TOPOLOGY

retraction: $r: X \to X$ s.t. $r(X) = A$, $r|_A = \mathrm{id}_A$

deformation retraction: homotopy from id_X to r. s.t. $f_t|_A = \mathrm{id}_A$ $\forall t$.

· Homotopy relative to A: $f_t: X \to Y$ s.t. $f_t|_A = \mathrm{id}$ $\forall t$

Homotopy equivalence: $f: X \to Y$, $g: Y \to X$ s.t. $fg \simeq \mathrm{id}$, $gf \simeq \mathrm{id}$.

· Loop concatenation: $f, g: I \to X$ loops based at x_0, then

$$f \cdot g(s) = \begin{cases} f(2s) & 0 \le s \le \frac{1}{2} \\ g(2s-1) & 1/2 \le s \le 1 \end{cases}$$

$\pi_1(S^1) = \mathbb{Z}$

, Basepoint change: $\beta_h: \pi_1(X, x_1) \to \pi_1(X, x_0)$, $\beta_h[f] = [h \cdot f \cdot \bar{h}]$, is isom.

(· Borsuk-Ulam ($f: S^2 \to \mathbb{R}^2$, $\exists x \in S^2$ s.t. $f(x) = f(-x)$))

· $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$ · $i: X \to A$ induces $\begin{Bmatrix} \text{injection} \\ \text{isom.} \end{Bmatrix}$ if $A \begin{Bmatrix} \text{retract} \\ \text{def retract} \end{Bmatrix}$ of X

· Seifert-van Kampen: If $X = A \cup B$, $A, B, A \cap B$ path-connected, then $\pi_1(X) \cong \pi_1(A) * \pi_1(B) / \langle i_1^*(\alpha) i_2^*(\alpha)^{-1} : \alpha \in \pi_1(A \cap B) \rangle$

where $i_1: A \cap B \hookrightarrow A$, $i_2: A \cap B \hookrightarrow B$.

· Covering spaces: $p: \tilde{X} \to X$ s.t. $\exists \{U_\alpha\}$ open cover of X s.t. $p^{-1}(U_\alpha)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped homeomorphically onto U_α

· Homotopy lifting: $f_t: Y \to X$, $\tilde{f}_0: Y \to \tilde{X}$ lifting $f_0 \Rightarrow \exists!\, \tilde{f}_t: Y \to \tilde{X}$ of $\tilde{f}_0$ lifting f_t.

· p_* is injective and $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ = loops in X whose lifts are loops in $\tilde{X}$

$*$ $[\pi_1(X, x_0) : p_*(\pi_1(\tilde{X}, \tilde{x}_0))]$ = # sheets of covering space

$*$ Lifting Criterion: $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \iff$

$$\begin{array}{ccc} & & \tilde{X} \\ & \overset{\exists \tilde{f}}{\nearrow} & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

· Classification: X path-conn, loc. path-conn, semilocally simply-connected, then

$$\text{basepoint preserving} \begin{Bmatrix} \text{iso. classes} \\ \text{of } p: \tilde{X} \to X \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \text{Subgroups } H \le \pi_1(X) \end{Bmatrix}$$

$$p \longmapsto p_*(\pi_1(\tilde{X}))$$

- Deck transformations: isomorphisms $\tilde{X} \to \tilde{X}$; group $G(\tilde{X})$ under $\circ$.
  - normal covering: for each $\tilde{x}_1, \tilde{x}_2 \in p^{-1}(x_0)$, $\exists\ \varphi \in G(\tilde{X})$ s.t. $\varphi(\tilde{x}_1) = \tilde{x}_2$
  - $p: \tilde{X} \to X$ normal $\Leftrightarrow$ $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \trianglelefteq \pi_1(X, x_0)$
  - $G(\tilde{X}) \cong N_{\pi_1(X)}(p_*(\pi_1(\tilde{X}))) / p_*(\pi_1(\tilde{X}))$ (i.e., $G(\tilde{X}) = \pi_1(X) / p_*(\pi_1(\tilde{X}))$ if $p: \tilde{X} \to X$ normal)
  - $p: \tilde{X} \to X$ universal $\Rightarrow$ $G(\tilde{X}) \cong \pi_1(X)$
- Actions: $G \curvearrowright Y$ s.t. every $y \in Y$ has nbhd s.t. $\{g(U)\}_{g \in G}$ disjoint (i.e. $g_1(U) \cap g_2(U) \neq \phi \Rightarrow g_1 = g_2$) $\rightsquigarrow$ free action

  Then: (1) quotient $p: Y \to Y/G$ normal covering
  (2) $G \cong G(Y)$
  (3) $G \cong \pi_1(Y/G) / p_*(\pi_1(Y))$
- [n-sheeted coverings: Classified by equiv. classes of homomorphisms $\pi_1(X) \longrightarrow S_n$.]

$\tilde{X}, X, Y$ path-connected, locally path-connected.

- $p: \pi_n(\tilde{X}) \to \pi_n(X)$ iso. for all $n \geq 2$.
- $H < F_{n+1}$; $[F_{n+1} : H] = k \Rightarrow H = F_{kn+1}$ (every sg. of free grp is free!)

2.

singular n-simplex: $\sigma : \Delta^n \to X$;

• $C_n(X) = \mathbb{Z}\langle n\text{-simplices in } X\rangle = \{\sum n_i \sigma_i : n_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X\}$

• boundary map: $\partial_n : C_n(X) \to C_{n-1}(X)$; $\partial_n(\sigma) = \sum_i (-1)^i \sigma|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$

→ So we have chain complex $C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \dots \to C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\partial} 0$

and may form Homology groups: $H_n(X) = \underbrace{\ker \partial_n}_{\text{cycles}} / \underbrace{\operatorname{Im} \partial_{n+1}}_{\text{Boundaries}}$

→ Properties:

(1) $X = \coprod_\alpha X_\alpha$, $\{X_\alpha\}$ path-components $\Rightarrow H_i(X) \cong \bigoplus_\alpha H_i(X_\alpha)$

(2) X path-connected $\Rightarrow H_0(X) \cong \mathbb{Z}$

• Reduced Homology: formed from augmented chain cplx $C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$

$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$ & $H_n(X) \cong \tilde{H}_n(X)$, $n > 0$

• Good pairs: (X, A) : $A \subseteq X$ closed, def retract of some nbhd in X; then

$\dots \to \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \dots \to \tilde{H}_0(X/A) \to 0$

(Rmk: (X, A), X CW-cplx, A subcplx is good pair!)

• Relative Homology: homology of chain complex $C_*(X, A) := C_*(X)/C_*(A)$

$\dots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to \dots \to H_0(X, A) \to 0$

Excision ① $Z \subseteq A \subseteq X$ s.t. $\bar{Z} \subseteq A^\circ$, then:

$H_n(X \setminus Z, A \setminus Z) \cong H_n(X, A)$ for all n

② A, B s.t. $A^\circ \cup B^\circ = X$,

$H_n(B, A \cap B) \cong H_n(X, A)$ for all n

induced from inclusion

• $H_n(X, A) \cong \tilde{H}_n(X/A)$ $\forall n$ for good pairs (X, A) ← induced from quotient map

$\tilde{H}_n(\bigvee_\alpha X_\alpha) \cong \bigoplus_\alpha \tilde{H}_n(X_\alpha)$ provided the pairs (X_α, x_α) are good

· **Five Lemma**:

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \varepsilon\downarrow \\ A' & \xrightarrow[i']{} & B' & \xrightarrow[j']{} & C' & \xrightarrow[k']{} & D' & \xrightarrow[l']{} & E' \end{array}$$

Commutative diagram, rows exact, $\alpha, \beta, \delta, \varepsilon$ iso $\Rightarrow$ γ iso

· **Degree**:
- $\deg f = 0$ if f not surjective.
- $\deg \alpha = (-1)^{n+1}$ for $\alpha : S^n \to S^n$ antipodal
- $\deg fg = \deg f \deg g$
- f no fixed pts $\Rightarrow \deg f = \deg \alpha$

· **Cellular Boundary Formula**: $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta}\, e^{n-1}_\beta$ Boundary of e^n_α?

where $d_{\alpha\beta} = \deg\left(S^{n-1}_\alpha \xrightarrow[\sigma^n_\alpha]{} X^{n-1} \xrightarrow[q_\beta]{} S^{n-1}_\beta\right)$

where σ^n_α attaching map for e^n_α + $q_\beta : X^{n-1} \to X^{n-1}/(X^{n-1} \setminus e^{n-1}_\beta)$

(X^{n-1} the $(n-1)$-skeleton)

· **Euler characteristic**: $\chi(X) = \sum_n (-1)^n \operatorname{rk}(H_n(X))$

[**Splitting Lemma**: $0 \to A \xrightarrow[i]{} B \xrightarrow[j]{} C \to 0$ short exact

TFAE: ① i has inverse (left) ③ $B \cong A \oplus C$

② j has inverse (right)]

[**Mayer-Vietoris**: A, B with $i : A\cap B \hookrightarrow A$, $j : A \cap B \hookrightarrow B$ + $k : A \hookrightarrow A\cup B$, $l : B \hookrightarrow A \cup B$

$$\cdots \to H_n(A\cap B) \xrightarrow{(i_*, j_*)} H_n(A) \oplus H_n(B) \xrightarrow{k^* - l^*} H_n(A^\circ \cup B^\circ) \xrightarrow{\partial} H_{n-1}(A\cap B)$$

$$\to \cdots \to H_0(X) \to 0$$]

Poincaré duality: M oriented, cpt, n-mfld, $\partial M = \phi$,

Then $H^k(M) \cong H_{n-k}(M)$

de Rham theorem: $H^k_{dR}(M) \cong H^k(M) \otimes_{\mathbb{Z}} \mathbb{R}$ i.e., $H^k(M; \mathbb{R})$

Geometry

M mfld, dim m

- Atlas: $\mathfrak{A} = \{(U,\varphi)\}$ for manifold M s.t.:
  (1) $U \subseteq M$ open & $M = \bigcup_{U \in \mathfrak{A}} U$
  (2) $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^m$ homeomorphism
  (3) $\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ diffeomorphism
- Differentiability: $f: M \to N$ diff. if $\Psi_j \circ f \circ \varphi_i^{-1}$ diff
- Quotient spaces: M/G is orbit space of $G \curvearrowright M$; mfld if action free & discont
  free: $\gamma x \neq x$ $\forall x \in M$, $\gamma \in G \setminus \mathrm{id}$; discont: $K \subseteq M$ cpt, $|\{\gamma \in G : K \cap \gamma K \neq \emptyset\}| < \infty$

- Tangent map: $T_p f : T_p M \to T_{f(p)} N$ • $\mathbb{R}^n$ case, $T_p f = \left(\frac{\partial f_j}{\partial x_i}(p)\right)$ (Jacobian)
  - $v \in T_p M$, α rep. curve $\Rightarrow T_p f(v) = (f \circ \alpha)'(0)$
  - Chain rule: $T_p(g \circ f) = T_{f(p)} g \circ T_p f$

[• SARD $f: M \to N$ regular vals of f have full measure in N]

[• Inverse Function Theorem: $f: \mathbb{R}^n \to \mathbb{R}^n$, $T_p f : \mathbb{R}^n \to \mathbb{R}^n$ iso $\Rightarrow f$ local diffeo at p]

- Reg/Crit vals: • Reg pt: $T_p f$ surj.; Crit pt: $T_p f$ not surj.
  • Reg/crit value: image of reg/crit pt. (resp.)

[Regular Value Theorem ① $f: M \to N$, then $f^{-1}(y)$ mfld if y reg value
② If $P = f^{-1}(y)$, then $T_x P = \ker T_x f$]

- Tangent bundle: $TM = \coprod_{p \in M} T_p M = \{(p,v) : p \in M, v \in T_p M\}$
- Vector field: $X : M \to TM$, $p \mapsto X_p \in T_p M$ → section of TM: $\pi \circ X = \mathrm{id}_M$ ($\pi: TM \to M$; $\pi(p,v) = p$)

[• Lie Bracket: $[X,Y] = \sum_{j=1}^n \left(\sum_{i=1}^n X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$]
{ • bilinear • antisymm/alt • Jacobi } → $[X,Y]_j = XY^j - YX^j$

- F-relatedness: If $F_* Y_p = X_{F(p)}$, then X, Y F-related (i.e. $F_* Y = X$)
- Naturality: $F_* X_1 = Y_1$, $F_* X_2 = Y_2 \Rightarrow F_*[X_1, X_2] = [F_* X_1, F_* X_2] = [Y_1, Y_2]$

- Immersion: $T_p f$ injective $\forall p \in M$
- Submersion: $T_p f$ surjective $\forall p \in M$

• embedding: immersion that is homeomorphism onto image

- k-linear form: $\omega : V^k \to \mathbb{R}$, i.e. $\omega \in V^* \otimes \cdots \otimes V^*$;
  - alternating: $\omega(v_1, \ldots, v_k) = 0$ if $v_i = v_j$ some $i \neq j$.
  - $\Lambda^k V = \mathrm{Alt}^k V$ = set of alternating k-linear forms.

wedge product: $\alpha \in \mathrm{Alt}^p(V)$, $\beta \in \mathrm{Alt}^q(V) \Rightarrow \alpha \wedge \beta \in \mathrm{Alt}^{p+q}(V)$

given by $\alpha \wedge \beta = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} \mathrm{sgn}(\sigma)\, \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(p)})\, \beta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})$

→ Properties: (1) bilinear + assoc. (2) $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$

(3) For $w_1, \ldots, w_k \in \mathrm{Alt}^1(V) \cong V^*$, $w_1 \wedge \cdots \wedge w_k(X_1, \ldots, X_k) = \det(w_j(X_i))$

→ Basis for $\mathrm{Alt}^p(V)$: $\{e_1, \ldots, e_n\}$ basis for V, $\{e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* : 1 \leq i_1 < \cdots < i_p \leq n\}$ is basis for $\mathrm{Alt}^p(V)$. $\Rightarrow \dim \mathrm{Alt}^p(V) = \binom{n}{p}$

→ Induced maps: $f_p^* : \mathrm{Alt}^p(W) \to \mathrm{Alt}^p(V)$, $f_p^*(\omega)(v_1, \ldots, v_p) = \omega(f(v_1), \ldots, f(v_p))$

Props: ① $(g \circ f)^* = g^* \circ f^*$ ② $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$

- Differential forms: $p \in M$, $\omega_p \in \mathrm{Alt}^k(T_pM)$ a diff. k-form; set $\Omega^k M$
  - wedge product: $\alpha \wedge \beta \in \Omega^{p+q}(M)$ given by $(\alpha \wedge \beta)_x = \alpha_x \wedge \beta_x \in \mathrm{Alt}^{p+q}(T_xM)$
- Pullbacks: $f : M \to N \rightsquigarrow f^* : \Omega^k(N) \to \Omega^k(M)$

$$f^*(\omega)_x = (T_xf)_k^*(\omega_{f(x)}) \ (= \mathrm{Alt}^k(T_xf)(\omega_{f(x)}))$$

→ Props
- $(f \circ g)^* = f^* \circ g^*$
- $f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$

- Exterior Derivative: $d : \Omega^p(M) \to \Omega^{p+1}(M)$

(1) linear (2) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$

(3) $d \circ f^* = f^* \circ d$ (commutes w/ pullback!) (4) $d \circ d = 0$

(5) $df_x = T_xf$

- ω is closed if $d\omega = 0$; ω is exact if $\exists \beta$ s.t. $d\beta = \omega$

- Given the cochain cpx $0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \Omega^3(M) \longrightarrow \cdots$,
we form the <u>p^{th} de Rham cohomology group</u>: $H^p(M) := \ker d_p / \mathrm{im}\, d_{p-1}$

closed p-forms; exact p-forms

  - pullback sends closed $\to$ closed, exact $\to$ exact $\Rightarrow$ induces map <u>$f^*: H^p(N) \to H^p(M)$</u>
  - $[\omega] = 0$ in $H^p(M) \Rightarrow \omega$ exact p-form.

2

- <u>Orientation</u>: $\{(U_i, \varphi_i)\}$ oriented if $\det(T_p(\varphi_j \circ \varphi_i^{-1})) > 0$ for all p, φ_i
  - <u>volume form</u>: top-dimensional form that is $\neq 0$ $\forall p \in M$

    $\rightsquigarrow$ <u>M admits vol. form $\Leftrightarrow$ M orientable</u>
  - <u>Integration</u>: $\dim M = m$, oriented. Then $\int_M \omega = \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \omega$ when $\mathrm{supp}\,\omega \subseteq U_i$

- $\boxed{\text{Stokes}: \int_M d\omega = \int_{\partial M} \omega}$ $\left(= \int_{\partial M} i^*(\omega) \text{ where } i: \partial M \hookrightarrow M \right)$

- <u>Top-Dimensional de Rham Cohomology</u>: M oriented, $\dim m$, $\partial M = \emptyset$.
  Then: $I: H^m_c(M) \to \mathbb{R}$, $\omega \longmapsto \int_M \omega$ is an isomorphism

  & $H^m_c(M)$ is generated by the <u>volume form</u>

- $\boxed{\text{Pullback-Pushforward}: f^*(\omega)_p(v_1, \ldots, v_k) = \omega_{f(p)}(T_pf(v_1), \ldots, T_pf(v_k))}$

<u>**Degree**</u>: M, N oriented, connected mflds, $\dim M = \dim N = n$

Then $f^*: H^m_c(N) \cong \mathbb{R} \longrightarrow H^m_c(M) \cong \mathbb{R}$ is multiplication by real #
$\alpha \longmapsto (\deg f)\alpha$ — the <u>degree of f</u>

<u>Geometric degree</u>: $f: M \to N$

$\deg_x f = \mathrm{sign}(\det T_x f) = \begin{cases} +1 & \text{ornt pres} \\ -1 & \text{ornt rev} \end{cases}$ for x reg. point.

$\deg_y f = \sum_{x \in f^{-1}(y)} \deg_x f$ for y reg. value. <u>FACT</u>: $\deg_y f = \deg f$ at every reg. value y

<u>THMS</u>: ① f: cpt $\to$ non cpt $\Rightarrow \deg f = 0$ ② $\deg \alpha = (-1)^{n+1}$ for $\alpha: S^n \to S^n$ ③ no fixed pts $\Rightarrow f \simeq \alpha \Rightarrow$

• Interior product: $i_X : \Omega^k(M) \to \Omega^{k-1}(M)$, $X \in \mathfrak{X}(M)$

(1) $\omega \in \Omega^1 \Rightarrow i_X(\omega) = \omega(X)$

(2) $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_X\beta$ ⟵ similar to exterior derivative

• Lie derivative: $\mathcal{L}_X = d \circ i_X + i_X \circ d$

(1) $\mathcal{L}_X f = X(f)$ (X considered as a derivation)

(2) $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X \omega)$

(3) $\mathcal{L}_X \alpha \wedge \beta = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X(\beta)$ (Leibniz rule)

• Gauss-Bonet:

M cpt, orit 2-mfld, $\partial M = \emptyset \Rightarrow \int_M K\,dA = 2\pi \chi(M)$

• Euler characteristic:

M CW-complex $\Rightarrow \chi(M) = k_0 - k_1 + k_2 - k_3 \cdots$, $k_i = \#$ i-cells

M_g genus $g \Rightarrow \chi(M_g) = 2 - 2g$

N_g genus g non-orit $\Rightarrow \chi(N_g) = 2 - g$ (e.g. N_2 = Klein bottle)

• Curvature:

~~Each point~~ on surface M has principal curvatures θ_1, θ_2

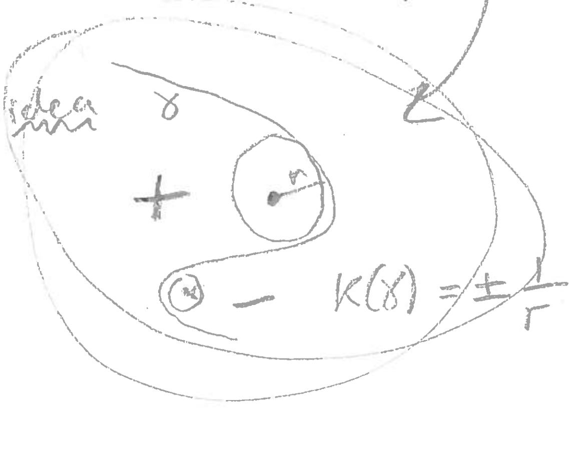
$\partial M \cap P = \gamma_P$

Find the curvature of γ_P, call it K_P

$\theta_1 = \min_P K_P$, $\theta_2 = \max_P K_P$

$\Rightarrow K = \theta_1 \theta_2$ is Gaussian curvature

(integrand of Gauss-Bonnet)



$K(\gamma) = \pm \frac{1}{r}$

Theorema egregium: K does not depend on embedding of M into $\mathbb{R}^3$

Homology computations

$$H_i(\Sigma_g) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}^{2g} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}, \quad H_i(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}_2 & 0<i<n \text{ odd} \\ \mathbb{Z} & i=n \\ 0 & \text{otherwise} \end{cases} \text{ (n odd)} \quad \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}_2 & 0<i<n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \text{ (n even)}$$

$$H_i(K) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z}_2 & i=1 \\ 0 & i>1 \end{cases}, \quad H_i(S^n) \cong \begin{cases} \mathbb{Z} & i=0,n \\ 0 & i\neq 0,n \end{cases}$$

$$H_i(X\times S^1) \cong H_i(X)\oplus H_{i-1}(X)$$

Fundamental Groups

$\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$ for $n>1$ $\qquad \pi_1(S^1)\cong\mathbb{Z}$

$\pi_1(\Sigma_g) \cong \langle a_1,b_1,\dots,a_g,b_g : [a_1,b_1]\cdots[a_g,b_g]=e\rangle$

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etc.

- Euler char. for covering spaces: $\chi(\tilde{X}) = n\,\chi(X)$ if $p:\tilde{X}\to X$ is n-sheeted.
- Semilocally simply connected: each $x\in X$ has nbhd U s.t. $i^*:\pi_1(U,x)\to\pi_1(X,x)$ is trivial. (locally simply conn $\Rightarrow$ semi)

locally ?

- Cramer's rule: $A \in GL_n$; then A^{-1} given by:

$$(A^{-1})_{ij} = \frac{1}{\det A}(-1)^{i+j} \det A_{ji}$$

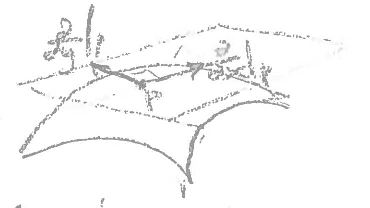
- genus: $g(\Sigma) = \frac{1}{2} \dim_{\mathbb{R}} H^1_{dR}(\Sigma)$

Geometry :

$f : M \to N$

$df_p = T_pf : T_pM \to T_{f(p)}N$; tangent bundle: $TM = \coprod_{p\in M} T_pM = \{(p,v) : p\in M, v\in T_pM\}$

push-forward = $f_* : TM \to TN$
$(p,v) \mapsto (f(p), T_pf(v))$



- Basis of T_pM is $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\} = e_i = (1,0,\dots)$ ~ tangent vectors
- Basis of $(T_pM)^*$ is $(dx_i)_p = e_i^*$ ~ cotangent vectors
- $X : M \to TM$ vector field = section of tangent bundle
  $p \mapsto (p, X_p)$
  Set of vector fields = $\mathcal{X}(M)$
  - Basis for $\mathcal{X}(M)$ is $\left\{ \frac{\partial}{\partial x_i} \right\}$
- $\omega : M \to T^*M$ covector field
  $p \mapsto (p, \omega_p)$ diff 1-form
  = section of cotangent bundle
  Cotangent bundle $T^*M = \coprod_{p\in M} (T_pM)^*$
  Set of one-forms = $\Omega^1(M)$
- Basis for $\Omega^1(M)$ is $\{dx_i\}$

$(dx_i)_p \left(\frac{\partial}{\partial x_j}\Big|_p \right) = \delta_{ij}$

$\frac{\partial}{\partial x_i}\Big|_p = e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in T_p\mathbb{R}^n \cong \mathbb{R}^n$

$(dx_i)_p = e_i^* \in (T_p\mathbb{R}^n)^* \cong (\mathbb{R}^n)^*$, i.e. $e_i^*(e_j) = \delta_{ij}$

$\boxed{\text{Tangent vectors}}$: $\left.\frac{\partial}{\partial x_i}\right|_p$ = tangent at p in the x_i direction

$= x_i\ (e_i)$-directional derivative at p.

$= D_{e_i}$ at p.

iso: $T_x M \xrightarrow{\sim}$ Derivations $\{D : C(M) \to \mathbb{R}\}$

$v \longmapsto D_v$, directional derivative in the $v \in T_x M$ direction

<u>Derivation</u> : $D(fg) = D(f)g(x) + f(x)D(g)$

• $Alt^p(V) = \{$ p-linear alternating forms $\underline{\omega : V^p \to \mathbb{R}}\}$ → $\omega(v_1, \dots, v_p) = 0$ if $v_i = v_j$ some $i \neq j$ (2-form: $\omega(x,x) = 0$)

→ $Alt^1(V) = V^*$

→ $Alt^2(V) = \{$ alternating bilinear forms $\langle , \rangle : V \otimes V \to \mathbb{R}\}$

• <u>wedge</u> : $\alpha \in Alt^p(V)$, $\beta \in Alt^q(V)$: $\alpha \wedge \beta \in Alt^{p+q}(V)$

given by:

$$\alpha \wedge \beta(v_1, \dots, v_{p+q}) = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} sgn(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(p)})\, \beta(v_{\sigma(p+1)}, \dots, v_{\sigma(p+q)})$$

• <u>basis for $Alt^p(V)$</u> : basis $\{e_1, \dots, e_n\}$ for V, $\{e_1^*, \dots, e_n^*\}$ for V^*

$\underline{\{e_{i_1}^* \wedge \dots \wedge e_{i_p}^* : 1 \le i_1 < i_2 < \dots < i_p \le n\}}$ basis

e.g. $Alt^2(V)$ has basis $\{e_i^* \wedge e_j^* : 1 \le i, j \le n\}$

• <u>Differential p-form</u> : $\omega : M \to Alt^p(T_x M)$ $\boxed{\text{set: } \Omega^p(M)}$

$\omega_x \in Alt^p(T_x M) \Rightarrow \omega_x : (T_x M \otimes \dots \otimes T_x M) \to \mathbb{R}$

$\omega_x \in Alt^1(T_x M) \Rightarrow \omega_x : T_x M \to \mathbb{R}$

Exterior derivative

sections of the cotangent bundle: have basis dx_i:

$(dx_i)_p \in (T_pM)^*$

$dx_i : M \longrightarrow T^*M$

$p \longmapsto (dx_i)_p \in (T_pM)^*$

s.t. $(dx_i)_p\left(\frac{\partial}{\partial x_i}\Big|_p\right) = 1$

$d : \Omega^0(\mathbb{R}) \longrightarrow \Omega^1(\mathbb{R})$

real-valued smooth fcn. $\longrightarrow$ $f \longmapsto df$

is defined as $df_p = T_pf$

$df : M \longrightarrow T^*M$

$p \longmapsto (p, T_pf)$

$T_pf : T_pM \longrightarrow \mathbb{R} \in (T_pM)^*$

$$v = a_1 \frac{\partial}{\partial x_1}\Big|_p + \dots + a_n \frac{\partial}{\partial x_n}\Big|_p \Rightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad v$$

$$T_pf(v) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} a_i$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i^*(v)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i(v)$$

$$\Rightarrow \underline{T_pf = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i^*}$$

$d : \Omega^p(M) \to \Omega^{p+1}(M)$

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coefficient functions

$$\text{If} \quad \omega = \sum_{i_1 < \dots < i_p} \omega_{i_1 i_2 \dots i_p} dx_{i_1} \wedge \dots \wedge dx_{i_p} \in \Omega^p(M)$$

$$\Rightarrow d\omega = \sum_{i_1 < \dots < i_p} (d\omega_{i_1 \dots i_p}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} \in \Omega^{p+1}(M)$$

pullback :

$Alt^1(N)$ $Alt^1(M)$
$\qquad$ '' $\qquad$ ''

$f: M \to N$, $T_p f: M \to N$, $(T_p f)^*: N^* \to M^*$

$\rightsquigarrow (T_p f)^* = Alt^k(T_p f): Alt^k(N) \to Alt^k(M)$

$\Rightarrow f^*: \Omega^k(N) \to \Omega^k(M)$

where $f^*(\omega)_p = (T_p f)^*(\omega_{f(p)}) \in Alt^k(M)$
$\qquad\qquad\qquad\qquad \omega_{f(p)} \in Alt^k(N)$

recall
$\omega \in \Omega^k(M)$

$\Rightarrow \omega: M \to Alt^k(T_p M)$
$\qquad p \mapsto \omega_p$

$\rightsquigarrow$ $\boxed{\begin{array}{l} \omega_p \in Alt^k(N) \\ f^*(\omega)_p \in Alt^k(M) \end{array}}$

$f^*: \Omega^1(N) \to \Omega^1(M)$

$$f^*(dy_j) = \sum_{i=1}^{m} \frac{\partial f_j}{\partial x_i} dx_i$$

1 forms :

- $f^*(\omega)(X) = \omega(f^* X)$

$\rightsquigarrow f^*(\omega)(X_p) = \omega(f^*(X)_p)$

$\rightsquigarrow f^*(\omega)_p(v) = \omega_{f(p)}(T_p f(v))$

$\rightsquigarrow f^*(dg) = d(g \circ f)$

$\rightsquigarrow f^*(g\omega) = (g \circ f) f^*(\omega)$

$f^*: \Omega^k(N) \to \Omega^k(M)$

$X \in \mathfrak{X}(M)$

$f^*(X) \in \mathfrak{X}(N)$

$\underline{f^*(g) = g \circ f}$

Lie Bracket:

$$[X,Y]_j = XY^j - YX^j$$
$$= \left(\sum X^i \frac{\partial}{\partial x_i}\right) Y^j - \left(\sum Y^i \frac{\partial}{\partial x_i}\right) X^j$$
$$= \sum \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i}\right)$$

$$[X,Y] = \sum_j [X,Y]_j \frac{\partial}{\partial x_j}$$