Geometry/Topology Qualifying Exam

Spring 2012

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

Tangent map it would be surj by dimension, image is compact so ∃ a critical pt.

1. (10 pts) Prove that a compact smooth manifold of dimension n cannot be immersed in $\mathbb{R}^n$.

2. (10 pts) Let $\Sigma_{1,1}$ be the compact oriented surface with boundary, obtained from $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x, y) by removing a small disk $\{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{100}\}$.
   (a) Compute the homology of $\Sigma_{1,1}$.
   (b) Let Σ_2 denote a closed oriented surface of genus 2. Use your answer from (a) to compute the homology of Σ_2.

3. (10 pts) Let S be an oriented embedded surface in $\mathbb{R}^3$ and ω be an area form on S which satisfies $\omega(p)(e_1, e_2) = 1$ for all $p \in S$ and any orthonormal basis (e_1, e_2) of T_pS with respect to the standard Euclidean metric on $\mathbb{R}^3$. If (n_1, n_2, n_3) is the unit normal vector field of S, then prove that

$$\omega = n_1 dy \wedge dz - n_2 dx \wedge dz + n_3 dx \wedge dy,$$

where (x, y, z) are the standard Euclidean coordinates on $\mathbb{R}^3$.

4. (10 pts) Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a *Möbius band* is the quotient space $([-1,1]\times[-1,1])/((1,y) \sim (-1,-y))$.
   (a) Determine the fundamental group of X.
   (b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

5. (10 pts) Determine all the connected covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$.

6. (10 pts) Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that $f^*(\mathcal{L}_Y\omega) = \mathcal{L}_X(f^*\omega)$, where ω is a 1-form on N. Here $\mathcal{L}$ denotes the Lie derivative.

7. (10 pts) Consider the linearly independent vector fields on $\mathbb{R}^4 - \{0\}$ given by:

$$X(x_1,x_2,x_3,x_4) = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}$$
$$Y(x_1,x_2,x_3,x_4) = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}.$$

Is the rank 2 distribution orthogonal to these two vector fields integrable? Here orthogonality is measured with respect to the standard Euclidean metric on $\mathbb{R}^4$.

Geo/Top. Sp'12: 1.

(1.) M cpt, n-dim'l, show cannot be immersed in $\mathbb{R}^n$

(*) Suppose that $f: M \to \mathbb{R}^n$ is an immersion. Then $T_pf: T_pM \to T_{f(p)}\mathbb{R}^n$ is injective, i.e. $T_pf: \mathbb{R}^n \to \mathbb{R}^n$ is injective, hence surjective since dimension of domain equals to dim. of codomain, hence f has only reg. vals

Now, since M compact, im f is compact, hence closed & bdd since im $f \subseteq \mathbb{R}^n$.

Thus for $x \in \text{im} f$, $x = (x_1, \ldots, x_n)$ and $|x_i| \leq b_i$ for all x_i. Now, since im f is compact, we may choose $p_0 \in M$ s.t. $f(p_0) = (b_1, x_2, \ldots, x_n)$, i.e. p_0 such that f_1 assumes its maximum at p_0;

Since $f_1(p_0) = b_1 = \sup_{p \in M} f_1$, we know that all partial derivatives of f_1 are zero, hence the matrix $T_{p_0}f$ has a row of zeros, hence not surjective, hence p_0 is a critical point.

So we've shown f must have a critical pt, which is a contradiction to (*), hence no such immersion exists.

② $\Sigma_{1,1} = T^2 \setminus \{pt\}$

(a) $H_i(\Sigma_{1,1})$

(b) Find $H_i(\Sigma_2)$ with part (a)

(a) Apply Mayer-Vietoris:

$A = \Sigma_{1,1}$, $B = D^2 \simeq pt$, $A \cup B = T^2$, $A \cap B = S^1$:

$H_2(S^1) \to H_2(\Sigma_{1,1}) \oplus H_2(pt) \to H_2(T^2) \xrightarrow{\partial} H_1(S^1) \xrightarrow{i_*} H_1(\Sigma_{1,1}) \oplus H_1(pt)$

$\to H_1(T^2) \to H_0(S^1) \to H_0(\Sigma_{1,1}) \oplus H_0(pt) \to H_0(T^2) \to 0$

$\Rightarrow 0 \to H_2(\Sigma_{1,1}) \xrightarrow{j^*} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i_*} H_1(\Sigma_{1,1}) \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

- $H_0(\Sigma_{1,1}) \cong \mathbb{Z}$ since $\Sigma_{1,1}$ is path-connected
- See that the inclusion induced homomorphism $i^*: H_p(S^1) \to H_p(\Sigma_{1,1})$ must be the zero map since S^1 is the boundary of $\Sigma_{1,1}$,

hence $\operatorname{im} i_* = 0 \Rightarrow \ker i_* = \mathbb{Z} \Rightarrow \operatorname{im} \partial = \mathbb{Z} \Rightarrow \ker \partial = 0$

$\Rightarrow \operatorname{im} j^* = 0$, hence $H_2(\Sigma_{1,1}) = 0$ since j^* must be injective

: Finally, α surj. by exactness, hence $\ker \alpha = \mathbb{Z}$, hence $\operatorname{im} \beta = \mathbb{Z}$, hence $\ker \beta = 0 \Rightarrow \operatorname{im} \gamma = 0 \Rightarrow \ker \gamma = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \operatorname{im} \delta = \mathbb{Z} \oplus \mathbb{Z}$,

but $\operatorname{im} i^* = 0$, hence $\ker \delta = 0$, hence $H_1(\Sigma_{1,1}) = \mathbb{Z} \oplus \mathbb{Z}$

2(b): See that $\Sigma_2 = \Sigma_{1,1} \sqcup \Sigma_{1,1} /$ boundaries identified :

So apply Mayer-Vietoris with $A = B = \Sigma_{1,1}$, $A \cap B = S^1$, $A \cup B = \Sigma_2$:

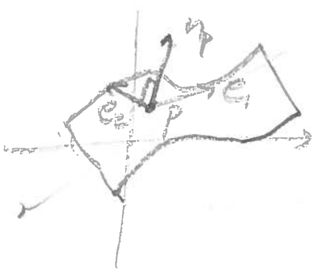
$$0 \to H_2(S^1) \to H_2(\Sigma_{1,1}) \oplus H_2(\Sigma_{1,1}) \to H_2(\Sigma_2) \to H_1(S^1)$$
$$\to H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1}) \to H_1(\Sigma_2) \to H_0(S^1) \to H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1})$$
$$\to H_0(\Sigma_2) \to 0$$

Clearly Σ_2 path-connected, hence $\underline{H_0(\Sigma_2) = \mathbb{Z}}$:

$$0 \to H_2(\Sigma_2) \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} H_1(\Sigma_2) \xrightarrow{\beta} \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \oplus \mathbb{Z}$$
$$\xrightarrow{\delta} \mathbb{Z} \to 0$$

- Consider map $i_* : H_1(S^1) \to H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$; clearly both inclusions send the generator of $H_1(S^1)$ to 0 since S^1 is a boundary in $\Sigma_{1,1}$. So i_* is zero map $\Rightarrow$ im $i_* = 0$ $\Rightarrow \ker i_* = \mathbb{Z} \Rightarrow \underline{H_2(\Sigma_2) = \mathbb{Z}}$ since $H_2(\Sigma_2) \to \mathbb{Z}$ injective by exactness.
- im $i_* = 0 \Rightarrow \ker \alpha = 0 \Rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \text{im}\,\alpha = \ker \beta$
  and im $\delta = \mathbb{Z} \Rightarrow \ker \delta = \mathbb{Z} \Rightarrow$ im $\gamma = \mathbb{Z} \Rightarrow \ker \gamma = 0 \Rightarrow$ im $\beta = 0$
  $\Rightarrow \ker \beta = H_1(\Sigma_2) \Rightarrow \underline{H_1(\Sigma_2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}$

(3.) $S \subseteq \mathbb{R}^3$ ornt, embedded surface.
$\omega \in \Omega^2(S)$ s.t. $\omega_p(e_1, e_2) = 1$ for all $p \in S$, and all o.n. bases (e_1, e_2) of T_pS.
Let $n = (n_1, n_2, n_3)$ be the unit normal v.f. of S and show $\omega = n_1\, dy \wedge dz - n_2\, dx \wedge dz + n_3\, dx \wedge dy$



If (e_1, e_2) is o.n. basis for T_pS, note that $e_1 \perp n_p$ & $e_2 \perp n_p$, and furthermore, $e_1 \times e_2 = n_p = (n_1, n_2, n_3)$

Let $e_1 = (a_1, b_1, c_1)$ be an o.n. basis for $T_pS \subseteq T_p\mathbb{R}^3$
$e_2 = (a_2, b_2, c_2)$

Then $e_1 \times e_2 = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = i(b_1c_2 - c_1b_2) - j(a_1c_2 - c_1a_2) + k(a_1b_2 - b_1a_2)$

$\Rightarrow n_1 = b_1c_2 - c_1b_2,\ -n_2 = a_1c_2 - c_1a_2,\ n_3 = a_1b_2 - b_1a_2$.

So now see that $\omega_p \in \mathrm{Alt}^2(T_pS)$, hence

$$\omega_p = \alpha_p (dy \wedge dz)_p + \beta_p (dx \wedge dz)_p + \gamma_p (dx \wedge dy)_p$$

and then:

$$\begin{aligned} 1 = \omega_p(e_1, e_2) &= \alpha_p (dy \wedge dz)_p(e_1, e_2) + \beta_p (dx \wedge dz)_p(e_1, e_2) + \gamma_p (dx \wedge dy)_p(e_1, e_2) \\ &= \alpha_p \left(dy_p(e_1)dz_p(e_2) - dy(e_2)dz(e_1)\right) + \beta_p\left(dx(e_1)dz(e_2) - dx(e_2)dz(e_1)\right) \\ &\quad + \gamma_p\left(dx(e_1)dy(e_2) - dx(e_2)dy(e_1)\right) \\ &= \alpha_p(b_1c_2 - b_2c_1) + \beta_p(a_1c_2 - a_2c_1) + \gamma_p(a_1b_2 - a_2b_1) \\ &= \alpha_p(n_1) + \beta_p(-n_2) + \gamma_p(n_3) \end{aligned}$$

Recall $\|n\| = \sqrt{n_1^2 + n_2^2 + n_3^2} = 1 \Rightarrow n_1^2 + n_2^2 + n_3^2 = 1$

$$\Rightarrow \underline{\alpha_p = n_1,\ \beta_p = -n_2,\ \gamma_p = n_3}$$

④ $X = M_1 \sqcup M_2 / \partial M_1 \sim \partial M_2$

(a) $\pi_1(X)$?

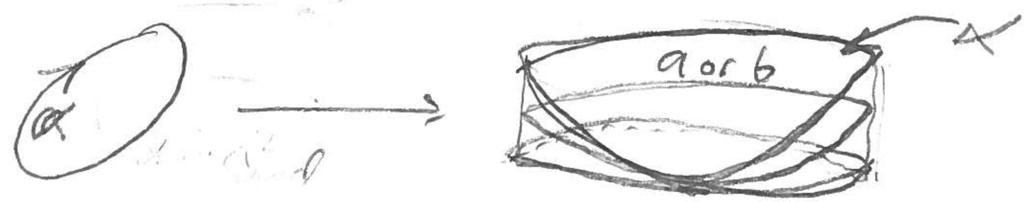
(b) $X \simeq \Sigma_g$ some g ?

(a) Let $A = M_1$, $B = M_2$, $A \cap B = \partial M_1 \cong S^1$, $A \cup B = X$.

Now consider the inclusions : $i_1 : A \cap B \hookrightarrow A$, $i_2\ A \cap B \hookrightarrow B$

See that $\pi_1(A \cap B) \cong \pi_1(S^1) = \langle \alpha \rangle$, so :

$i_1^* : \langle \alpha \rangle \longrightarrow \pi_1(A) = \mathbb{Z}\langle a \rangle$ $\qquad$ $i_2^* : \langle \alpha \rangle \longrightarrow \pi_1(B) = \mathbb{Z}\langle b \rangle$



once around α in M_i is twice around a or b

Hence, $i_1^*(\alpha) = a^2$, $i_2^*(\alpha) = b^2$ and now apply Seifert-van Kampen

$$\pi_1(X) \cong \pi_1(A) * \pi_1(B) / \langle a^2 b^{-2} \rangle$$

$$\cong \langle a \rangle * \langle b \rangle / \langle a^2 b^{-2} \rangle \cong \underline{\langle a, b : a^2 = b^2 \rangle}$$

(b) $H_1(X) = \langle a, b : a^2 = b^2 \rangle / \langle aba^{-1}b^{-1} \rangle$

$$= \langle a, b : a^2 b^{-2} = 1,\ aba^{-1}b^{-1} = 1 \rangle$$

Note that $(ab^{-1})^2 = a^2 b^{-2} = b^2 b^{-2} = 1$, hence (ab^{-1}) torsion order 2 ; and $(ab^{-1})b = a$, so group generated by $\langle ab^{-1}, b \rangle = \langle c, b : c^2 = 1,\ cbc^{-1}b^{-1} = 1 \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}$

$\rightsquigarrow$ but $H_1(M_g)$ has no torsion

(5.) Determine connected covering spaces of $\mathbb{R}P^{14} \vee \mathbb{R}P^{15}$

By the classification of connected covering spaces, the iso. classes of conn.-covers are in 1-1 correspondence with the subgroups of $\pi_1(\mathbb{R}P^{14} \vee \mathbb{R}P^{15})$.

See that $\pi_1(\mathbb{R}P^{14} \vee \mathbb{R}P^{15}) = \langle a, b : a^2 = b^2 = 1 \rangle$ by Seifert-van Kampen (i.e. $\mathbb{Z}_2 * \mathbb{Z}_2$).

Recall the universal cover of $\mathbb{R}P^n$, $n>1$, is $\pi: S^n \to \mathbb{R}P^n$.

So now:

Subgroup	Cover
$\langle e \rangle$	$S^{14} \vee S^{15}$
$\langle a \rangle$	$\mathbb{R}P^{14} \vee S^{15}$
$\langle b \rangle$	$S^{14} \vee \mathbb{R}P^{15}$
$\langle ab \rangle$	

7:

Frobenius: distribution involutive $\Rightarrow$ integrable

WTS: $D = \amalg D_p = \amalg \operatorname{span}(X_p, Y_p)^{\perp}$ involutive $\quad X_p, Y_p \in T_p(\mathbb{R}^4 \setminus \{0\})$

[$D \subseteq TM$ involutive $\Leftrightarrow$ ~~If $\eta \in \Omega^1(M)$ annihilates D on an open $U \subseteq M$, then $d\eta$ also annihilates D on U.~~]

If $V, W \in D$, then $[V,W] \in D$ $\rightsquigarrow$ D involutive.

$X = (x_1, x_2, x_3, x_4) \qquad X_1 = x_1,\ X_2 = x_2,\ X_3 = x_3,\ X_4 = x_4$

$Y = (-x_2, x_1, -x_4, x_3) \qquad Y_1 = -x_2,\ Y_2 = x_1,\ Y_3 = -x_4,\ Y_4 = x_3$

$$[X,Y] = \sum_{j=1}^{4}\left(\sum_{i=1}^{4}\left[X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i}\right]\right)\frac{\partial}{\partial x_j}$$

$$[X,Y]_j = \underbrace{\left(x_1 \frac{\partial Y^j}{\partial x_1} + x_2 \frac{\partial X^j}{\partial x_1}\right)}_{i=1} + \underbrace{\left(x_2 \frac{\partial Y^j}{\partial x_2} - x_1 \frac{\partial X^j}{\partial x_2}\right)}_{i=2} + \left(x_3 \frac{\partial Y^j}{\partial x_3} + x_4 \frac{\partial X^j}{\partial x_3}\right)$$

$$+ \left(x_4 \frac{\partial Y^j}{\partial x_4} - x_3 \frac{\partial X^j}{\partial x_4}\right)$$

$[X,Y]_1 = (x_1(0) + x_2(1)) + (x_2(-1) - x_1(0)) + (x_3(0) + x_4(0)) + 0 = x_2 - x_2 = 0$

$[X,Y]_2 = (x_1(1) + x_2(0)) + (0 - x_1(1)) + (0) + (0) = x_1 - x_1 = 0$

$[X,Y]_3 = (0) + (0) + (x_3(0) + x_4(1)) + (x_4(-1) + 0) = x_4 - x_4 = 0$

$[X,Y]_4 = (0) + (0) + (x_3(1) + 0) + (x_4(0) - x_3(1)) = x_3 - x_3 = 0$

so $[X,Y] = 0$.

(6.)

$f: M \to N$, $f_* X = Y$, φ

<u>Show</u>: $f^*(\mathcal{L}_Y \omega) = \mathcal{L}_X (f^* \omega)$ $\quad (f^*\omega = \omega \circ f_*)$

$$
\begin{aligned}
f^*(\mathcal{L}_Y \omega) &= f^*(d \circ i_Y(\omega) + i_Y \circ d\omega) \\
&= f^*(d \circ i_Y(\omega)) + f^*(i_Y \circ d\omega) \\
&= d(f^*(\omega(Y))) + \cancel{f^*(i_Y \circ d\omega)} \\
&= \underline{d(\omega(Y) \circ f)} + f^* \cancel{(i_{f_* X}(d\omega))}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_X (f^*\omega) &= d \circ i_X(f^*\omega) + i_X \circ d(f^*\omega) \\
&= d \circ (f^*\omega(X)) + \cancel{i_X \circ d(f^*\omega)} \\
&= d \circ (\omega(f_* X) \circ f) + \cancel{i_X \circ d(f^*\omega)} \\
&= \underline{d(\omega(Y) \circ f)} + i_X \cancel{\circ (f^*(d\omega))}
\end{aligned}
$$

$$
\begin{aligned}
f^*(i_Y \circ d\omega)(V) &= i_Y \circ d\omega(f_* V) \\
&= d\omega(Y, f_* V) - d\omega(f_* V, Y)
\end{aligned}
$$

$$
\begin{aligned}
i_X \circ d(f^*\omega)(V) &= i_X(f^* d\omega)(V) \\
&= f^* d\omega(X, V) - f^* d\omega(V, X) \\
&= d\omega(f_* X, f_* V) - d\omega(f_* V, f_* X) \\
&= d\omega(Y, f_* V) - d\omega(f_* V, Y)
\end{aligned}
$$

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}$$

$$\frac{\partial}{\partial x_{ij}} (\det X) = \frac{\partial}{\partial x_{ij}} \left(\sum_k (-1)^{i+k} x_{ik} \cdot \det(A_{ik}) \right)$$

$$= (-1)^{i+j} \det(A_{ij})$$

cofactor

$$(A^{-1})_{ij} = \frac{(-1)^{i+j}}{\det A} \det(A_{ji})$$

Geometry/Topology Qualifying Exam

Spring 2011

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. (10 pts) Let $S^3 = \{x \in \mathbb{R}^4 \mid \|x\| = 1\}$ be the 3-dimensional sphere, oriented as the boundary of the unit ball B^4 in $\mathbb{R}^4$ with the standard orientation. Compute $\int_{S^3} \omega$, where

$$\omega = x_1 dx_2 \wedge dx_3 \wedge dx_4 + x_2 dx_1 \wedge dx_3 \wedge dx_4 + x_3 dx_1 \wedge dx_2 \wedge dx_4.$$

(You may leave your answer in terms of volumes $vol(S^n)$ and $vol(B^n)$.)

2. (10 pts) Let $M = \{(x, y) \mid x, y \in \mathbb{R}^3, \|x\| = 1, \|y\| = 1, \langle x, y\rangle = 0\}$, where $\langle x, y\rangle$ is the standard inner product on $\mathbb{R}^3$. Show that M is a smooth compact embedded submanifold of $\mathbb{R}^6$ and explain how M can be identified with the unit tangent bundle of S^2.

check

3. (20 pts) Let $\mathbb{RP}^n$ be the real projective space given by $S^n/\sim$, where $S^n = \{\|x\| = 1\} \subset \mathbb{R}^{n+1}$ and $x \sim -x$ for all $x \in S^n$.
   (a) (5 pts) Use covering spaces to compute $\pi_1(\mathbb{RP}^n)$.
   (b) (5 pts) Give a cell (CW) decomposition of $\mathbb{RP}^n$ for $n \geq 1$.
   (c) (5 pts) Use the cell decomposition to compute the homology groups $H_k(\mathbb{RP}^n)$, $k \geq 0$.
   (d) (5 pts) For which values of $n \geq 1$ is $\mathbb{RP}^n$ orientable? Explain.

4. (10 pts) Given a continuous map $f : X \to Y$ between topological spaces, define

$$C_f = \left((X \times [0, 1]) \coprod Y \right) / \sim,$$

where $(x, 1) \sim f(x)$ for all $x \in X$ and $(x, 0) \sim (x', 0)$ for all $x, x' \in X$. Here $\coprod$ is the disjoint union. Then prove that there is a long exact sequence

$$\cdots \to H_{i+1}(X) \xrightarrow{f_*} H_{i+1}(Y) \to \tilde{H}_{i+1}(C_f) \to H_i(X) \xrightarrow{f_*} H_i(Y) \to \cdots,$$

where f_* is the map on homology induced from f and $\tilde{H}_i$ denotes the ith reduced homology group.

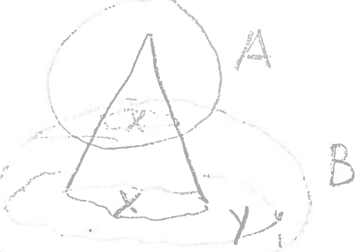
5. (10 pts) Prove that the fundamental group of a connected Lie group G is abelian. (A *Lie group* G is a smooth manifold which is also a group, and whose group operations multiplication and inverse are smooth maps.) [Hint: One possible way of proving this is to find an explicit homotopy between fg and gf, where f and g are loops in G.]

6. (10 pts) Let $M \subset \mathbb{R}^3$ be an embedded compact oriented surface (without boundary) of genus $g \geq 1$. Show that the Gaussian curvature κ of M must vanish somewhere on M.

→ means we have pt. of zero curvature

(g>0, cpt, orntble)

4 =



→ $A \cap B \simeq X$

$A \simeq pt$

$B \simeq Y$

$A \cup B \simeq C_f$

1. Stokes
2. Regular Value Theorem
3. Cellular Homology
4. Snake Lemma (short exact seq of chain complex $\rightsquigarrow$ long exact seq of homology ; Hatcher p. 116–117)
5. Let basepoint be the identity. Use $I \times I \to G$ trick
6. Gauss-Bonnett, must have a point of positive curvature.

Geometry/Topology - spring '11 :

① $S^3 \subseteq \mathbb{R}^4$ oriented as ∂B^4. Compute $\int_{S^3} \omega$ where
$\omega = x_1\, dx_2 \wedge dx_3 \wedge dx_4 + x_2\, dx_1 \wedge dx_3 \wedge dx_4 + x_3\, dx_1 \wedge dx_2 \wedge dx_4$

$$d\omega = d(x_1 dx_2 \wedge dx_3 \wedge dx_4) + d(x_2 dx_1 \wedge dx_3 \wedge dx_4) + d(x_3 dx_1 \wedge dx_2 \wedge dx_4)$$
$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + \underline{dx_2 \wedge dx_1} \wedge dx_3 \wedge dx_4 + \underline{dx_3 \wedge dx_1 \wedge dx_2} \wedge dx_4$$
$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 - \underline{dx_1 \wedge dx_2} \wedge dx_3 \wedge dx_4 + dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$
$$= dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$$

Now apply Stokes:

$$\int_{S^3} \omega = \int_{\partial B^4} \omega = \int_{B^4} d\omega = \int_{B^4} dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = \underline{\mathrm{Vol}(B^4)}.$$

② $M = \{(x,y) : x, y \in \mathbb{R}^3, \|x\| = \|y\| = 1, \langle x, y\rangle = 0\}$
Show that M is a smooth, compact, embedded submanifold of $\mathbb{R}^6$ and show that we can identify $M \sim T^1S^2$, unit tan. bundle of S^2

Consider the map $f: S^2 \times S^2 \longrightarrow \mathbb{R}$
$(x,y) \longmapsto \langle x,y\rangle$

Clearly we have $f^{-1}(0) = M$, hence M is closed subspace of compact space, hence compact.

Now apply the Regular Value Theorem; if 0 is regular value then $f^{-1}(0)$ is embedded submfld. 0 is regular if T_pf surjective $\forall p \in f^{-1}(0)$

Consider f on $\mathbb{R}^3 \times \mathbb{R}^3$ and it is given by $f(x,y) = x_1y_1 + x_2y_2 + x_3y_3$.
hence $T_pf = [y_1\ y_2\ y_3\ x_1\ x_2\ x_3]$ where $p = (x_1, x_2, x_3, y_1, y_2, y_3)$.

Recall that $p \in f^{-1}(0) \subseteq S^2 \times S^2 \Rightarrow \exists i,j$ s.t. $x_i \neq 0, y_j \neq 0$,
hence $T_pf \not\equiv 0$, hence T_pf is surjective by linearity & codim $= \mathbb{R}$

$\rightarrow$ we may make the identification $M \cong T^1S^2$ by looking at the defns:

$T^1S^2 = \{(x,v) : x \in S^2, v \in T_xS^2 \subseteq \mathbb{R}^3, \|v\| = 1\}$
$M = \{(x,y) : x, y \in \mathbb{R}^3, \|x\| = \|y\| = 1, \langle x,y\rangle = 0\}$

- $x \in S^2, v \in T_xS^2 \Leftrightarrow x \perp v \Leftrightarrow \langle x, v\rangle = 0$
- $\|v\| = 1 \Leftrightarrow v \in S^2$

③ $\mathbb{R}P^n = S^n/\sim$ where $x \sim -x$ (antipodes identified)

(a) Use covering spaces to compute $\pi_1(\mathbb{R}P^n)$

• Case $n \geq 2$: The antipodal map $\alpha(x) = -x$ gives an action of $\mathbb{Z}_2$ on S^n (since $\langle \alpha \rangle \cong \mathbb{Z}_2$), hence $\mathbb{R}P^n = S^n/\mathbb{Z}_2$ the orbit space under this action.

→ Now, if each $x \in S^n$ has a nbhd $U \ni x$ st. $g(U)$ are disjoint for all $g \in \mathbb{Z}_2$, ("covering space action")

we get that:

(1) $\pi: S^n \to S^n/\mathbb{Z}_2$ normal covering space

(2) $\mathbb{Z}_2$ is the group of deck transformations of π

(3) $\mathbb{Z}_2 \cong \pi_1(S^n/\mathbb{Z}_2) / \pi_*(\pi_1(S^n))$ (since S^n path-conn, loc path-conn.)

→ Each hemisphere of S^n is disjoint from its image under α, the generator for $\mathbb{Z}_2$ (i.e. id(hemisphere) $\cap$ α(hemisphere) $= \emptyset$), hence we get statement (3), i.e. $\mathbb{Z}_2 \cong \pi_1(S^n/\mathbb{Z}_2) / \pi_*(\pi_1(S^n)) \cong \pi_1(\mathbb{R}P^n) / \pi_*(1) \cong \pi_1(\mathbb{R}P^n)$

$\Rightarrow \underline{\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2}$.

• Case $n = 1$: In this case $\mathbb{R}P^1$ & S^1 are homeomorphic, hence

$\pi_1(\mathbb{R}P^1) \cong \pi_1(S^1) \cong \mathbb{Z}$ $\Rightarrow$ $\underline{\pi_1(\mathbb{R}P^1) \cong \mathbb{Z}}$

ALT. ▷ explicit version for $n \geq 2$:

Let g be a path in S^n connecting x and $\alpha(x)$; recall $\pi: S^n \to \mathbb{R}P^n$

then define $\gamma = \pi \circ g$, a loop in $\mathbb{R}P^n$ (since $\alpha(x) \sim x$ here) that generates $\pi_1(\mathbb{R}P^n)$

Now consider γ^2 and lift it to S^n:

$$\begin{tikzcd} & S^n \arrow[d, "\pi"] \\ {[0,2]} \arrow[ur, "\tilde{\gamma}^2"] \arrow[r, "\gamma^2"'] & \mathbb{R}P^n \end{tikzcd}$$

$\tilde{\gamma}^2$ is a loop in S^n since it goes from x to $\alpha(x)$ to $\alpha(\alpha(x)) = x$, hence it is homotopic to const. since $\pi_1(S^n) = 1$, hence

By the uniqueness of lifting, γ^2 must also be homotopic to const,

hence $\gamma^2 \simeq$ const $\Rightarrow$ γ order 2 $\Rightarrow \pi_1(\mathbb{R}P^n) \cong \mathbb{Z}_2$

(b) Give cell decomposition of $\mathbb{R}P^n$ for $n \geq 1$.

$\mathbb{R}P^n$ has a cell structure with 1 cell of each dimension $\leq n$.
See that $\mathbb{R}P^n = \mathbb{R}P^{n-1} \cup_p D^n$ with attaching map the two-sheeted covering $p : \partial D^n = S^{n-1} \to \mathbb{R}P^{n-1}$

(c) Use the cell decomposition to compute $H_k(\mathbb{R}P^n)$, $k \geq 0$

The cellular chain complex $C_k(\mathbb{R}P^n)$ has one cell e^k in each dim, $k \leq n$
hence $C_k(\mathbb{R}P^n) \cong \mathbb{Z}\{e_k\}$ for each $0 \leq k \leq n$
we know e^k has attaching map $p_k : \underbrace{S^{k-1}}_{\text{boundary of } e^k} \to \mathbb{R}P^{k-1}$ (2-sheeted cover)

→ apply Cellular Boundary Formula:
$d_k(e^k) = \sum_\beta d_\beta e_\beta^{n-1}$ where $d_\beta = \deg(\partial(e^k) = S^{k-1} \xrightarrow{\text{attaching map of } e^k} \underbrace{X^{k-1}}_{(k-1)\text{skeleton}} \xrightarrow{\text{quotient sending } X^{k-1} - e_\beta^{k-1} \text{ to pt}} S_\beta^{k-1} = e_\beta^{k-1})$

Recall that $\mathbb{R}P^n$ has k-skeleton the $\mathbb{R}P^k$, $k \leq n$.
So now to find $d_k(e^k) = d_\beta e^{k-1}$, we need to find d_β,
the degree of $S^{k-1} \xrightarrow{p_k} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k-1}/\mathbb{R}P^{k-2} \simeq S^{k-1}$

attaching map of e^k

quotient sending $X^{k-1} - e^{k-1} \cong \mathbb{R}P^{k-1} - e^{k-1} \cong \mathbb{R}P^{k-2}$ to point

Now let $N, S \subseteq S^{k-1}$ be the hemispheres of S^{k-1}.
Then $q \circ p_k|_N$ & $q \circ p_k|_S$ are homeomorphisms such that $q \circ p_k \circ \alpha|_N = q \circ p_k|_S$, (antipode of S^{k-1})
hence $\deg(q \circ p_k) = \deg(q \circ p_k|_N) + \deg(q \circ p_k|_S) = \deg(\mathrm{id}) + \deg(\alpha)$
$= (-1) + (-1)^k$

degree computed from local degree ~> Prop 2.30 Hatcher

$$\Rightarrow d_k(e^k) = \begin{cases} 2e^{k-1}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

Fact: $\deg \alpha = (-1)^k$ for S^{k-1}

So we have the chain complexes:

$$n \text{ even} \quad 0 \to \mathbb{Z} \xrightarrow{\cdot 2 = d_n} \mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} 0$$

$$n \text{ odd} \quad 0 \to \mathbb{Z} \xrightarrow{\cdot 0 = d_n} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \cdots \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 0} 0$$

$$\Rightarrow \boxed{H_k(\mathbb{RP}^n) = \ker d_k / \operatorname{im} d_{k+1} \cong \begin{cases} \mathbb{Z} & k=0 \text{ or } n \text{ odd} \\ \mathbb{Z}_2 & k \text{ odd}, \ 0<k<n \\ 0 & \text{otherwise} \end{cases}}$$

(d) Prove $\mathbb{RP}^n$ orientable $\Leftrightarrow$ n odd.

First, note that $\alpha: \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ (antipode) is given by $\alpha = \operatorname{diag}(-1)$, hence orientation preserving $\Leftrightarrow$ $0 < \det \alpha = (-1)^{n+1} \Leftrightarrow n$ odd

($\Rightarrow$) $\mathbb{RP}^n$ orientable: Recall the quotient map $\pi: S^n \to \mathbb{RP}^n$

The orientation of $\mathbb{RP}^n$ induces orientation on S^n via π ...

$(v_1, \dots, v_n) \subseteq T_p S^n$ positively oriented basis if $(T_p\pi(v_1), \dots, T_p\pi(v_n)) \subseteq T_{\pi(p)}\mathbb{RP}^n$ positively oriented.

Suppose n even; recall that α orientation reversing in this case, but now see that

$(T_p\pi(v_1), \dots, T_p\pi(v_n))$ and $(T_p\pi(\alpha(v_1)), \dots, T_p\pi(\alpha(v_n)))$

have the same orientation since $\alpha = id$ on $\mathbb{RP}^n$.

Therefore $(\alpha(v_1), \dots, \alpha(v_n))$ & $(v_1, \dots, v_n)$ have same orientation, which is a contradiction since n is even; thus n must be odd.

($\Leftarrow$) n odd; now we want to put an orientation on $\mathbb{RP}^n$

Define $(w_1, \dots, w_n) \subseteq T_{\pi(p)}\mathbb{RP}^n$ to be positively-oriented if $\exists$ positively-oriented basis $(v_1, \dots, v_n) \subseteq T_p S^n$ such that $(T_p\pi(v_1), \dots, T_p\pi(v_n)) = (w_1, \dots, w_n)$.

This is well-defd: Let $(v_1, \dots, v_n)$ & $(v_1', \dots, v_n')$ be orient. bases for T_pS^n, $T_{-p}S^n$ (resp.) in the same orient. class. Now recall $\pi \circ \alpha = \pi$, hence:

$$T_p\pi(v_i) = T_p(\pi \circ \alpha)(v_i) = T_{\alpha(p)}\pi \circ T_p\alpha(v_i) = T_{-p}\pi \circ T_p\alpha(v_i).$$

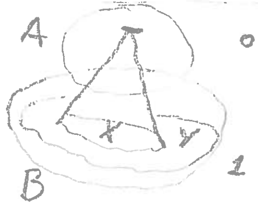
n odd $\Rightarrow$ α orient. pres. $\Rightarrow$ $T_p\alpha(v_i)$ & (v_i') same orient. class on $T_{-p}S^n$

$\Rightarrow T_p\pi(v_i)$ & $T_{-p}\pi(v_i')$ same orient. class on $T_{\pi(p)}\mathbb{RP}^n$.

④ $f: X \to Y$; $C_f = (X \times [0,1]) \amalg Y / \sim$ where $(x,1) \sim f(x)$, $(x,0) \sim (x',0)$ $\forall x, x' \in X$

Show $\exists$ long exact sequence

$$\cdots \to H_{i+1}(X) \xrightarrow[f_*]{} H_{i+1}(Y) \longrightarrow \tilde{H}_{i+1}(C_f) \longrightarrow H_i(X) \xrightarrow[f_*]{} H_i(Y) \longrightarrow \cdots$$

$C_f =$ 

Let $A = X \times [0, \frac{1}{2}+\varepsilon] / \sim$ Then $A \cup B = C_f$

$B = (X \times [\frac{1}{2}-\varepsilon, 1] \amalg_f Y)$

Hence $A \cap B = X \times [-\varepsilon, \varepsilon] \simeq X$ and $A \simeq X \times \{0\} \simeq pt.$ by $\sim$.

Also, $B \simeq (X \times \{1\}) \amalg Y \simeq (f(X)) \amalg Y \simeq Y$ since $f(X) \subseteq Y$.

Now apply Mayer-Vietoris:

$$\cdots \to H_i(A \cap B) \to H_i(A) \oplus H_i(B) \to H_i(A \cup B) \to H_{i-1}(A \cap B) \to \cdots$$

$$\Rightarrow \begin{cases} \cdots \to H_i(X) \to H_i(pt.) \oplus H_i(Y) \to H_i(C_f) \to H_{i-1}(X) \to \cdots \\ \cdots \to H_0(X) \to H_0(pt.) \oplus H_0(Y) \to H_0(C_f) \to 0 \end{cases}$$

$$\Rightarrow \begin{cases} \cdots \to H_i(X) \to H_i(Y) \to H_i(C_f) \to H_{i-1}(X) \to \cdots \\ \cdots \to H_0(X) \to \mathbb{Z} \oplus H_0(Y) \to H_0(C_f) \to 0 \end{cases}$$

$$\Rightarrow \begin{cases} \cdots \to H_i(X) \to H_i(Y) \to \tilde{H}_i(C_f) \to H_{i-1}(X) \to \cdots \\ \cdots \to H_0(X) \to H_0(Y) \to \tilde{H}_0(C_f) \to 0 \end{cases}$$

recall: $H_i(C_f) \cong \tilde{H}_i(C_f)$, $i > 0$

$H_0(C_f) = \tilde{H}_0(C_f) \oplus \mathbb{Z}$.

and furthermore:

$$\cdots \longrightarrow H_i(A\cup B) \xrightarrow{i^*} \cancel{H_i(A)}^{\nearrow 0} \oplus H_i(B) \longrightarrow H_i(A\cup B) \longrightarrow \cdots$$

where $i : A\cup B \hookrightarrow B$.

Now note that $A \cup B$ def. retracts to X and B def. retracts to Y.

So then:

$$\begin{tikzcd} A\cup B \arrow[r, hook] \arrow[d, "r_1"'] & B \arrow[d, "r_2"] \\ X \arrow[r] & Y \end{tikzcd}$$

will have commutative induced homology diagram since r_i are def. retractions (i.e. r_i^* are isomophisms):

$$\begin{tikzcd} H_i(A\cup B) \arrow[r, "i^*"] \arrow[d, "r_1^*"', "\wr"] & H_i(B) \arrow[d, "r_2^*", "\wr"'] \\ H_i(X) \arrow[r, "f^*"'] & H_i(Y) \end{tikzcd}$$

Hence $(r_2^*)^{-1} \circ f \circ r_1^* = i^*$, hence these maps agree since r_i^* are isomophisms and $Y \subseteq B$,

hence

$$\cdots \longrightarrow H_i(X) \xrightarrow{f^*} H_i(Y) \longrightarrow \tilde{H}_i(C_f) \longrightarrow \cdots$$

as was desired.

(5) Prove that the fundamental group of a connected Lie group G is abelian:

(WLOG)

Let f, g be loops in G based at e, the identity elt.
We will show that $fg \simeq gf$ (homotopic) by writing an explicit homotopy.

Recall $f, g : I \to G$; now define $F : I \times I \to G$, $(s,t) \mapsto f(s)g(t)$ ← group multiplication is continuous, hence so is F.

See that
$$F(t,0) = F(t,1) = f(t)$$
$$F(1,t) = F(0,t) = g(t)$$

So we want to construct a homotopy from $F(t,0)\cdot F(1,t) = f\cdot g(t)$ to $F(0,t)\cdot F(t,1) = g\cdot f(t)$:

$$f\cdot g(t) = F(t,0)\cdot F(1,t) = \begin{cases} F(2t,0), & 0 \le t \le \frac{1}{2} \\ F(1, 2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

$$g\cdot f(t) = F(0,t)\cdot F(t,1) = \begin{cases} F(0,2t), & 0 \le t \le \frac{1}{2} \\ F(2t-1, 1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Then see that
$$G_s(t) = \begin{cases} F(2t(1+s), 2ts), & 0 \le t \le \frac{1}{2} \\ F((1-s) + (2t-1)s, (1-s)(2t-1) + s), & \frac{1}{2} \le t \le 1 \end{cases}$$
a homotopy between the two. - ?

~~or more simply: $G_s(t) = F((1-s)t, st)\cdot F((1-s)+st, s+(1-s)t)$,~~
~~which gives $G_0(t) = F(t,0)\cdot F(1,t) = f\cdot g$~~
~~$G_1(t) = F(0,t)\cdot F(t,1) = g\cdot f$~~

ok but this is loop composition

6.) $M \subseteq \mathbb{R}^3$ embedded, cpt, oriented surface, genus $g \geq 1$.
Show that Gaussian curvature K of M must be 0 somewhere on M:

Since M is cpt, orientable, positive genus, $\exists$ pt of positive curvature, i.e. $K > 0$ somewhere on M.
Now apply Gauss-Bonnet:

$$\int_M K\,dA = 2\pi\chi(M) = 2\pi(2-2g)$$
$$= 4\pi(1-g)$$
$$\leq 0$$

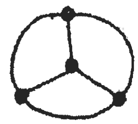
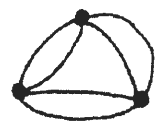
since $\chi(M) = 2-2g$ for genus g surface.

Hence K must be negative for some part of M, and then by the Intermediate Value Theorem, $\exists p \in M$ s.t. $K(p) = 0$.

Incomplete
3.

Geometry and Topology Graduate Exam
Fall 2010

Problem 1. Compute the fundamental groups of the following two graphs:

$X_1 =$  $X_2 =$ 

Problem 2. Let P_1, P_2, P_3 be three distinct points in the sphere S^2, and let X be the topological space obtained from S^2 by gluing these three points together. Compute all homology groups $H_n(X;\mathbb{Z})$.

0

Problem 3. Define the Gaussian (or scalar) curvature $\kappa(p)$ of an immersed surface Σ in $\mathbb{R}^3$ at the point p. Does there exist a compact immersed surface Σ without boundary in $\mathbb{R}^3$ which has $\kappa(p) = -1$ for all $p \in \Sigma$?

Problem 4. Let $M_n(\mathbb{R})$ be the set of $n \times n$ matrices with real entries. Prove that the *orthogonal group* $O(n) = \{A \in M_n(\mathbb{R}) | AA^T = \mathrm{id}\}$ is a smooth manifold. What is its dimension?

Problem 5. Let $\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ be a differential form such that

$$d\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$$

where $x_1, x_2, \ldots, x_n$ are the standard coordinates of $\mathbb{R}^n$. Show that, for every $p \in \mathbb{R}$, the differential form

$$\alpha = \frac{1}{(x_1^2 + x_2^2 + \cdots + x_n^2)^p}\omega \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$$

is not exact. Possible hint: S^{n-1}.

Problem 6. Consider the 2-form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$ with coordinates x_1, $y_1, \ldots, x_n, y_n$. If f is a smooth function on $\mathbb{R}^{2n}$, find the vector field X such that $i_X\omega = df$, where i_X denotes the interior product. Then compute the Lie derivative $\mathcal{L}_X\omega$.

Problem 7. Let X be a topological space such that the homology group $H_p(X;\mathbb{Z})$ is finite and such that the cohomology group $H^{p+1}(X;\mathbb{Q})$ is equal to 0. Let $u \in C^{p+1}(X;\mathbb{Z}) = \mathrm{Hom}(C_{p+1}(X;\mathbb{Z}),\mathbb{Z})$ be a cochain with $du = 0$.

a. Show that, for every $\alpha \in C_p(X,\mathbb{Z})$ with $\partial\alpha = 0$, there exists $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$.

b. Show that there exists a homomorphism

$$L_u : H_p(X;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

such that

$$L_u([\alpha]) = \frac{1}{k}u(\beta)$$

for every $k \in \mathbb{Z} - \{0\}$ and $\beta \in C_{p+1}(X;\mathbb{Z})$ with $k\alpha = \partial\beta$. Namely, show that $L_u([\alpha])$ is independent of k, β and of the representative α of $[\alpha] \in H_p(X;\mathbb{Z})$.

$\partial\alpha = 0$
cycle
$\exists\beta$ s.t. $\partial\beta = \alpha$
boundary.

Check

Q

1. Quotienting by contractible subcomplexes is a homotopy equivalence, so
$X_1 \sim S^1 \vee S^1 \vee S^1$ and $X_2 \sim S^1 \vee S^1 \vee S^1 \vee S^1$

2. Same as 1. $X \sim S^2 \vee S^1 \vee S^1$.

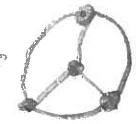
3. Product of principal curvatures (intersecting planes containing normal vector)
Gauss Bonnet.

4. Regular Value Theorem

5. Top dimensional De Rham Cohomology isomorphism. Stokes.

6. $i_X(\alpha) = \alpha(X)$ for 1-forms. $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_X \beta$.

7. $u = \delta v$ some $v \in C^p(X;\mathbb{Z})$. Just unwind cohomology definitions.
$k = |H_p(X;\mathbb{Z})|$.

① Find $\pi_1(X_1)$ & $\pi_1(X_2)$ where $X_1 =$ , $X_2 =$ 

Recall that the quotient under a contractible subcomplex is homotopy equivalent to original space.

So then: $X_1 \simeq X_1 /$  $\simeq$  $= S^1 \vee S^1 \vee S^1$.

$\Rightarrow \pi_1(X_1) \cong \pi_1(S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

and: $X_2 \simeq X_2 /$  $\simeq$  $\simeq$  $= S^1 \vee S^1 \vee S^1 \vee S^1$

$\Rightarrow \pi_1(X_2) \cong \pi_1(S^1 \vee S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

(2) $P_1, P_2, P_3 \in S^2$, $X = S^2/_{P_1 \sim P_2 \sim P_3}$. Find $H_n(X)$:

$X =$  $\simeq$  $= S^2 \vee S^1 \vee S^1$

So we want $H_n(S^2 \vee S^1 \vee S^1)$; we'll apply Mayer-Vietoris:

Let $A \simeq S^2$, $B \simeq S^1 \vee S^1$, $A \cap B \simeq pt$, $A \cup B = S^2 \vee S^1 \vee S^1$.

Then we have:

$$H_n(A) \cong H_n(S^2) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases} \quad \text{and} \quad H_n(B) \cong H_n(S^1 \vee S^1),$$

hence $H_n(B) = 0$ for $n > 1$, $H_1(B) = \pi_1(B)/[\pi_1(B), \pi_1(B)] = \mathbb{Z} * \mathbb{Z}/(\mathbb{Z} * \mathbb{Z})' \cong \mathbb{Z} \oplus \mathbb{Z}$

and $H_0(B) \cong \mathbb{Z}$ since path-connected.

Now we have exact sequence:

$H_2(A \cap B) \to H_2(A) \oplus H_2(B) \to H_2(A \cup B) \to H_1(A \cap B) \to H_1(A) \oplus H_1(B)$

$\to H_1(A \cup B) \to H_0(A \cap B) \to H_0(A) \oplus H_0(B) \to H_0(A \cup B) \to 0$

$\Rightarrow \underline{0 \to \mathbb{Z} \to H_2(X) \to 0} \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} H_1(X) \xrightarrow{\beta} \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta} H_0(X) \to 0$

$\Rightarrow \underline{H_2(X) \cong \mathbb{Z}}$ by exactness

$\to \mathbb{Z} \oplus \mathbb{Z} = \operatorname{im} \alpha = \ker \beta$

and $\underline{\mathbb{Z} \cong H_0(X)} = \operatorname{im} \delta \Rightarrow \mathbb{Z} = \ker \delta = \operatorname{im} \gamma \Rightarrow 0 = \ker \gamma = \operatorname{im} \beta$

by path-connectedness

$\Rightarrow \underline{H_1(X) = \mathbb{Z} \oplus \mathbb{Z}}$

(can also get H_1 via $\pi_1 \ldots$)

(4.) Show $O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : AA^T = I_n\}$ smooth mfld. Dimension?

We'll apply regular value theorem:

Consider $f: M_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

$$A \longmapsto AA^T$$

Clearly $f^{-1}(I_n) = O_n(\mathbb{R})$, hence we need to show I_n a regular value;

First note that $X \in \operatorname{im} f$ has $X = AA^T \Rightarrow X^T = (AA^T)^T = A^{TT}A^T = AA^T = X$, hence $\operatorname{im} f \subseteq Sym_n(\mathbb{R})$, so restrict $f: M_n(\mathbb{R}) \to Sym_n(\mathbb{R})$ and choose $A \in f^{-1}(I_n)$ and consider:

$$T_A f : T_A M_n(\mathbb{R}) \to T_{I_n}(Sym_n(\mathbb{R}))$$

Let $B \in T_A M_n(\mathbb{R})$ and $\gamma(t) = A + tB$. Then:

$$T_A f(B) = (f \circ \gamma)'(0) = \frac{d}{dt}(f(\gamma(t)))\Big|_{t=0} = \frac{d}{dt} f(A+tB)\Big|_{t=0}$$

$$= \frac{d}{dt}\left((A+tB)^T(A+tB)\right)\Big|_{t=0} = \frac{d}{dt}\left(I_n + tA^TB + tB^TA + t^2B^TB\right)\Big|_{t=0}$$

$$= A^TB + B^TA + 2tB^TB\Big|_{t=0} = A^TB + B^TA$$

Recall that $T_{I_n}(Sym_n(\mathbb{R})) \cong Sym_n\mathbb{R}$, and if $C \in Sym_n\mathbb{R}$, we have:

$A^TA = I_n$ since $A \in f^{-1}(I_n)$

$$C = \frac{1}{2}C + \frac{1}{2}C = \frac{1}{2}C^T + \frac{1}{2}C = \frac{1}{2}C^TA^TA + \frac{1}{2}A^TAC$$

$$= \left(\frac{1}{2}C^TA^T\right)A + A^T\left(\frac{1}{2}AC\right)$$

$$= T_Af\left(\frac{1}{2}AC\right)$$

hence $\operatorname{im} T_Af = Sym_n\mathbb{R}$,

hence T_Af surjective onto $Sym_n\mathbb{R}$, hence $A \in f^{-1}(I_n)$ regular point, hence I_n reg. value, hence $f^{-1}(I_n) = O_n$ submfld.

Now recall that elements of $\mathrm{Sym}_n\mathbb{R}$ are determined by entries on & above main diagonal, i.e.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \text{ entries, i.e. } \dim \mathrm{Sym}_n\mathbb{R} = \frac{n(n+1)}{2}$$

Hence

$$\dim O_n(\mathbb{R}) = \overset{\mathrm{dom}(f)}{\dim M_n(\mathbb{R})} - \overset{\mathrm{codom}(f)}{\dim \mathrm{Sym}_n\mathbb{R}}$$

$$= n^2 - \frac{n(n+1)}{2} = \frac{2n^2 - n^2 - n}{2}$$

$$= \frac{n^2 - n}{2} = \frac{n(n-1)}{2}$$

(5.) $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ s.t. $d\omega = dx_1 \wedge \cdots \wedge dx_n$, $p \in \mathbb{R}$.

Show that $\alpha = \dfrac{\omega}{(x_1^2 + \cdots + x_n^2)^p} \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ not exact.

Consider α restricted to $S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$

Then we see that $\alpha|_{S^{n-1}} = \omega|_{S^{n-1}} \in \Omega^{n-1}(S^{n-1})$.

Then $d\alpha \in \Omega^n(S^{n-1}) = 0$, hence $d\alpha = 0$, hence α closed on S^{n-1}.

Thus we may consider the class $[\alpha] \in H^{n-1}(S^{n-1})$ and apply the top-dim de Rham isomophism:

$$I: H^{n-1}(S^{n-1}) \longrightarrow \mathbb{R}$$
$$[\alpha] \longmapsto \int_{S^{n-1}} \alpha$$

and see that:

Stokes

$$\int_{S^{n-1}} \alpha = \int_{S^{n-1}} \omega = \int_{\partial B^n} \omega = \int_{B^n} d\omega = \int_{B^n} dx_1 \wedge \cdots \wedge dx_n$$

$$= \mathrm{vol}(B^n) \neq 0.$$

hence $[\alpha] \neq 0$ in $H^{n-1}(S^{n-1})$, !

hence α restricted to S^{n-1} is not exact.

hence α is not exact on all of $\mathbb{R}^n \setminus \{0\}$.

(6.) $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \in \Omega^2(\mathbb{R}^{2n})$ where $(x_1, y_1, \dots, x_n, y_n)$ coord. on $\mathbb{R}^{2n}$. If $f: \mathbb{R}^{2n} \to \mathbb{R}$ smooth, find $X \in \mathfrak{X}(\mathbb{R}^{2n})$ such that $i_X\omega = df$, where i_X interior product.

Then find Lie derivative $\mathcal{L}_X\omega$?

The interior product $i_X : \Omega^k(\mathbb{R}^{2n}) \to \Omega^{k-1}(\mathbb{R}^{2n})$ has the properties: (1) for $\omega \in \Omega^1(\mathbb{R}^{2n})$, $i_X(\omega) = \omega(X)$

(2) $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge i_X\beta$.

See that $i_X(dx_i \wedge dy_i) = i_X(dx_i) \wedge dy_i - dx_i \wedge i_X dy_i$

$= dx_i(X)\, dy_i - dy_i(X)\, dx_i$

Now let

$$X = \sum_{i=1}^{n} \left(a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i} \right)$$

and see that $dx_j(X) = \sum_{i=1}^{n} \left(a_i\, dx_j\left(\frac{\partial}{\partial x_i}\right) + b_i\, dx_j\left(\frac{\partial}{\partial y_i}\right) \right)$

$= a_j$ and similarly, $dy_j(X) = b_j$

So we have:

$i_X(dx_i \wedge dy_i) = a_i\, dy_i - b_i\, dx_i$

$= -b_i\, dx_i + a_i\, dy_i$.

Hence $i_X(\omega) = i_X\left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = \sum_{i=1}^{n} (-b_i\, dx_i + a_i\, dy_i)$

$df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i \right)$

hence $\frac{\partial f}{\partial x_i} = -b_i$

$\frac{\partial f}{\partial y_i} = a_i$ $\Rightarrow$ $X = \sum_{i=1}^{n} \left[\left(\frac{\partial f}{\partial y_i}\right) \frac{\partial}{\partial x_i} - \left(\frac{\partial f}{\partial x_i}\right) \frac{\partial}{\partial y_i} \right]$

Now $X = \sum_{i=1}^{n} \left[\left(\frac{\partial f}{\partial y_i}\right)\frac{\partial}{\partial x_i} - \left(\frac{\partial f}{\partial x_i}\right)\frac{\partial}{\partial y_i}\right]$

$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \in \Omega^2(\mathbb{R}^{2n})$

and recall $\mathcal{L}_X : \Omega^2(\mathbb{R}^{2n}) \to \Omega^2(\mathbb{R}^{2n})$

$\omega \longmapsto (d \circ i_X + i_X \circ d)(\omega)$

So $\mathcal{L}_X(\omega) = d \circ i_X(\omega) + i_X \circ d(\omega)$

$= d(\cancel{df})^{0} + i_X(d\omega) = i_X(d\omega)$

$= \cancel{d\omega(X)}$ ← only true for ω is 1-form - interior.

But $d\omega = d\left(\sum_{i=1}^{n} dx_i \wedge dy_i\right) = \sum_{i=1}^{n} d(dx_i \wedge dy_i)$

$= \sum_{i=1}^{n} \overset{\to 0}{d(dx_i)} \wedge dy_i - dx_i \wedge \overset{\to 0}{d(dy_i)}$

$= 0$

hence $\underline{\mathcal{L}_X(\omega) = 0}$.

tangent vector ⟷ vector field, same as derivation

$\boxed{\text{\underline{note}: Lie derivative } \mathcal{L}_X : \Omega^k(M) \to \Omega^k(M) \text{ naturally extends derivation } X : \Omega^0(M) \to \Omega^0(M)}$

⑦ X space s.t. $|H_p(X,\mathbb{Z})| < \infty$ & $H^{p+1}(X,\mathbb{Q}) = 0$.
Let $u \in C^{p+1}(X,\mathbb{Q}) = \mathrm{Hom}(C_{p+1}(X,\mathbb{Z}),\mathbb{Q})$ cochain with $du = 0$:

(a) Show that, for every $\alpha \in C_p(X,\mathbb{Z})$ with $\partial\alpha = 0$, $\exists\, k \in \mathbb{Z}\setminus\{0\}$ and $\beta \in C_{p+1}(X,\mathbb{Z})$ with $k\alpha = \partial\beta$.

If $\alpha \in C_p(X)$ has $\partial\alpha = 0$, the homology class $[\alpha] \in H_p(X)$ is well defined.
Now, since $|H_p(X)| < \infty$ it is a torsion $\mathbb{Z}$-module, hence $\exists\, k$ s.t. $k[\alpha] = 0$, namely $k = |H_p(X)|$.
That means that $k[\alpha] = [k\alpha]$ is a boundary, namely $\exists\, \beta \in C_{p+1}(X)$ s.t. $\partial\beta = k\alpha$.

(b) Show $\exists$ hom. $L_u : H_p(X,\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$ with $L_u([\alpha]) = \frac{1}{k}u(\beta)$ for every $k \in \mathbb{Z}\setminus\{0\}$, $\beta \in C_{p+1}(X)$ with $\partial\beta = k\alpha$

---

we're given that $du = 0$, hence $[u] \in H^{p+1}(X,\mathbb{Q}) = 0$ is well defined, but clearly $[u] = 0$, hence u is a coboundary, hence $\exists\, v \in C^p(X)$ such that $dv = u$.
Now, since $d = \partial^*$, we have:

$$L_u([\alpha]) = \frac{u(\beta)}{k} = \frac{dv(\beta)}{k} = \frac{v(\partial\beta)}{k} = \frac{v(k\alpha)}{k} = v(\alpha) \in \mathbb{Q}$$

But see that $nk(L_u([\alpha])) = L_u(nk[\alpha]) = L_u(0) = 0$,
hence $L_u([\alpha]) \in \mathbb{Q}/\mathbb{Z}$.

Incomplete: 7.

Geometry and Topology Graduate Exam
Fall 2009

Problem 1. Let $f : M \to N$ be a map between two compact oriented manifolds of the same dimension. Suppose that the subgroup $f^*(\pi_1(M))$ has finite index in $\pi_1(N)$.

a. Show that the index $[\pi_1(N) : f^*(\pi_1(M))]$ divides the degree of f.

b. Give an example where $[\pi_1(N) : f^*(\pi_1(M))]$ is different from the degree of f.

Problem 2. Is there a differentiable map $\mathbb{R}^2 \to \mathbb{R}^2$ that sends the vector field $\frac{\partial}{\partial x}$ to the vector field $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and sends the vector field $\frac{\partial}{\partial y}$ to the vector field $Y = -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$?

Problem 3. Let $f : S^n \to S^n$ be a degree 5 map from the sphere S^n to itself.

a. Show that there exists $x_1 \in S^n$ such that $f(x_1) = -x_1$.

b. Show that there exists $x_2 \in S^n$ such that $f(x_2) = x_2$.

Problem 4. Let M be a compact submanifold of $\mathbb{R}^n$, of dimension at most $n-3$, and let $f : B^2 \to \mathbb{R}^n$ be a differentiable map from the 2-dimensional ball (or disk) B^2 to $\mathbb{R}^n$. Let $T_v : \mathbb{R}^n \to \mathbb{R}^n$ denote the translation along the vector $v \in \mathbb{R}^n$.

a. Show that there exists arbitrarily small vectors $v \in \mathbb{R}^n$ such that the image of $T_v \circ f$ is disjoint from M.

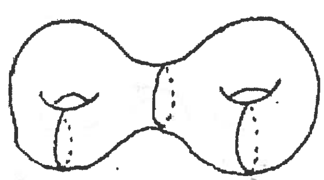
b. Conclude that the complement $\mathbb{R}^n - M$ is simply connected.

Problem 5. Let ω be a closed form of degree n on $\mathbb{R}^{n+1} - \{0\}$. Show that, for any two differentiable maps $f, g : S^n \to \mathbb{R}^{n+1} - \{0\}$, the ratio

$$\frac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)}$$

is a rational number when the denominator is not 0.

Problem 6. Let S be the standard surface of genus 2 in $\mathbb{R}^3$ as in the picture below, and let W be the closure of the bounded component of $\mathbb{R}^3 - S$. Compute the relative homology groups $H_n(W, S)$.



Problem 7. Let M be a compact connected submanifold of an oriented manifold N, with $\dim M = \dim N - 1$. Show that M is orientable if and only if it admits arbitrarily small connected neighborhoods U such that $U - M$ is disconnected. Namely, if and only if, for every open subset $V \subset N$ containing M, there is a connected open subset $U \subset V$ such that $U - M$ is not connected.

1. Lifting Criterion, Multiplicativity of degrees, $p: \tilde{N} \to N$ finite sheeted cover and N compact $\Rightarrow$ $\tilde{N}$ compact.

2 f acts on a vector space by its tangent map.

3. If f fixes no point, f is homotopic to the antipode which has degree ± 1. If f sends no point to its antipode, $f \sim id$ so $\deg f = 1$.

4. Sard (Consider $F: M \times B^2 \to \mathbb{R}^n$ $F(x,y) = x - f(y)$)

5. Degree (Must define a retraction from $\mathbb{R}^{n+1} - \{0\}$ to S^n).

6. Relative Homology Long Exact Sequence.

7. Tubular neighborhoods ($T(M) \cong M \times I$ if M is oriented, so $T(M) - M \cong M \times (-\varepsilon, 0) \cup (0, \varepsilon)$)

Geometry / Topology - Fall 09:

① $f: M \to N$ map between compact, oriented manifolds, $\dim M = \dim N$.
Suppose $f^*(\pi_1(M)) \leq \pi_1(N)$ has finite index. So degree is well-defined. 10

(a) Show that $[\pi_1(N) : f^*(\pi_1(M))] \mid \deg f$.

Let $p: \tilde{N} \to N$ be the cover corresponding to the subgroup $f^*(\pi_1(M)) \leq \pi_1(N)$. By construction, the number of sheets of p is $[\pi_1(N) : f^*(\pi_1(M))]$, i.e. $\deg p = [\pi_1(N) : f^*(\pi_1(M))] < \infty$

Because of our construction we have $f_*(\pi_1(M)) = p_*(\pi_1(\tilde{N}))$, hence the lifting criterion is satisfied, and we therefore have $\tilde{f}: M \to \tilde{N}$ such that $f = p \circ \tilde{f}$.

If $\deg f = 0$, statement is trivial, so assume $\deg f \neq 0$.

Then $\deg f = \deg p \circ \tilde{f} = \deg p \cdot \deg \tilde{f} \Rightarrow \underbrace{[\pi_1(N) : f^*(\pi_1(M))]}_{= \text{ no. of sheets of } \tilde{N}} = \deg p \mid \deg f$

(b) Example where $[\pi_1(N) : f^*(\pi_1(M))] < \deg f$

(2) Is there a differentiable map relating $\frac{\partial}{\partial x}$ & X, $\frac{\partial}{\partial y}$ & Y where $X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ & $Y = -\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

By the naturality of the Lie bracket, if $\frac{\partial}{\partial x}$ & X, $\frac{\partial}{\partial y}$ & Y are F-related, then $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]$ & $[X, Y]$ are F-related.

Now, see that $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] = 0$ and:

$$[X,Y] = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\right)\frac{\partial}{\partial x^i} \quad \text{where } X_1 = x,\ X_2 = 1,\ Y_1 = -1,\ Y_2 = x$$

$$= \left(X^1\frac{\partial Y^1}{\partial x} - Y^1\frac{\partial X^1}{\partial x} + X^2\frac{\partial Y^1}{\partial y} - Y^2\frac{\partial X^1}{\partial y}\right)\frac{\partial}{\partial x}$$

$$+ \left(X^1\frac{\partial Y^2}{\partial x} - Y^1\frac{\partial X^2}{\partial x} + X^2\frac{\partial Y^2}{\partial y} - Y^2\frac{\partial X^2}{\partial y}\right)\frac{\partial}{\partial y}$$

$$= (x\cdot 0 + (+1)(1) + 0 - 0)\frac{\partial}{\partial x} + (x\cdot(1) - (-1)(0) + 0 - 0)\frac{\partial}{\partial y}$$

$$= \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \neq 0,$$

hence $[X,Y]$ & $[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]$ cannot be F-related.

hence neither may their constituents.

③ $f: S^n \to S^n$ degree 5

(a) show $\exists x_1 \in S^n$ such that $f(x_1) = -x_1$

(b) show $\exists x_2 \in S^n$ such that $f(x_2) = x_2$

(a) Suppose $f: S^n \to S^n$ sends no point to its antipode, i.e. $\nexists x \in S^n$ such that $f(x) = \alpha(x)$, i.e. $\alpha \circ f(x) \neq x$ for all $x \in S^n$.

Since $\alpha \circ f: S^n \to S^n$ has no fixed points, it is homotopic to the antipodal map $\alpha: S^n \to S^n$, hence $(-1)^{n+1} = \deg \alpha = \deg \alpha \circ f = \deg \alpha \cdot \deg f$

$\Rightarrow \deg f = 1$ ↯ contradiction since $\deg f = 5$

(b) Suppose $f: S^n \to S^n$ has no fixed points

Then f is homotopic to the antipodal map, hence $\deg f = \deg \alpha = (-1)^{n+1}$ ↯ contradiction since $\deg f = 5$

---

④ $M \subseteq \mathbb{R}^n$ cpt submfld, $\dim M \leq n-3$, $f: B^2 \to \mathbb{R}^n$ diff map, $T_v: \mathbb{R}^n \to \mathbb{R}^n$, $w \mapsto w+v$, translation along $v \in \mathbb{R}^n$.

(a) Show $\exists$ arbitrarily small $v \in \mathbb{R}^n$ s.t. $T_v(\operatorname{im} f) \cap M = \phi$

(b) Show $\mathbb{R}^n \setminus M$ is simply-connected.

(a) Consider the set $\{v \in \mathbb{R}^n : T_v(\operatorname{im} f) \cap M \neq \emptyset\}$

Now see that: $\{v \in \mathbb{R}^n : T_v(\operatorname{im} f) \cap M \neq 0\} =$

$= \{v \in \mathbb{R}^n : \exists y \in B^2, x \in M \text{ s.t. } T_v(f(y)) = x\}$

$= \{v \in \mathbb{R}^n : \exists y \in B^2, x \in M \text{ s.t. } f(y) + v = x\}$

$= \{v \in \mathbb{R}^n : \exists y \in B^2, x \in M \text{ s.t. } v = x - f(y)\}$

Now define the map $F: M \times B^2 \to \mathbb{R}^n$, $(x, y) \mapsto x - f(y)$, hence ~~we~~ have $= \operatorname{im} F$

Then see that $T_pF: T_p(M \times B^2) \to T_p\mathbb{R}^n$ is a map from an at most $(n-3)+2 = n-1$ dimensional space to an n-dim space, hence cannot be surjective, hence $\operatorname{im} F$ consists of critical values, hence $\operatorname{im} F$ measure 0 by Sard

$\Rightarrow \{v \in \mathbb{R}^n : T_v(\operatorname{im} f) \cap M \neq 0\}$ measure 0

$\Rightarrow \{v \in \mathbb{R}^n : T_v(\operatorname{im} f) \cap M = 0\}$ full measure $\Rightarrow v$ can be arb. small

4 cont'd:

(b) By part (a), for any differentiable map $f: B^2 \to \mathbb{R}^n$ we can get $\text{im}(T_v \circ f) \cap M = \phi$ for arbitrarily small v.

Consider the restriction $f: S^1 \to \mathbb{R}^n$, which is a loop in $\mathbb{R}^n$. By part (a), we may translate by small v and get a loop in $\mathbb{R}^n \setminus M$, i.e. $T_v \circ f$ a loop in $\mathbb{R}^n \setminus M$. But we have actually translated all of $\text{im} f = f(B^2)$, i.e. the interior of the loop is also in $\mathbb{R}^n \setminus M$, hence it is nullhomotopic.

Since every loop is just a map $f: S^1 \to \mathbb{R}^n$ we have they are all nullhomotopic $\Rightarrow$ $\underline{\pi_1(M \setminus \mathbb{R}^n) = 1}$

(5) $\omega \in \Omega^n(\mathbb{R}^{n+1} \setminus \{0\})$ closed. 3.

Show that, for any two differentiable maps $f, g : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$, the ratio: $\dfrac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)}$ is rational when the denom $\neq 0$.

Let $r : \mathbb{R}^{n+1} \setminus \{0\} \to S^n$, $x \longmapsto \frac{x}{\|x\|}$ be the deformation retraction of $\mathbb{R}^{n+1} \setminus \{0\}$ onto S^n. Then $r \circ f, r \circ g : S^n \to S^n$, hence degree is well-defined, i.e.:

$$(r\circ f)^* : H^n(S^n) \to H^n(S^n) \quad \& \quad (r \circ g)^* : H^n(S^n) \to H^n(S^n)$$
$$[\omega] \longmapsto a[\omega] \qquad\qquad [\omega] \longmapsto b[\omega]$$

where a and b are integers.

Furthermore, we have that $(r\circ f)^*(\omega) = r^* \circ f^*(\omega) = a\omega$
$$(r \circ g)^*(\omega) = r^* \circ g^*(\omega) = b\omega$$

Recall that since r is deformation retraction, r^* is isomorphism; i.e. it is invertible; hence:

$$f^*(\omega) = (r^*)^{-1}(a\omega) = a(r^*)^{-1}(\omega)$$
$$g^*(\omega) = (r^*)^{-1}(b\omega) = b(r^*)^{-1}(\omega)$$

since homomorphism, and then:

$$\frac{\int_{S^n} f^*(\omega)}{\int_{S^n} g^*(\omega)} = \frac{\int_{S^n} a(r^*)^{-1}(\omega)}{\int_{S^n} b(r^*)^{-1}(\omega)} = \frac{a \cancel{\int_{S^n}(r^*)^{-1}(\omega)}}{b \cancel{\int_{S^n}(r^*)^{-1}(\omega)}} = \frac{a}{b}$$

a rational number since $a, b \in \mathbb{Z}$.

⑥ Let S be the surface of genus 2 and let W be the solid version. Compute $H_n(W,S)$

$S =$  , $W =$  solid

Recall the Long Exact Sequence for Relative Homology:

$$0 \to H_3(S) \to H_3(W) \to H_3(W,S) \to H_2(S) \to H_2(W) \to H_2(W,S)$$
$$\to H_1(S) \to H_1(W) \to H_1(W,S) \to H_0(S) \to H_0(W) \to H_0(W,S) \to 0$$

now, see that W = solid double torus $\simeq \infty = S^1 \vee S^1$, therefore:

$$H_n(W) \cong H_n(S^1 \vee S^1) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ 0 & n>1 \end{cases}$$

and S is the surface of genus 2, hence:

$$H_n(S) \cong H_n(M_2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases}$$

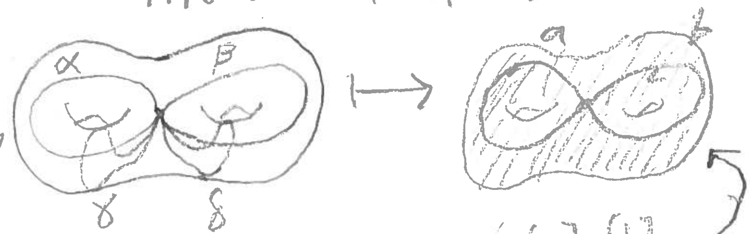
So our sequence reduces to:

$$\underline{0 \to H_3(W,S) \xrightarrow{\partial_3} \mathbb{Z} \to 0} \to H_2(W,S) \xrightarrow{\partial_2} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}\oplus\mathbb{Z}$$
$$\to H_1(W,S) \xrightarrow{\partial_1} \mathbb{Z} \to \mathbb{Z} \to H_0(W,S) \xrightarrow{\partial_0} 0$$

By exactness, $\underline{H_3(W,S) \cong \mathbb{Z}}$.

Now consider the induced map of inclusion on the first homology groups:

$$H_1(S) \xrightarrow{i_*} H_1(W)$$



α, β, γ, δ the generators of $H_1(S)$

[a], [b] generators of $H_1(W)$

see that i sends
$\alpha \mapsto a$
$\beta \mapsto b$
and γ and δ to nullhomotopic loops (these circles are included in solid double-torus)
$\Rightarrow \gamma \mapsto 1,\ \beta \mapsto 1$

Therefore we see that $\ker i_* \cong \mathbb{Z}\oplus\mathbb{Z}$ and $\operatorname{im} i_* \cong \mathbb{Z}\oplus\mathbb{Z}$

Now, by exactness, we see from $0 \to H_2(W,S) \xrightarrow{\partial_2} H_1(S)$ that ∂_2 is injective, hence $H_2(W,S) \cong \operatorname{im}\partial_2 \cong \ker i_* \cong \mathbb{Z}\oplus\mathbb{Z}$

$\Rightarrow \underline{H_2(W,S) \cong \mathbb{Z}\oplus\mathbb{Z}}$

Now consider $\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{i_*} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{j_*} H_1(W,S) \xrightarrow{\partial_1} \underset{H_0(S)}{\mathbb{Z}} \xrightarrow{i_*} \underset{H_0(W)}{\mathbb{Z}}$

We have now $\mathbb{Z}\oplus\mathbb{Z} \cong \operatorname{im} i_* \cong \ker j_* \Rightarrow 0 \cong \operatorname{im} j_* \cong \ker \partial_1$
$\Rightarrow \partial_1$ injective,
hence $H_1(W,S) \cong \operatorname{im}\partial$

Now, the inclusion map induced on the 0^{th} homology groups must be injective since both are path-connected, hence $\ker(H_0(S) \xrightarrow{i_*} H_0(W)) \cong 0$,
i.e. $H_1(W,S) \cong \operatorname{im}\partial \cong \ker i_* \cong 0$, so we have $\underline{H_1(W,S) \cong 0}$

Finally, we have:

$H_1(W,S)=0 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \xrightarrow{j_*} H_0(W,S) \to 0,$

which has i_* injective, j_* surjective by exactness.

See that i_* inj. $\Rightarrow \mathbb{Z} \cong \operatorname{im} i_* \cong \ker j_*$
$\Rightarrow \operatorname{im} j_* = 0 \Rightarrow \underline{H_0(W,S) \cong 0}$ by surjectivity of j_*

$$\text{So: } H_n(W,S) \cong \begin{cases} 0 & n>3 \\ \mathbb{Z} & n=3 \\ \mathbb{Z}\oplus\mathbb{Z} & n=2 \\ 0 & n=1 \\ 0 & n=0 \end{cases}$$

Geometry/Topology Qualifying Exam

Spring 2009

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. Let $S^2 = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Prove that the map
$$f : S^2 \to \mathbb{R}^4, \; f(x_1, x_2, x_3) = (x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3)$$
is an immersion and that $f(S^2)$ is diffeomorphic to the projective plane $\mathbb{RP}^2$.

2. Let ω be a closed n-form on $\mathbb{R}^{n+1} - \{0\}$. Prove that ω is exact if and only if $\int_{S^n} \omega = 0$, where S^n is the unit sphere in $\mathbb{R}^{n+1}$.

3. Find all vector fields Z on $\mathbb{R}^2$ which satisfy $[X, Z] = 0$ and $[Y, Z] = 0$, where $X = e^x \frac{\partial}{\partial x}$ and $Y = \frac{\partial}{\partial y}$ are vector fields defined on all of $\mathbb{R}^2$.

4. Compute $\pi_n(T^p)$ for all $n \geq 1$, where $T^p = S^1 \times \cdots \times S^1$ (p times) is the p-dimensional torus.

5. Compute $\pi_1(\mathbb{R}^3 - K)$, where $K \subset \mathbb{R}^3$ is the union of the vertical axis $\{x = 0, y = 0\}$ and the unit circle $\{x^2 + y^2 = 1, z = 0\}$.

6. Let X be a compact, oriented surface of genus 2 (without boundary), and let A be a simple closed curve which separates the surface X into two punctured tori, as given in Figure 1 below. Then compute the relative homology groups $H_n(X, A)$ for all $n \geq 0$.

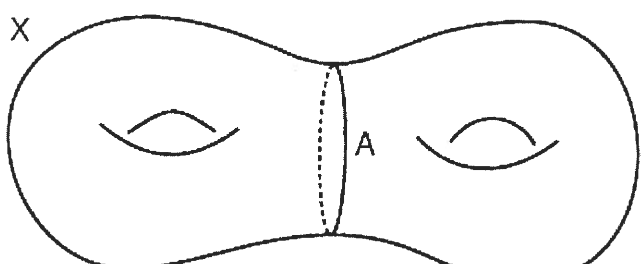


FIGURE 1

1. Solve Equations, use $\mathbb{RP}^2 = S^2/\sim$.

2. $I : H^n_{dR}(S^n) \to \mathbb{R}$, $I(\omega) = \int_{S^n} \omega$ is an isomorphism. Stokes, $\partial S^n = \emptyset$.

3. $[A,B] = AB - BA$, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ is a basis for vector fields in $\mathbb{R}^2$.

4. If $p : \tilde{X} \to X$ is a covering space, $p^*(\pi_n(\tilde{X}, \tilde{x})) \to \pi_n(X)$ is an isomorphism $\forall\ n \geq 2$.

5. $\mathbb{R}^3 - K \overset{h.e.}{\cong} T^2$ so $\pi_1(\mathbb{R}^3 - K) = \mathbb{Z} \times \mathbb{Z}$

6. For a good pair (X, A), $H_n(X, A) \simeq \tilde{H}_n(X/A)$ $\forall$ n.

① $f: S^2 \to \mathbb{R}^4$ given by $f(x_1,x_2,x_3) = (x_1^2-x_2^2, x_1x_2, x_1x_3, x_2x_3)$
Show f immersion and $f(S^2)$ diffeomorphic to $\mathbb{R}P^2$

Consider the tangent map extended to $\mathbb{R}^3$, i.e. $T_pf: T_p\mathbb{R}^3 \to T_{f(p)}\mathbb{R}^4$ given by the matrix $\left(\frac{\partial f_i}{\partial x_j}\right)$ where $f_1 = x_1^2-x_2^2$, $f_2 = x_1x_2$, $f_3 = x_1x_3$, $f_4 = x_2x_3$
i.e. :

$$T_pf = \begin{pmatrix} 2x_1 & -2x_2 & 0 \\ x_2 & x_1 & 0 \\ x_3 & 0 & x_1 \\ 0 & x_3 & x_2 \end{pmatrix}\Bigg|_{p=(x_1,x_2,x_3)}$$

Now, we want to show that $T_pf: T_pS^2 \to T_{f(p)}\mathbb{R}^4$ is injective, so restrict to $T_pS^2 = \{v \in T_p\mathbb{R}^3 : \langle p, v\rangle = 0\}$; now suppose $v \in T_pS^2$ such that $T_pf(v) = 0$. This yields the system (where $v=(a,b,c)$) :

$$\begin{cases} 2ax_1 - 2bx_2 = 0 \\ ax_2 + bx_1 = 0 \\ ax_3 + cx_1 = 0 \\ bx_3 + cx_2 = 0 \\ x_1^2+x_2^2+x_3^2 = 1 \text{ (since } p \in S^2) \\ ax_1+bx_2+cx_3 = 0 \text{ (since } v \in T_pS^2) \end{cases} \Rightarrow \begin{cases} ax_1 = bx_2 \\ ax_2 = -bx_1 \\ ax_3 = -cx_1 \\ bx_3 = -cx_2 \end{cases}$$

The equation $x_1^2+x_2^2+x_3^2 = 1$ implies one of the x_i must be nonzero.

- case $x_1 \neq 0$: $a = \frac{bx_2}{x_1}$ & $b = -\frac{ax_2}{x_1}$ $\Rightarrow a = -a\frac{x_2^2}{x_1^2} = a\left(-\frac{x_2^2}{x_1^2}\right) \Rightarrow \underline{a=0}$
So now $-bx_1 = 0 \Rightarrow \underline{b=0}$ since $x_1 \neq 0$. Finally, $-cx_1 = 0 \Rightarrow c=0$ since $x_1 \neq 0$
So here we have $v = (0,0,0)$.
- Case $x_2 \neq 0$: $b = a\frac{x_1}{x_2}$ & $a = -\frac{bx_1}{x_2}$ $\Rightarrow b = -b\frac{x_1^2}{x_2^2} \Rightarrow \underline{b=0}$; then $ax_2 = 0$ & $-cx_2 = 0$ $\Rightarrow a=c=0$ since $x_2 \neq 0 \Rightarrow \underline{v=(0,0,0)}$
- Case $x_3 \neq 0$: $b = -\frac{cx_2}{x_3}$ & $a = -\frac{cx_1}{x_3}$ $\Rightarrow -c\frac{x_1^2}{x_3} = -c\frac{x_2^2}{x_3}$

Suppose $c \neq 0$. Then $x_1^2 = x_2^2$. If $x_1 \neq 0$ or $x_2 \neq 0$, we are back to the previous case and get $v = \vec{0}$.

So suppose $x_1^2 = x_2^2 = 0$, hence by $ax_1 + bx_2 + cx_3 = 0$, we get $cx_3 = 0 \Rightarrow c = 0$ since $x_3 \neq 0$, which contradicts our assumption, hence $c = 0$.

With $c = 0$, we get $ax_3 = 0 = bx_3 \Rightarrow a, b = 0$ since $x_3 \neq 0$, hence again $v = (0,0,0)$.

$\rightsquigarrow$ So we've shown $\ker(T_pf : T_pS^2 \to T_{f(p)}\mathbb{R}^4) = 0$, hence $f : S^2 \to \mathbb{R}^4$ is an <u>immersion</u>.

Recall that the projective plane $\mathbb{RP}^2$ is obtained via identifying antipodal points of S^2; we have the standard quotient map: $\pi : S^2 \to \mathbb{RP}^2$ s.t. $\pi(\alpha(p)) = \pi(p)$.

Now note that $\underline{f(\alpha(p))} = f(-x_1, -x_2, -x_3)$

$$= f(x_1^2 - x_2^2, (-x_1)(-x_2), (-x_1)(-x_3), (-x_2)(-x_3))$$

$$= f(x_1^2 - x_2^2, x_1x_2, x_1x_3, x_2x_3) = f(x_1, x_2, x_3) = \underline{f(p)}$$

Hence f also identifies antipodal points.

$\rightarrow$ Therefore, f will be injective on $\mathbb{RP}^2$, and an injective immersion on a compact space is an embedding.

<u>Check</u>

② $\omega \in \Omega^n(\mathbb{R}^{n+1}\setminus\{0\})$ closed
Prove ω exact $\iff \int_{S^n}\omega = 0$?

($\Rightarrow$) Suppose ω exact, i.e. $\exists \alpha \in \Omega^{n-1}(\mathbb{R}^{n+1}\setminus\{0\})$ such that $d\alpha = \omega$.
Then we may apply Stokes:

$$\int_{S^n}\omega = \int_{S^n} d\alpha = \int_{\partial S^n}\alpha = \int_{\emptyset}\alpha = 0$$

($\Leftarrow$) Suppose $\int_{S^n}\omega = 0$

Recall that $S^n \simeq \mathbb{R}^{n+1}\setminus\{0\}$ orientable, connected, compact.
So we get an isomorphism $I: H^n(S^n) \longrightarrow \mathbb{R}$, i.e. $H^n(\mathbb{R}^{n+1}\setminus\{0\}) \cong \mathbb{R}$
$[\omega] \longmapsto \int_{S^n}\omega$

Now, we have $0 = \int_{S^n}\omega = I([\omega]) \Rightarrow [\omega] = I^{-1}(0) = 0$,
hence the class $[\omega]$ is zero, hence ω is exact.

③ Find all vector fields Z on $\mathbb{R}^2$ satisfying $[X,Z] = 0 = [Y,Z]$
where $X = e^x\frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} \in \mathfrak{X}(\mathbb{R}^2)$.

Recall
$$\left.\begin{aligned} X &= X^1\tfrac{\partial}{\partial x} + X^2\tfrac{\partial}{\partial y} = e^x\tfrac{\partial}{\partial x} \\ Y &= Y^1\tfrac{\partial}{\partial x} + Y^2\tfrac{\partial}{\partial y} = \tfrac{\partial}{\partial y}\end{aligned}\right\} \Rightarrow \begin{aligned} X^1 &= e^x, \; X^2 = 0 \\ Y^1 &= 0, \; Y^2 = 1\end{aligned}$$

and Recall:
$$[V,W] = \left(V^1\frac{\partial W^1}{\partial x} - W^1\frac{\partial V^1}{\partial x} + V^2\frac{\partial W^1}{\partial y} - W^2\frac{\partial V^1}{\partial y}\right)\frac{\partial}{\partial x} + \left(V^1\frac{\partial W^2}{\partial x} - W^1\frac{\partial V^2}{\partial x} + V^2\frac{\partial W^2}{\partial y} - W^2\frac{\partial V^2}{\partial y}\right)\frac{\partial}{\partial y}$$

and then
$$0 = [X,Z] = \left(e^x\frac{\partial Z^1}{\partial x} - Z^1 e^x + 0 - Z^2(0)\right)\frac{\partial}{\partial x} + \left(e^x\frac{\partial Z^2}{\partial x} - Z^1(0) + (0) - Z^2(0)\right)\frac{\partial}{\partial y}$$
$$= \left(e^x\frac{\partial Z^1}{\partial x} - Z^1 e^x\right)\frac{\partial}{\partial x} + \left(e^x\frac{\partial Z^2}{\partial x}\right)\frac{\partial}{\partial y}$$

&
$$0 = [Y,Z] = \left(0 - Z^1(0) + \frac{\partial Z^1}{\partial y} - Z^2(0)\right)\frac{\partial}{\partial x} + \left(0 - Z^1(0) + (1)\frac{\partial Z^2}{\partial y} - Z^2(0)\right)\frac{\partial}{\partial y}$$
$$= \left(\frac{\partial Z^1}{\partial y}\right)\frac{\partial}{\partial x} + \left(\frac{\partial Z^2}{\partial y}\right)\frac{\partial}{\partial y}$$

So then we have

$$\begin{cases} e^x \dfrac{\partial z^1}{\partial x} - z^1 e^x = 0 \Rightarrow e^x \dfrac{\partial z^1}{\partial x} = z^1 e^x \Rightarrow \dfrac{\partial z^1}{\partial x} = z^1 \\ e^x \dfrac{\partial z^2}{\partial x} = 0 \Rightarrow \dfrac{\partial z^2}{\partial x} = 0 \end{cases}$$

Since $e^x \neq 0$ for all x.

$\dfrac{\partial z^1}{\partial y} = \dfrac{\partial z^2}{\partial y} = 0 \Rightarrow$ no occurrence of y

Now, $\dfrac{\partial z^1}{\partial x} = z^1 \Rightarrow z^1 = Ae^x$

and $\dfrac{\partial z^2}{\partial x} = \dfrac{\partial z^2}{\partial y} = 0 \Rightarrow z^2 = k$ constant.

Hence $\boxed{z = Ae^x \dfrac{\partial}{\partial x} + k \dfrac{\partial}{\partial y}}$

---

(4) Find $\pi_n(T^p)$ for all $n \geq 1$ where $T^p = S^1 \times \cdots \times S^1$ p times.

First, see that $\pi_1(T^p) = \pi_1(S^1 \times \cdots \times S^1) = \pi_1(S^1) \times \cdots \times \pi_1(S^1)$

$= \mathbb{Z} \times \cdots \times \mathbb{Z} = \underline{\mathbb{Z}^p}$

Similarly, we have $\pi_n(T^p) = \pi_n(S^1) \times \cdots \times \pi_n(S^1)$

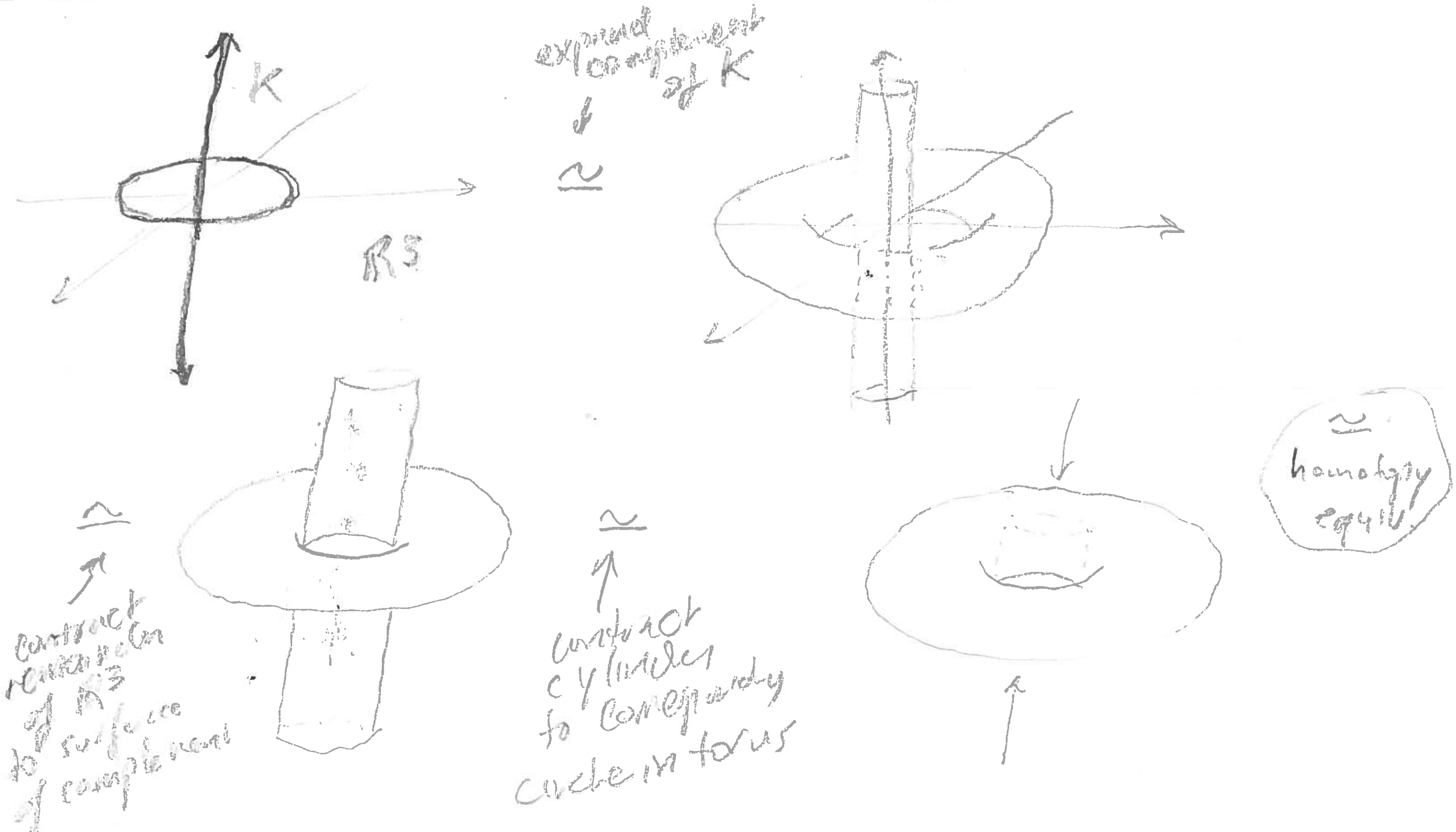
Recall that $p : \mathbb{R} \to S^1$ is the universal cover and that for $n \geq 2$, $p_* : \pi_n(\mathbb{R}) \to \pi_n(S^1)$ is an isomorph. (for covering spaces).

Hence: $\pi_n(S^1) \cong p_*(\pi_n(\mathbb{R})) \cong p_*(0) \cong 0$,

i.e. $\pi_n(S^1)$ trivial, hence $\pi_n(T^p) = 0^p = 0$ trivial as well

$$\Rightarrow \pi_n(T^p) = \begin{cases} \mathbb{Z}^p & n = 1 \\ 0 & n > 1 \end{cases}$$

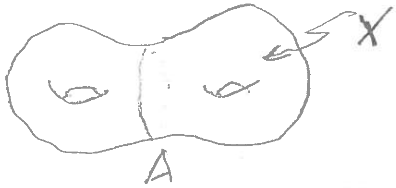
⑤ Compute $\pi_1(\mathbb{R}^3 \setminus K)$ where $K = \{x=0, y=0\} \cup \{x^2+y^2=1, z=0\}$.
i.e. K = (vertical axis) $\cup$ (unit circle in z-plane)



Hence we see that $\mathbb{R}^3 \setminus K \simeq T^2$

$\Rightarrow \underline{\pi_1(\mathbb{R}^3 \setminus K) \cong \pi_1(T^2) \cong \mathbb{Z} \times \mathbb{Z}}$.

⑥ $X = M_2 =$  and A the curve pictured:



Compute $H_n(X,A)$ for $n \geq 0$

Recall that, for good pair (X,A), we have

$H_n(X,A) \cong \tilde{H}_n(X/A)$.

Now, see that $X/A \cong$  $= T^2 \vee T^2$

Consider the decomposition of $T^2 \vee T^2$ where

$A \simeq T^2$, $B \simeq T^2$, $A \cup B = T^2 \vee T^2$, $A \cap B = pt$.

Then apply Mayer-Vietoris:

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \cdots$$

$$\Rightarrow \cdots \to H_n(pt) \to H_n(T^2) \oplus H_n(T^2) \to H_n(T^2 \vee T^2) \to H_{n-1}(pt) \to \cdots$$

$$\Rightarrow \cdots \to 0 \to H_n(T^2) \oplus H_n(T^2) \to H_n(T^2 \vee T^2) \to 0 \to \cdots$$

$\Rightarrow H_n(T^2) \oplus H_n(T^2) \cong H_n(T^2 \vee T^2)$ by exactness.

Now we know $H_n(T^2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases} \Rightarrow H_n(T^2 \vee T^2) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & n=0 \\ \mathbb{Z}^4 & n=1 \\ \mathbb{Z}\oplus\mathbb{Z} & n=2 \\ 0 & n>2 \end{cases}$

and recall that $H_n(T^2 \vee T^2) \cong \tilde{H}_n(T^2 \vee T^2)$, $n > 0$

$H_0(T^2 \vee T^2) \cong \tilde{H}_0(T^2 \vee T^2) \oplus \mathbb{Z}$, hence:

$$\begin{array}{c} H_n(X,A) \\ \wr\wr \\ \tilde{H}_n(X/A) \end{array} \cong \tilde{H}_n(T^2 \vee T^2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}^4 & n=1 \\ \mathbb{Z}\oplus\mathbb{Z} & n=2 \\ 0 & n>2 \end{cases} = H_n(X,A)$$

Geometry/Topology Qualifying Exam

Fall 2008

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

✓ 1. Consider the map $d_f : \Omega^i(M) \to \Omega^{i+1}(M)$ given by $\omega \mapsto d\omega + df \wedge \omega$, where M is a smooth manifold, $\Omega^i(M)$ is the set of smooth i-forms on M, and f is a smooth function on M.

   (a) Show that d_f is a cochain map, i.e., $d_f \circ d_f = 0$.

   (b) Let $H^i_f(M)$ be the ith cohomology group of the cochain complex $(\Omega^i(M), d_f)$. Show that $H^0_f(M) \cong \mathbb{R}$ when M is the real line $\mathbb{R}$.

✓ 2. Show that, when $m, n > 0$, the homomorphism $f^* : H^k_{dR}(S^m \times S^n) \to H^k_{dR}(S^{m+n})$ induced in de Rham cohomology by $f : S^{m+n} \to S^m \times S^n$ is trivial for all $k > 0$. Here S^n is the n-dimensional sphere. [Possible hint: Construct a volume form for $S^m \times S^n$ from a volume form on S^m and a volume form on S^n.]

✓ 3. Prove that the set $C = \{(x, y) \mid y^2 - x^3 = 0\}$ is not a smooth submanifold of the plane. [Hint: What is the space of tangent vectors in $T_{(0,0)}\mathbb{R}^2$ which are tangent to C?]

✓ 4. Let T be the surface obtained by revolving the circle $\{(x, y, z) \mid z = 0, (x - R)^2 + y^2 = r^2\}$ around the y-axis, where $R > r$. Compute the integral

$$\int_T x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy.$$

✓ 5. Let B^3 be the (closed) 3-dimensional ball, and let K be a closed, connected 1-dimensional submanifold of B^3 with $\partial K = K \cap \partial B^3 = 2$ points. Compute the homology of the complement $B^3 - K$ (= an apple minus a wormhole).

✓ 6. Recall that two covering spaces $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X$ are *isomorphic* if there exists a homeomorphism $\tilde{\phi} : \tilde{X} \xrightarrow{\sim} \tilde{X}'$ such that $p' \circ \tilde{\phi} = p$. Consider the covering spaces $p : \tilde{X} \to X$ of the torus $X = S^1 \times S^1$ whose fiber $p^{-1}(x_0)$ at any point $x_0 \in X$ consists of 3 points. How many distinct isomorphism classes of such coverings are there?

1. a) $d \circ d = 0$, $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\operatorname{ord} \alpha} \alpha \wedge d\beta$

   b) $\ker d_p = \{ c e^{f(x)} \in \Omega^0(\mathbb{R}) \} \cong \mathbb{R}$.

2. Pull volume forms on S^n, S^m back to S^{n+m}. Use de Rham of S^{m+n} to conclude they must be exact.

3. Analysis: $\gamma_1(t) = v_1 t + \hat{\gamma}_1(t)$ etc...

4. Stokes

5. $\cancel{\mathbb{R}^3 - K \overset{h.e.}{\cong} S^1}$ Inverse Mayer Vietoris and Tubular neighborhoods

6. Classification of n-sheeted covering spaces.

1. Consider $d_f : \Omega^i(M) \to \Omega^{i+1}(M)$ where M smooth mfld,
$\omega \longmapsto d\omega + df \wedge \omega$ f smooth fn.

(a) Show d_f is a cochain map, i.e. $d_f \circ d_f = 0$
(b) Let $H^i_f(M)$ be the i^{th} cohomology group of the cochain complex (Ω^i, d_f). Show $H^0_f(M) \cong \mathbb{R}$ when $M = \mathbb{R}$..!

(a) Recall that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta$.

$$\begin{aligned}
d_f \circ d_f(\omega) &= d_f(d\omega + df \wedge \omega) \\
&= d(d\omega + df \wedge \omega) + df \wedge (d\omega + df \wedge \omega) \\
&= \underbrace{d(d\omega)}_{0} + d(df \wedge \omega) + df \wedge (d\omega + df \wedge \omega) \\
&= \underbrace{d(df)}_{0} \wedge \omega + (-1)^{\deg(df)} df \wedge d\omega + df \wedge (d\omega + df \wedge \omega) \\
&= \underbrace{\cancel{-df \wedge d\omega + df \wedge d\omega}}_{0} + df \wedge df \wedge \omega \\
&= df \wedge df \wedge \omega = 0 \quad \text{since } df \wedge df = -df \wedge df \\
&\qquad\qquad\qquad\qquad \Rightarrow df \wedge df = 0 \\
&\qquad\qquad\qquad\qquad (df \text{ 1-form})
\end{aligned}$$

(b) Consider the cochain complex:

$$0 \longrightarrow \Omega^0(\mathbb{R}) \xrightarrow{d_f} \Omega^1(\mathbb{R}) \Rightarrow H^0_f(\mathbb{R}) \cong \ker d_f.$$

So consider this kernel: $\ker d_f = \{ g \in \Omega^0(\mathbb{R}) : d_f g = 0 \}$

$$\Rightarrow 0 = d_f(g) = dg + df \wedge g \Rightarrow dg = -df \wedge g = g \wedge df$$
$$= g\, df$$
$$\Rightarrow g = Ae^f,$$

hence $\ker d_f = \{ A e^{f(x)} \}_{A \in \mathbb{R}}$
$\cong \mathbb{R}$.

$$\Rightarrow H^0_f(\mathbb{R}) \cong \mathbb{R}$$

(2.) Show that the homomorphism $f^*: H^k_{dR}(S^m \times S^n) \to H^k_{dR}(S^{m+n})$ induced in de Rham cohomology by $f: S^{m+n} \to S^m \times S^n$ is trivial for all $k>0$, $m,n>0$.

First, note that $H^k(S^{m+n}) \equiv 0$ for all $k < m+n$, hence we only need to be concerned with the map:

$$f^*: H^{m+n}(S^m \times S^n) \longrightarrow H^{m+n}(S^{m+n})$$

Recall that both S^n and S^m have volume forms: $\begin{cases} \omega_m \in \Omega^m(S^m) \\ \omega_n \in \Omega^n(S^n) \end{cases}$, which we know generate $H^m(S^m)$, $H^n(S^n)$ (resp.) by the top-dim'l de Rham theorem. Now consider their pullbacks under the respective projection maps $\pi_n: S^n \times S^m \to S^n$, $\pi_m: S^n \times S^m \to S^m$:

$$\pi_n^*: H^n(S^n) \longrightarrow H^n(S^n \times S^m), \quad \pi_m^*: H^m(S^m) \longrightarrow H^m(S^n \times S^m)$$
$$\omega_n \longmapsto \pi_n^*(\omega_n) \qquad\qquad \omega_m \longmapsto \pi_m^*(\omega_m)$$

and take the wedge product of the two:

$$\pi_n^*(\omega_n) \wedge \pi_m^*(\omega_m) \in H^{n+m}(S^n \times S^m)$$

See now that ω_n is of the form $\omega_n = f\,\underbrace{dx_1 \wedge \dots \wedge dx_n}_{\text{for coord. } (x_i) \text{ on } S^n}$, $f \neq 0$, and so $\pi_n^*(f\,dx_1 \wedge \dots \wedge dx_n) = \pi_n^*(f)\,\pi_n^*(dx_1) \wedge \dots \wedge \pi_n^*(dx_n)$
(?) $= (f \circ \pi_n)\,dx_1 \wedge \dots \wedge dx_n$

and similarly for ω_m and $\pi_m^*(\omega_m)$, hence their wedge product is of the form $F\,dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_m$ with $F \neq 0$, hence $\pi_n^*(\omega_n) \wedge \pi_m^*(\omega_m)$ is a volume form on $\underbrace{S^n \times S^m}_{\text{coords } (x_i, y_j)}$.

for coord. (y_i) on S^m

2 cont'd :

So we have a volume form $\pi_n^*(\omega_n) \wedge \pi_m^*(\omega_m)] \in H^{n+m}(S^n \times S^m)$, i.e. a generator of $H^{n+m}(S^n \times S^m)$.

Now see that:

$$f^*(\pi_n^*(\omega_n) \wedge \pi_m^*(\omega_m)) = f^*\pi_n^*(\omega_n) \wedge f^*\pi_m^*(\omega_m)$$

$$= 0 \wedge 0 = 0$$

Since $f^*: H^n(S^n \times S^m) \to H^n(S^{n+m}) \cong 0$

$f^*: H^m(S^n \times S^m) \to H^m(S^{n+m}) \cong 0$,

hence $f^*: H^{n+m}(S^n \times S^m) \to H^{n+m}(S^{n+m})$ is the zero map, as was desired.

③ ~~Prove $C = \{(x,y) : y^2 - x^3 = 0\}$ is not a smooth submfld of $\mathbb{R}^2$.~~

~~First, notice that C is a curve in $\mathbb{R}^2$, hence $\dim C = 1$. Now, if C were a submanifold we would have $\dim T_pC = 1$ for all points $p \in C$. Consider now the tangent space at $p = (0,0)$:~~

See next page

③ Show that $C = \{(x,y) : y^2 = x^3\}$ is not a smooth submfld of $\mathbb{R}^2$:

Consider a curve $\alpha : (-\varepsilon, \varepsilon) \to C$ with $\alpha(0) = (0,0)$.

Then $\alpha(t) = (x(t), y(t))$; hence $x(t)^3 = y(t)^2$.

These constituent functions are smooth, hence representable as Taylor series:

Suppose $y(t) = ct + O(t^2)$ and $x(t) = O(t)$ (since $x(0) = 0 = y(0)$)

Then: $x(t)^3 = y(t)^2 \Rightarrow O(t^3) = (ct + O(t^2))^2$

$$= c^2t^2 + O(t^3)$$

$$\Rightarrow O(t^3) = c^2t^2 \Rightarrow c^2 = 0 \Rightarrow c = 0$$

Hence $y(t) = O(t^2) \Rightarrow y'(t) = O(t)$

$$\Rightarrow \underline{y'(0) = 0}$$

So we've shown $y(t) = O(t^2)$.

Now suppose $x(t) = dt + O(t^2)$. Then:

$$(dt + O(t^2))^3 = (O(t^2))^2$$

$$\Rightarrow d^3t^3 + O(t^4) = O(t^4) \Rightarrow d^3t^3 = O(t^4)$$

$$\Rightarrow d^3 = 0 \Rightarrow d = 0$$

So then $x(t) = O(t^2) \Rightarrow x'(t) = O(t) \Rightarrow \underline{x'(0) = 0}$

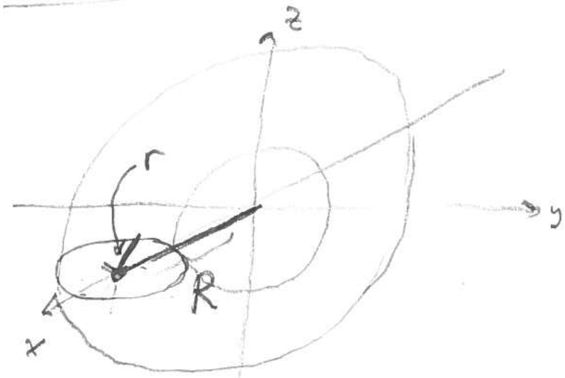
Hence we've shown the tangent vector to any smooth curve thru $(0,0)$ in C is $(0,0)$, i.e.

$T_{(0,0)}C = 0 \Rightarrow \dim T_{(0,0)}C = 0$, a contradiction

since if C submfld, $\dim T_{(0,0)} = 1$

(4.) T = revolve $\{(x,y,z): z=0, (x-R)^2+y^2=r^2\}$ around y-axis, and $R>r$. Compute:

$$\int_T x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy$$



T is a torus.

Let K = solid torus s.t. $\partial K = T$.

Now see that $d(x\,dy\wedge dz - y\,dx\wedge dz + x\,dx\wedge dy)$

$$= dx\wedge dy\wedge dz - dy\wedge dx\wedge dz + dz\wedge dx\wedge dy$$
$$= dx\wedge dy\wedge dz + dx\wedge dy\wedge dz + dx\wedge dy\wedge dz$$
$$= 3\,dx\wedge dy\wedge dz$$

and apply Stokes:

$$\int_T (x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) = \int_{\partial K} (x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)$$

$$= \int_K d(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) = \int_K 3\,dx\wedge dy\wedge dz$$

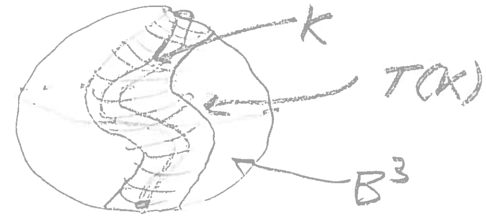
$$= 3\int_K dx\wedge dy\wedge dz = 3\,\mathrm{Vol}(K) = (\pi r^2)(2\pi R) = \underline{2\pi^2 R r^2}$$

(5) B^3 closed ball, $K \subseteq B^3$ closed, connected, 1-dim submfld such that $\partial K = K \cap \partial B^3 = 2$ pts (apple minus wormhole)

Compute $H_i(B^3 \setminus K)$:

Let $A =$ tubular nbhd of $K := T(K)$

$B = B^3 \setminus K$



Now note that K is a submanifold, hence it will not have any self-intersections since those would not be locally Euclidean (i.e. $\nexists$ homeo. )

Therefore K can be contracted to a point (as a submfld of $\mathbb{R}^3$), hence $T(K) \simeq pt$.

Now, by construction, $K \subseteq T(K)$ such that $T(K) \setminus K \simeq$ cylinder $\simeq S^1$, hence

$$A \cap B \simeq T(K) \setminus K \simeq S^1$$

Finally, clearly $A \cup B = B^3 \simeq pt$.

So we have:

$$H_i(A) \cong H_i(pt) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases} \qquad H_i(A \cap B) \cong H_i(S^1) \cong \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & i>1 \end{cases}$$

$$H_i(B) \cong H_i(B^3 \setminus K) \qquad H_i(A \cup B) \cong H_i(pt) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$$

Now apply Mayer-Vietoris:

$$0 \to H_2(A\cap B) \to H_2(A) \oplus H_2(B) \to H_2(A\cup B) \to H_1(A\cap B)$$
$$\to H_1(A) \oplus H_1(B) \to H_1(A\cup B) \to H_0(A\cap B) \to H_0(A) \oplus H_0(B)$$
$$\to H_0(A\cup B) \to 0$$

$$\Rightarrow \quad 0 \to 0 \to 0 \oplus H_2(B^3\setminus K) \to 0 \to \mathbb{Z} \to 0 \oplus H_1(B^3\setminus K)$$
$$\to 0 \to \mathbb{Z} \to \mathbb{Z} \oplus H_0(B^3\setminus K) \to \mathbb{Z} \to 0$$

$\Rightarrow$ $H_2(B^3\setminus K) \cong 0$, $H_1(B^3\setminus K) \cong \mathbb{Z}$ by exactness

$H_0(B^3\setminus K) \cong \mathbb{Z}$ by path-connectedness.

(6.) $p: \tilde{X} \to X$, $p': \tilde{X}' \to X$ isomorphic if $\exists$ homeo. $\phi: \tilde{X} \xrightarrow{\sim} \tilde{X}'$ s.t. $p' \circ \phi = p$.

Consider the covering space $p: \tilde{X} \to T^2$ of the torus whose fibers $p^{-1}(x_0)$ at any pt. $x_0 \in T^2$ consists of 3 points.

How many isomorphism classes of such a covering?

Recall that n-sheeted covering spaces of X are classified by equivalence classes of homomorphisms $\pi_1(X) \to S_n$.

Here, $X = T^2$ and $n = 3$, hence consider homomorphisms

$$\pi_1(T^2) \to S_3, \text{ i.e. } \mathbb{Z} \oplus \mathbb{Z} \to S_3.$$

See that $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\}$, i.e. has two generators a, b; the group is free abelian, hence the generators commute: $ab = ba$, hence for any hom. $f: \mathbb{Z} \oplus \mathbb{Z} \to S_3$, $f(a)f(b) = f(ab) = f(ba) = f(b)f(a)$, hence f maps the generators to commuting pairs $\sigma, \tau \in S_3$.

- Case $f(a) = f(b)$: Clearly two equal permutations commute; there are 6 elts in S_3, so this case represents 6 iso. classes.
- Commuting pairs: the following are the commuting pairs in S_3:

$$\begin{cases} (id, (12)) \\ (id, (13)) \\ (id, (23)) \\ (id, (123)) \\ (id, (132)) \end{cases} \text{ are clear.}$$

- Also see that $(123) = (132)^2$, hence $(123)(132) = (132)^2(132) = id = (132)(132)' = (132)(123)$.

So we also have $((123), (132))$.

For a total of 6 commuting pairs, and 12 total cases since pairs may $= (f(a), f(b))$ or $= (f(b), f(a))$. Thus $12 + 6 = 18$ iso. classes

Incomplete

2(b) ?

Geometry and Topology Graduate Exam

February 2008

1. Let $p : \tilde{X} \to X$ be a covering with path connected base X, and let G be its automorphism group, consisting of those homeomorphisms $\varphi : \tilde{X} \to \tilde{X}$ such that $p \circ \varphi = \varphi$. Pick base points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$. Suppose that, for any two $\tilde{x}_0', \tilde{x}_0'' \in p^{-1}(x_0)$, there exists $\varphi \in G$ such that $\varphi(\tilde{x}_0'') = \tilde{x}_0'$. Show that there is an exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}; \tilde{x}_0) \xrightarrow{p_*} \pi_1(X; x_0) \longrightarrow G \longrightarrow 1.$$

2. Consider on $\mathbb{R}^n$ the standard inner product $\langle \vec{a}, \vec{b} \rangle = \sum_{i=1}^n a_i b_i$, when $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$. Let V be a vector subspace of $\mathbb{R}^n$, and let $\pi : \mathbb{R}^n \to V$ be the orthogonal projection with respect to the above inner product. If M is a submanifold of $\mathbb{R}^n$, show that the restriction $\pi_{|M} : M \to V$ is an immersion if and only if $T_x M \cap V^\perp = \{0\}$ for every $x \in M$.

3. Let $f : X \to X$ be a map homotopic to a constant map, and let $M_f = X \times [0,1]/\sim$ where the equivalence relation $\sim$ identifies $(x, 0)$ to $(f(x), 1)$. Compute the homology groups of M_f.

4. Consider a differentiable map $f : S^{2n-1} \to S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n on S^n such that $\int_{S^n} \alpha = 1$, let $f^*(\alpha) \in \Omega^n(S^{2n-1})$ be its pull-back under the map f.

a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.

b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of β and α. It may be useful to remember that the map $H^n(S^n) \to \mathbb{R}$ defined by $\gamma \mapsto \int_{S^n} \gamma$ is an isomorphism.

5. Let $\omega \in \Omega^2(S^2)$ be the restriction of the 2-form

$$x\,dy \wedge dz + z\,dx \wedge dy + y\,dz \wedge dx$$

to the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$. Compute the integral $\int_{S^2} \omega$.

6. Recall that the 1-dimensional projective space $\mathbb{RP}^1$ consists of all lines in $\mathbb{R}^2$ passing through the origin. Let $f : \mathbb{R} \to \mathbb{RP}^1$ associate to $x \in \mathbb{R}$ the line passing through $(x, 1)$ and the origin. Finally, let $P(x)$ be a polynomial function of the variable x.

a) Show that there is no differential form ω on $\mathbb{RP}^1$ such that $f^*(\omega) = P(x)\,dx$.

b) Show that there exists a vector field V on $\mathbb{RP}^1$ such that $f^*(V) = P(x)\frac{\partial}{\partial x}$ if and only if the degree of $P(x)$ is ≤ 2.

notation?

7. Let M be a compact differentiable manifold, and let $C^\infty(M)$ be the algebra of all differentiable functions $M \to \mathbb{R}$. Let $\mathcal{I}$ be a maximal ideal of $C^\infty(M)$. Show that there is a point $x_0 \in M$ such that $\mathcal{I} = \{f \in C^\infty(M); f(x_0) = 0\}$. (Possible hint: Suppose that the property is not true and show that, for every $x \in M$, there exists a non-negative function $f \in \mathcal{I}$ such that $f(x) > 0$.)

1. Theorem in Hatcher. Uses lifting criterion and classification of connected covering spaces.

2. Def of immersion, $T_x \pi = \pi \ \forall x \in M$.

3. Mayer Vietoris. Answer $H_i(M_f) = H_i(X) \oplus H_i(S^1)$

4. Solved if n is even

5. Stokes

6. Use x as a coordinate for $\mathbb{R}$ and θ as a coordinate for $\mathbb{RP}^1$

7. Show hint using maximality of I, then construct an invertible function in I (everywhere positive) using compactness.

Geometry / Topology - Spring '08 :

① $p: \tilde{X} \to X$, X path-connected, $G = G(\tilde{X}) = \{\varphi \in Homeo(\tilde{X}) : p\circ\varphi = \varphi\}$ (= Deck transf. group). Choose $x_0 \in X$, $\tilde{x}_0 \in \tilde{X}$ s.t. $p(\tilde{x}_0) = x_0$

Suppose that for any 2 $\tilde{x}_0, \tilde{x}_0' \in p^{-1}(x_0)$, $\exists \varphi \in G$ s.t. $\varphi(\tilde{x}_0') = \tilde{x}_0$ (i.e. that p is normal covering)

Show that we have short exact sequence:

$$1 \longrightarrow \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow[p_*]{} \pi_1(X, x_0) \longrightarrow G \longrightarrow 1$$

Let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$

- Step 1 : We'll show p normal $\Rightarrow$ $H \trianglelefteq \pi_1(X, x_0)$.

Recall that p_* is injective, hence $H \cong \pi_1(\tilde{X}, \tilde{x}_0)$, and recall that changing the basepoint of $\tilde{X}$ from $\tilde{x}_0 \in p^{-1}(x_0)$ to $\tilde{x}_1 \in p^{-1}(x_0)$ corresponds to conjugating $\pi_1(\tilde{X}, \tilde{x}_0)$ by a path from $\tilde{x}_0$ to $\tilde{x}_1$, i.e. conjugating $H \leq \pi_1(X, x_0)$ by element $\gamma \in \pi_1(X, x_0)$ that lifts to path $\tilde{\gamma}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$:

$\leadsto$ γ loop in X s.t. $\tilde{\gamma}$ path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$, hence for $\alpha \in \pi_1(\tilde{X}, \tilde{x}_0)$, we get $\tilde{\gamma}^{-1}\alpha\tilde{\gamma} \in \pi_1(\tilde{X}, \tilde{x}_1)$.

Thus $\gamma \in N_{\pi_1(X,x_0)}(H) \iff \gamma^{-1}H\gamma = H \iff \gamma^{-1}p_*(\pi_1(\tilde{X}, \tilde{x}_0))\gamma^{-1} = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$\iff p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

Prop 1.37 Hatcher $\iff \exists \varphi \in G$ s.t. $\varphi(\tilde{x}_1) = \tilde{x}_0$ ($\tilde{x}_i \in p^{-1}(x_0)$)

Recall that the last condition is satisfied for any pair of points in the fiber of x_0 since p normal, hence

$N_{\pi_1(X,x_0)}(H) = \pi_1(X, x_0) \Rightarrow \underline{H \trianglelefteq \pi_1(X, x_0)}$.

• Step 2: Define the following map:

$\Phi : \pi_1(X, x_0) = N_{\pi_1(X,x_0)}(H) \longrightarrow G$

γ with $\tilde{\gamma}$ path b/t $\tilde{x}_0$ & $\tilde{x}_1$ $\longmapsto$ τ sending $\tilde{x}_0$ to $\tilde{x}_1$

amongst pts in $p^{-1}(x_0)$

This map is surjective since base-point changes in $\tilde{X}$ correspond precisely to deck transformations via the classification of covering spaces.

Also, clearly $\ker \Phi = \{\gamma \text{ lifting to loops in } \tilde{X}\}$
$= p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = H$

So now we have the sequence:

$$1 \longrightarrow \pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow[p_*]{} \pi_1(X, x_0) \xrightarrow[\Phi]{} G \longrightarrow 1$$

→ p_* injective since p covering map
→ Φ surjective by above
→ $\ker \Phi = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \operatorname{im} p_*$

$\Rightarrow$ seq is short exact

[this problem is proven in Hatcher, Cf. Prop. 1.39]

(2) $V \subseteq \mathbb{R}^n$ subspace, $\pi: \mathbb{R}^n \to V$ orth. proj w.r.t. $\langle , \rangle$ (i.e. $\ker \pi = V^\perp$). If $M \subseteq \mathbb{R}^n$ submfld, show that $\pi|_M : M \to V$ immersion $\iff T_pM \cap V^\perp = \{0\}$ $\forall p \in M$.

Recall that $\pi: \mathbb{R}^n \to V$ has kernel $V^\perp$ & $\mathbb{R}^n = V \oplus V^\perp$, so we may write $\pi: V \oplus V^\perp \to V$ and we can see that, furthermore, we may write $T_p\pi : V \oplus V^\perp \to V$ since $T_p\mathbb{R}^n \cong \mathbb{R}^n \cong V \oplus V^\perp$ and $T_{\pi(p)} V \cong V$.

Now choose $v \in T_p\mathbb{R}^n \cong V \oplus V^\perp \Rightarrow v = v_1 + v_2$, $v_1 \in V$, $v_2 \in V^\perp$. Then let $\alpha(t)$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$; since $\alpha(t) \in \mathbb{R}^n$, we may also write $\alpha(t) = \alpha_1(t) + \alpha_2(t)$ with $\alpha_1(t) \in V_1$, $\alpha_2(t) \in V^\perp$ and then $\alpha_1'(0) = v_1$, $\alpha_2'(0) = v_2$ in the decomposition above.

So then we see $T_p\pi(v) = (\pi \circ \alpha)'(0) = (\alpha_1)'(0) = v_1$, hence $\ker T_p\pi = V^\perp$, hence $T_p\pi = \pi$ as maps from $\mathbb{R}^n \to V$. So now:

($\Leftarrow$) Suppose $T_pM \cap V^\perp = \{0\}$ for all $p \in M$.

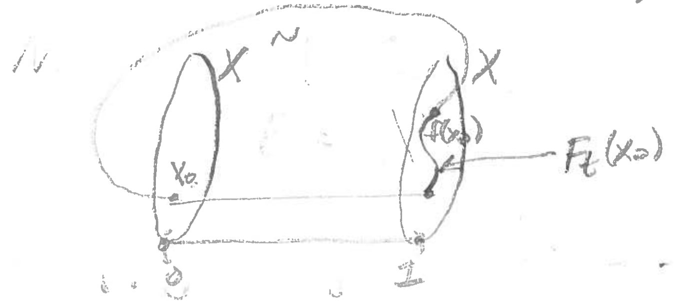
We have just shown that this means $\{0\} = T_pM \cap \ker(T_p\pi)$ for all p, hence $T_p\pi$ restricted to T_pM is injective, hence $\pi|_M : M \to V$ is an immersion.

($\Rightarrow$) Suppose $\pi|_M : M \to V$ is an immersion.

Then $T_p(\pi|_M)$ is an injection for all $p \in M$, hence $\ker T_p\pi \cap T_pM = \{0\}$, but we know $\ker T_p\pi = \ker \pi = V^\perp$, i.e. $V^\perp \cap T_pM = \{0\}$ for all $p \in M$.

(3) $f: X \to X$ homotopic to constant,
let $M_f = X \times [0,1]/\sim$ where $(x,0) \sim (f(x), 1)$.
Compute $H_n(M_f)$ for all n :

Since $f: X \to X$ homotopic to constant, $\exists F_t : X \to X$ with $F_0 = f$, $F_1 \equiv x_0$.
Therefore, for any $x \in X$, we have a path $F_t(x)$ from $F_0(x) = f(x)$ to $F_1(x) = x_0$, hence there exists a path from $(x_0, 1) \to (f(x_0), 1) \sim (x_0, 0)$, i.e. we have a path from $(x_0, 1) \to (x_0, 0)$ in M_f, hence $\{x_0\} \times [0,1] \simeq S^1$:



Now contract $X \times [0,1]$ to X and we have

$\simeq X \vee S^1$

So then, since each is a good pair w/ base point,

$\tilde{H}_i(M_f) \cong \tilde{H}_i(X \vee S^1) \cong \tilde{H}_i(X) \oplus \tilde{H}_i(S^1)$

Hence $H_i(M_f) \cong H_i(X) \oplus H_i(S^1)$ for $i > 0$

$\tilde{H}_0(M_f) \cong \tilde{H}_0(X) \oplus 0 \Rightarrow H_0(M_f) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$
$\cong \tilde{H}_0(X) \oplus H_0(S^1)$

So $\underline{H_i(M_f) \cong \tilde{H}_i(X) \oplus H_i(S^1)}$

3.

(4) $f: S^{2n-1} \to S^n$, $n \geq 2$ differentiable (?)

$\alpha \in \Omega^n(S^n)$ such that $\int_{S^n} \alpha = 1$, $f^*(\alpha) \in \Omega^n(S^{2n-1})$ pullback.

(a) Show $\exists \beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$ ($f^*(\alpha)$ exact)

(b) Show that $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of β & α:

(a) Step 1: $f^*(\alpha)$ is closed: $df^*(\alpha) = f^*(d\alpha)$, but see that $d\alpha \in \Omega^{n+1}(S^n) = 0$, hence $d\alpha = 0$, hence $f^*(d\alpha) = 0$, hence $df^*(\alpha) = 0$ & $f^*(\alpha)$ is closed

Step 2: Now we may consider the class of $f^*(\alpha)$ in $H^n(S^{2n-1})$, But $H^n(S^{2n-1}) = 0$ since $n \geq 2$, hence $[f^*(\alpha)] = 0$ in $H^n(S^{2n-1})$, hence $f^*(\alpha)$ is exact, i.e. $\exists \beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $f^*(\alpha) = d\beta$

(b) Consider $\beta' \in \Omega^{n-1}(S^{2n-1})$ such that $d\beta' = f^*(\alpha)$.

Then $d(\beta - \beta') = d\beta - d\beta' = f^*(\alpha) - f^*(\alpha) = 0$, hence we may consider the class $[\beta - \beta'] \in H^{n-1}(S^{2n-1}) = 0 \Rightarrow [\beta - \beta'] = 0$

$\Rightarrow \beta - \beta'$ exact $\Rightarrow \exists \omega \in \Omega^{n-2}(S^{2n-1})$ such that $d\omega = \beta - \beta'$

$\Rightarrow \beta' = \beta - d\omega$.

Now: $\int_{S^{2n-1}} \beta' \wedge d\beta' = \int_{S^{2n-1}} (\beta - d\omega) \wedge (d\beta + \cancel{d^2\omega}^{\,0})$

$= \int_{S^{2n-1}} (\beta - d\omega) \wedge d\beta = \int_{S^{2n-1}} \beta \wedge d\beta - \int_{S^{2n-1}} d\omega \wedge d\beta$

and see that

$d(\omega \wedge d\beta) = d\omega \wedge d\beta + \cancel{(-1)^{n-2} \omega \wedge d^2\beta}^{\,0} = d\omega \wedge d\beta$

and by Stokes: $\int_{S^{2n-1}} d\omega \wedge d\beta = \int_{\partial B^{2n}} d(\omega \wedge d\beta) = \int_{B^{2n}} d^2(\omega \wedge d\beta) = \int_{B^{2n}} 0 = 0$

hence $\int_{S^{2n-1}} \beta' \wedge d\beta' = \int_{S^{2n-1}} \beta \wedge d\beta$

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i.e. $I(f)$ is independent of choice of β.

(5) Let $w \in \Omega^2(S^2)$ be the restriction of
$x\,dy \wedge dz + x\,dz \wedge dy + y\,dz \wedge dx$
to the sphere $S^2 \subseteq \mathbb{R}^3$. Compute $\int_{S^2} w$.

$$d(x\,dy \wedge dz + x\,dz \wedge dy + y\,dz \wedge dx)$$
$$= d(x\,dy \wedge dz) + d(x\,dz \wedge dy) + d(y\,dz \wedge dx)$$
$$= dx \wedge dy \wedge dz + dx \wedge dz \wedge dy + dy \wedge dz \wedge dx$$
$$= dx \wedge dy \wedge dz - dx \wedge dy \wedge dz + dx \wedge dy \wedge dz$$
$$= dx \wedge dy \wedge dz$$

now apply Stokes: $\int_{S^2} w = \int_{\partial B^3} w = \int_{B^3} dw = \int_{B^3} dx \wedge dy \wedge dz$
$$= Vol(B^3)$$

6. Let $f: \mathbb{R} \to \mathbb{RP}^1$ be given by $f(x) = [x, 1]$.

(a) Show $\nexists\ \omega \in \Omega^1(\mathbb{RP}^1)$ such that $f^*(\omega) = p(x)dx$, p polynomial.

(b) Show $\exists\ V \in \mathfrak{X}(\mathbb{RP}^1)$ such that $f^*(V) = p(x)\frac{\partial}{\partial x} \iff \deg p \leq 2$.

(a) Consider the usual atlas on $\mathbb{RP}^1$: $\{(U,u),(V,v)\}$

where $U = \{[x,y] : x \neq 0\}$ and $u([x,y]) = \frac{y}{x}$

Now let $\omega \in \Omega^1(\mathbb{RP}^1)$ and consider in local coordinate u,

i.e. $\omega = g(u)\,du$. Then:

$f(x) \in \mathbb{RP}^1$ considered in coordinate u.

$$f^*(\omega) = f^*(g(u)du) = g(u(f(x)))\,d(u \circ f(x))$$

$$= g(u([x,1]))\,d(u([x,1]))$$

$$= g\left(\tfrac{1}{x}\right) d\left(\tfrac{1}{x}\right)$$

$$= -g\left(\tfrac{1}{x}\right) \cdot \frac{dx}{x^2}.$$

Now suppose that $g(\frac{1}{x})\frac{1}{x^2} = \sum_{i=0}^{n} a_i x^i$, a polynomial.

$$\Rightarrow \quad g\left(\tfrac{1}{x}\right) = \sum_{i=0}^{n} a_i x^{i+2}$$

$$\Rightarrow \quad g(u) = \sum_{i=0}^{n} a_i \frac{1}{u^{i+2}}$$, i.e. g is not smooth in coordinate u,

which contradicts our hypothesis.

Hence $f^*(\omega)$ cannot be a polynomial.

(7) M cpt, $C^\infty(M) = \{f : M \to \mathbb{R} \text{ smooth}\}$, $\mathcal{I} \subseteq C^\infty(M)$ max. ideal

Show: $\exists\, x_0 \in M$ s.t. $\mathcal{I} = \{f \in C^\infty(M) : f(x_0) = 0\}$

Suppose this is not true, i.e. for each $x \in M$, $\exists f_x \in \mathcal{I}$ such that $f_x(x) \neq 0$. The f_x are continuous, hence let U_x be the nbhd of x such that $f_x \neq 0$. By construction, we have that $\{U_x\}_{x \in M}$ an open cover of M. M is compact, so take a finite subcover:

$U_{x_1}, \dots, U_{x_k}$ and define $F = f_{x_1}^2 + \dots + f_{x_k}^2 \in \mathcal{I}$

Clearly $F(x) > 0$ for all $x \in \bigcup_{i=1}^{k} U_{x_i} = M$, hence $F > 0$ over all M, hence F is a unit in $C^\infty(M)$.

But, $\mathcal{I}$ is a proper ideal, hence $F \in \mathcal{I}$ is a contradiction since $\mathcal{I}$ may not contain a unit (o/w, $\mathcal{I} = C^\infty(M)$).

Therefore, there must be a point x_0 such that $f(x_0) = 0$ for all $f \in \mathcal{I}$.

Let $\mathcal{I}(x_0) = \{f \in C^\infty(M) : f(x_0) = 0\}$

$\Rightarrow \mathcal{I} \subseteq \mathcal{I}(x_0) \subsetneq C^\infty(M) \Rightarrow \mathcal{I} = \mathcal{I}(x_0)$ by maximality of $\mathcal{I}$.