$ small $(n-1)$-dim'l sphere $S^{n-1} \subset \phi(U)$ centered at 0.

Let this be the boundry of a small B^n. Then $\phi^{-1}(B^n)$ contains x in M with boundary $\phi^{-1}(S^{n-1})$.

Thus $\phi^{-1}(S^{n-1})$ is a $(n-1)$-dim'l oriented closed submfd by regular level set theorem wrt the radius function. Then $\int_{\phi^{-1}(S^{n-1})} \omega = \int_{\phi^{-1}(B^n)} d\omega = 0$

by Stokes. Since x arbitrary, $\exists$ a collection of such $\phi^{-1}(B^n)$ covering M. Hence $d\omega = 0$ on M, For if not, then $\exists$ $\phi^{-1}(B^n)$ on which $\int_{\phi^{-1}(B^n)} d\omega \neq 0$.

2015 Fall $\boxed{\#5}$

V.F. $v = \partial_x + xz\partial_z$, $w = \partial_y + yz\partial_z$ in $\mathbb{R}^3$.

$p \in \mathbb{R}^3$. $\exists$? local coordinate system in a nbd of p in which v and w?

I.e. is there a diffeomorphism $\phi : U \to V$ from a nbd U of p to an open $V \subset \mathbb{R}^3$ that sends v to ∂_x and w to ∂_y?

We need ϕ s.t. $d\phi_p(\partial_x + xz\partial_z) = \partial_x$

and $d\phi_p(\partial_y + yz\partial_z) = \partial_y$.

i.e.

$$\begin{pmatrix} \phi^1_x & \phi^1_y & \phi^1_z \\ \phi^2_x & \phi^2_y & \phi^2_z \\ \phi^3_x & \phi^3_y & \phi^3_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ xz \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \phi^1_x & \phi^1_y & \phi^1_z \\ \phi^2_x & \phi^2_y & \phi^2_z \\ \phi^3_x & \phi^3_y & \phi^3_z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ yz \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

So $\phi^1_x + xz\,\phi^1_z = 1$, $\phi^2_x + yz\,\phi^2_z = 1$

$$\phi_x^2 + xz\phi_z^2 = 0 \Rightarrow -xz\phi_z^2 + yz\phi_z^2 = 1$$

$$\Rightarrow \phi_z^2 = \frac{1}{z(y-x)}$$

$$\Rightarrow \phi^2 = \frac{1}{y-x}\ln(z) + C$$

$$\Rightarrow \phi_x^2 = -xz\phi_z^2 = -xz\frac{1}{z(y-x)} = -\frac{x}{y-x} = \frac{x}{x-y}$$

But $\phi_x^2 = -\frac{1}{(y-x)^2}\ln(z) \Rightarrow \Leftarrow$.

$$[V,W] = VW - WV = (\partial_x + xz\partial_z)(\partial_y + yz\partial_z) - WV$$

$$= \partial_x(\partial_y + yz\partial_z) + xz\partial_z(\partial_y + yz\partial_z) - WV$$

$$= \partial_{xy} + yz\partial_{xz} + xz\partial_{yz} + xyz\partial_z + xyz^2\partial_{z^2} - WV$$

$$= \partial_{xy} + yz\partial_{xz} + xz\partial_{yz} + xyz\partial_z + xyz^2\partial_{z^2}$$
$$- (\partial_y + yz\partial_z)(\partial_x + xz\partial_z)$$

$$= \cancel{\partial_{xy}} + \cancel{yz\partial_{xz}} + \cancel{xz\partial_{yz}} + \cancel{xyz\partial_z} + \cancel{xyz^2\partial_{z^2}}$$
$$\cancel{-\partial_{xy}} - \cancel{xz\partial_{yz}} - \cancel{yz\partial_{xz}} - \cancel{xyz\partial_z} - \cancel{xyz^2\partial_{z^2}}$$

$$= 0$$

2015 Fall $\boxed{\#6}$

$f: \mathbb{C} \to \mathbb{C}$ polynomial of one complex variable.

One point compactification $\mathbb{C} \cup \{\infty\} \cong S^2$.

(a) Prove f extends to a continuous $\bar{f}: S^2 \to S^2$.

(b) Prove $\deg(\bar{f}) = \deg(f)$.

(a) extend to $\bar{f}$ by sending ∞ to ∞. We have to check that $\bar{f}$ is continuous.

$$\begin{tikzcd} \mathbb{C} \cup \{\infty\} \arrow[r, "{z \mapsto f(z)}", "{\infty \mapsto \infty}"'] \arrow[d, "\cong"] & \mathbb{C} \cup \{\infty\} \arrow[d, "\cong"] \\ S^2 \arrow[r, "\bar{f}"] & S^2 \end{tikzcd}$$

if $z \to \infty$ does $f(z) \to \infty$?

$f(z) = w_0 z^0 + w_1 z^1 + \cdots + w_d z^d$ for $d = \deg(f)$.

$$f(re^{i\theta}) = w_0 r + w_1 re^{i\theta} + \cdots + w_d r^d e^{i\theta d}$$

$$= r^d \left(w_0 \frac{1}{r^{d-1}} + w_1 \frac{1}{r^{d-2}} e^{i\theta} + \cdots + w_d e^{i\theta d} \right)$$

$\lim_{r \to \infty} f(re^{i\theta}) = \lim_{r \to \infty} r^d w_d e^{id\theta} = \infty$, So $\bar{f}$ Continuous.

(b) $\bar{f}_* : H_2(S^2) \to H_2(S^2)$ is multiplication by $n \in \mathbb{Z}$.

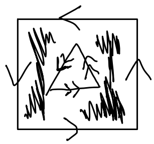
$\mathbb{Z} \longrightarrow \mathbb{Z}$ $\Rightarrow \deg(\bar{f}) = n$.

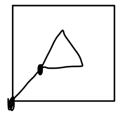
Geometrically, $\int_{S^2} \bar{f}^* \omega = n \int_{S^2} \omega$ for $\omega \in \Omega^2(S^2)$.

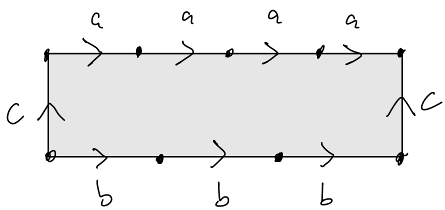
$$(\bar{f}^{*}\omega)_y(V_1, V_2) = \left(d\bar{f}_y(\omega)(v_1),\ d\bar{f}_y(\omega)(v_2) \right)$$

$$d\bar{f}_y = w_1 + 2w_2 y' + \cdots + d w_d{}^{\,d-1}$$

2016 Spring $\boxed{\#1}$

$X =$  Compute $\pi_1(X)$.

By making a cut from  the bottom left vertex of the square to the bottom left vertex of the triangle, and identifying the resulting edges, we get the diagram

$X =$ 

Hence $\pi_1(X) = \langle a, b, c : a^4 c^{-1} b^{-3} c = 1 \rangle$

2016 Spring $\boxed{\#2}$

X path connected $\pi_1(X; x_0) = \mathbb{Z}/5$

Covering space $\pi : \tilde{X} \to X$ 6-sheeted.

Claim: $\tilde{X}$ either has 2 or 6 connected components.

The connected covering spaces of X are classified by the subgps of $\pi_1(X; x_0) = \mathbb{Z}/5$. There are only two subgroups: the trivial subgp and the whole group, because $\langle 1 \rangle = \langle 2 \rangle = \langle 3 \rangle = \langle 4 \rangle = \mathbb{Z}/5$ since 5 prime.

Hence the trivial subgrp corresponds to a 5-sheeted covering space since the trivial subgp has index 5 in $\mathbb{Z}/5$. Similarly the subgroup $\mathbb{Z}/5 \subset \mathbb{Z}/5$ corresponds to a 1-sheeted covering space.

Therefore, a 6-sheeted covering space can only be made from six 1-sheeted covering spaces or one 5-sheeted and one 1-sheeted covering spaces.

i.e. $\tilde{X}$ either has 2 or 6 connected components.

Spring 2016 $\boxed{\#3}$

$X = S^1 \times S^n$, $n \geq 1$ compute $H_k(S^1 \times S^n; \mathbb{Z})$.

Let $A = \left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right) \times S^n$, $B = \left(\pi/4, \frac{7\pi}{4}\right) \times S^n$.

Then $A \cap B \simeq S^n \sqcup S^n$, $A, B \simeq S^n$, $A \cup B = X$.

By M.V. we have

$$\begin{array}{ccccccc} \cdots \longrightarrow & H_k(A \cap B) & \longrightarrow & H_k(A) \oplus H_k(B) & \longrightarrow & H_k(X) & \longrightarrow \cdots \\ & \wr\wr & & \wr\wr & & & \\ & H_k(S^n) \oplus H_k(S^n) & & H_k(S^n) \oplus H_k(S^n) & & & \end{array}$$

For $H_n(S^n) \oplus H_n(S^n) \longrightarrow H_n(S^n) \oplus H_n(S^n)$ we have

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

$$(a, b) \longmapsto (a+b, a+b)$$

$$\ker = \{(a, -a)\} \approx \mathbb{Z}, \quad im = \{(a+b, a+b)\} \approx \mathbb{Z}$$

Then

$$\begin{array}{ccccccc} 0 \longrightarrow & H_{n+1}(X) & \longrightarrow & H_n(S^n) \oplus H_n(S^n) & \longrightarrow & H_n(S^n) \oplus H_n(S^n) \\ & & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

$\Rightarrow H_{n+1}(X) \approx \mathbb{Z}$, the kernel of

and for $n \geq 2$,

$$\begin{array}{ccccccc} H_n(S^n) \oplus H_n(S^n) & \longrightarrow & H_n(S^n) \oplus H_n(S^n) & \longrightarrow & H_n(X) & \longrightarrow & 0 \\ \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & & & \end{array}$$

which gives $H_n(X) \approx \operatorname{im}\left(H_n(S^n) \oplus H_n(S^n) \longrightarrow H_n(X) \right) \approx \mathbb{Z}$

since the kernel of that map is $\mathbb{Z}$.

Sticking with $n \geq 2$, we have

$$0 \to H_1(X) \longrightarrow H_0(S^n) \oplus H_0(S^n) \longrightarrow H_0(S^n) \oplus H_0(S^n)$$

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

$$(u, v) \longmapsto (u+v, u+v)$$

So again $H_1(X) \approx \mathbb{Z}$, and $H_0(X) \approx \mathbb{Z}$ b/c X path connected.

In sum,

For $n \geq 2$

$$\boxed{H_k(S^1 \times S^n) = \begin{cases} \mathbb{Z} & k = 0, 1, n, n+1 \\ 0 & \text{else} \end{cases}}$$

and

$$\boxed{H_k(S^1 \times S^1) = H_k(T^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & k = 1 \\ 0 & \text{else.} \end{cases}}$$

Spring 2016 $\boxed{\#4}$

M cpct oriented n-dim'l mfd.
$f : M \to \mathbb{R}^n$ differentiable, $f(M)$ non-empty interior in $\mathbb{R}^n$.

(a) Claim: $\exists\, x \in M$ s.t. $\exists$ nbd U of x s.t.
$f|_U : U \to f(U)$ is a diffeomorphism.

Let V be an open n-dim'l ball contained in $f(M)$. Let $y \in f(M)$. Consider $f^{-1}(y) \subset M$. If df_x not surjective for any $x \in f^{-1}(y) \subset M$, then y is a critical value of f. By Sards theorem, the set of such y have measure 0 in $\mathbb{R}^n$. Then there must be some $y \in V$ s.t. $\exists\, x \in f^{-1}(y)$ with df_x surjective. By dimension df_x is an isomorphism. Let W be a chart containing x. Then, df_x can be viewed as a matrix, and since det is a continuous function we can find U with $x \in U \subset W$ s.t. $df_{\tilde{x}}$ isomorphic $\forall\, \tilde{x} \in U$, i.e. a diffeomorphism.

(b) Claim: $\exists$ at least two points $x, y \in M$ s.t. f is a local diffeo at x and y, f orientation preserving at x, and reversing at y.

Let $\omega \in \Omega^n(\mathbb{R}^n)$ s.t. ω vanishes on $f(M)$, is compactly supported and has $\int_{\mathbb{R}^n} \omega > 0$. Then

$$\deg(f)\int_{\mathbb{R}^n}\omega = \int_M f^*\omega = \int_M 0 = 0 \Rightarrow \deg(f)=0.$$

Since degree of $f = 0$, then

for some regular point $y \in f(M)$,

then $0 = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(x) \Rightarrow$ there must

be some x with $\operatorname{sgn}(x) = +1$ and

some with $\operatorname{sgn}(x) = -1$ in $f^{-1}(y)$.

This is to say (b).

Spring 2016 $\boxed{\#5}$

Is there an $\omega \in \Omega^n(\mathbb{RP}^n)$ s.t. $\omega(y) \neq 0$ at every $y \in \mathbb{RP}^n$?

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k=0 \text{ or } k=n \text{ odd} \\ \mathbb{Z}/_2 & k \text{ odd}, \quad 0<k<n \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$ $\mathbb{RP}^n$ orientable for n odd

Since $H_n(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & n \text{ odd} \\ 0 & \text{else} \end{cases}$

? Cellular homology

2016 Spring $\boxed{\#6}$

$$H^k(S^n) \approx \begin{cases} 0 & k \neq 0, n \\ \mathbb{R} & k = 0, n \end{cases}$$

$f: S^{2n-1} \to S^n$ with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ with $\int_{S^n} \alpha = 1$, then

(a) $\exists\ \beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $f^*(\alpha) = d\beta$.

(b) $I(f) := \int_{S^{2n-1}} \beta \wedge d\beta$ independent of α, β.

(a) α closed since $d\alpha \in \Omega^{n+1}(S^n)$ all zero.

$f^*(\alpha)$ closed since $d(f^*\alpha) = f^*(d\alpha) = 0$.

Then $[f^*(\alpha)] \in H^n(S^{2n-1}) = 0 \Rightarrow f^*(\alpha)$ exact.

(b) This is equivalent to the claim that $[\beta \wedge d\beta] = [\beta' \wedge d\beta']$ for any α', β' satisfying the same conditions.

$$f_*([\beta \wedge d\beta]) = 0 \in H^{2n-1}(S^n) = 0$$

?

Let α', β' be other forms satisfying the above conditions. Then WTS

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} \beta' \wedge d\beta'$$

$$d\beta' = f^*\alpha', \quad \int_{S^n} \alpha' = \int_{S^n} \alpha = 1$$

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n}} \beta' \wedge d\beta$$

$$\int_{S^{2n-1}} \beta \wedge d\beta - \beta' \wedge d\beta = 0$$

i.e. $\beta \wedge d\beta - \beta' \wedge d\beta$ exact

$(\beta - \beta') \wedge d\beta$ exact?

$\beta - \beta'$ closed $\Rightarrow$ exact $\Rightarrow$

$\beta - \beta' = d\gamma \quad \beta = d\gamma - \beta'$

$d\gamma \wedge d\beta$

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} (d\gamma - \beta) \wedge d\beta$$

$$= \int d\gamma \wedge d\beta +$$

$$\int_{S^{2n-1}} d(\gamma \wedge d\beta) = \int_{S^{2n-1}} d(\gamma \wedge d\beta) =$$

$$= \int_{B^{2n}} d^2(\gamma \wedge d\beta) =$$

α, α' $\quad \int_{S^n} \alpha = \int_{S^n} \alpha' = 1.$

$\exists\ \beta, \beta'$ with $d\beta, d\beta' = f^*(\alpha), f^*(\alpha')$

NTS $\int_{S^{2n-1}} \beta \wedge d\beta \overset{?}{=} \int_{S^{2n-1}} \beta' \wedge d\beta'$

$$\int_{S^{2n-1}} \beta \wedge f^*(\alpha) \overset{?}{=} \int_{S^{2n-1}} \beta' \wedge f^*(\alpha')$$

Stokes

$$= \int_{B^{2n}} d\beta \wedge f^*(\alpha) \overset{?}{=} \int_{B^{2n}} d\beta' \wedge f^*(\alpha')$$

$$= \int_{B^{2n}}$$