Graduate Exam

Geometry and Topology

Fall 2007

check ① ✓

Problem 1. Let X be a path connected space such that $H_p(X, \mathbb{Z}) = 0$ for every p with $0 < p \leq n$. If $X \times S^n$ denotes the product of X with the n dimensional sphere S^n, compute the homology groups $H_p(X \times S^n; \mathbb{Z})$ for every p with $0 < p \leq n$.

✓

Problem 2. Let C_1 and C_2 be two disjoint circles in $\mathbb{R}^3$, and let $A = S^1 \times [0, 1]$ denote the cylinder. Let X be the space obtained from the disjoint union $\mathbb{R}^3$ ~~X~~ $\sqcup A$ by gluing the boundary component $S^1 \times \{0\}$ of A to the circle C_1 by a homeomorphism, and by gluing the other boundary component $S^1 \times \{1\}$ to C_2 by another homeomorphism. Compute the fundamental group of the space X so obtained.

✓

Problem 3. Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ matrices with coefficients in $\mathbb{R}$, and consider the determinant function $det : M_n(\mathbb{R}) \to \mathbb{R}$, which to a matrix A associates its determinant $det(A)$. Compute the differential map (also called tangent map) of the function det at the identity matrix $I_n \in M_n(\mathbb{R})$.

✓ ① ✓

Problem ④. Let M be a compact orientable n dimensional manifold whose boundary ∂M is homeomorphism to the sphere $S^{n-1} \subset \mathbb{R}^n$ by a homeomorphism $f : \partial M \to S^{n-1}$. Let F be a continuous map $F : M \to \mathbb{R}^n$ whose restriction to the boundary ∂M coincides with f. Show that the image $F(M)$ necessarily contains the center O of the sphere S^{n-1}.

✓

Problem 5. Let Ω be the open shell in $\mathbb{R}^2$ consisting of those $(x, y) \in \mathbb{R}^2$ such that $1 < x^2 + y^2 < 10$, and consider the 1-form

$$\omega = \frac{x\,dy - y\,dx}{4x^2 + y^2}$$

a) Show that ω is closed in Ω.

b) Show that ω is not ~~closed~~ in Ω. (Possible hint: consider an ellipse of equation $4x^2 + y^2 =$ constant). exact

typo?

check ⑥ ✓ surjectivity argument

Problem ⑥. Let $\mathbb{RP}^2$ denote the real projective plane of dimension 2. Consider the map $\varphi : \mathbb{R}^2 \to \mathbb{RP}^2$ which to $(x, y) \in \mathbb{R}^2$ associates the element of $\mathbb{RP}^2$ represented by the line passing through the point $(x, y, 1)$. (Recall that $\mathbb{RP}^2$ is the space of lines passing through the origin in $\mathbb{R}^3$.) If $C = \{(x, y) \in \mathbb{R}^2; y^2 = x^3 - x\}$, show that the closure $\overline{\varphi(C)}$ of $\varphi(C)$ in $\mathbb{RP}^2$ is a differentiable submanifold of $\mathbb{RP}^2$.

✓

Problem ⑦. Let M and N be two compact connected manifolds of the same dimension n, and let $f : M \to N$ be a continuous map. Suppose that the homomorphism $H_n(f) : H_n(M; \mathbb{Z}) \to H_n(\mathbb{Z})$ induced by f is not 0. If $f_* : \pi_1(M, x_0) \to \pi_1(N, f(x_0))$ is the homomorphism induced by f between the fundamental groups, show that its image $f_*(\pi(M; x_0))$ has finite index in $\pi(N; f(x_0))$. (Possible hint: Consider a suitable covering of N.)

1. Mayer Vietoris

2. $X \overset{h.e.}{\simeq} S^1$

3. Definition of det: $T_I \det(A) = Tr(A)$.

4. $\partial M \xrightarrow{i} M \xrightarrow{F} \mathbb{R}^n - \{0\} \xrightarrow{\frac{x}{|x|}} S^{n-1} \xrightarrow{f^{-1}} \partial M$ is identity

5. Exterior derivative, Stokes

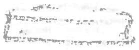
6. Consider the image of the curve as projected on the upper hemisphere of S^2.
It must be smooth at $(1,0,0)$.

7. Lifting criterion, one-to-one correspondence between connected covers and subgroups of π_1, if $\tilde{N}$ is non-compact, $\deg \tilde{f} = 0$, a contradiction as $\deg f \neq 0$.

Geo/Top Fall 07:

① X path-connected, $H_i(X)=0$ for all $0<i\le n$.
Compute $H_i(X\times S^n)$ for all i, $0<i\le n$

Let $A = X\times$ , $B = X\times$  so that

$A\cap B = X\times$  $\simeq X\times S^{n-1}$, and $A\cup B = X\times S^n$.

Now apply Mayer-Vietoris:

- Case $i>1$:

$\dashrightarrow H_i(A)\oplus H_i(B)\to H_i(X\times S^n)\to H_{i-1}(A\cap B)$
$\to H_{i-1}(A)\oplus H_{i-1}(B)\dashrightarrow$

$\Rightarrow \dashrightarrow \underline{\overset{0}{H_i(X)\oplus H_i(X)}}\to H_i(X\times S^n)\to H_{i-1}(X\times S^{n-1})$
$\to \overset{0}{H_{i-1}(X)\oplus H_{i-1}(X)}$ $\Rightarrow$ $\underline{H_i(X\times S^n)\cong H_{i-1}(X\times S^{n-1})}$ by exactness.

Case $i=1$:

$\to 0\to H_1(X\times S^n)\to H_0(X\times S^{n-1})\xrightarrow{(i^*,j^*)} H_0(X)\oplus H_0(X)$
$\to H_0(X\times S^n)\to 0$; X path connected, so $H_0(X)\cong H_0(X\times S^n)\cong\mathbb{Z}$,

so we have:

$\dashrightarrow 0\to H_1(X\times S^n)\to\mathbb{Z}\xrightarrow{(i^*,j^*)}\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\to 0.$

Now consider the inclusions:

$i: A\cap B\hookrightarrow A$
$X\times$  $\hookrightarrow X\times$ 

$j: A\cap B\hookrightarrow B$
$X\times$  $\hookrightarrow X\times$ 

$\rightsquigarrow$ $i^*: H_0(A\cap B)\to H_0(A)$
$\alpha\longmapsto\alpha$,
$j^*: H_0(A\cap B)\to H_0(B)$
$\alpha\longmapsto\alpha$

$\to$ Clearly injective since path connected & takes vertex to vertex.

So we've shown that $(i^*, j^*)_0 : H_0(X \times S^{n-1}) \longrightarrow H_0(X) \oplus H_0(X)$ is injective, hence:

$$0 \longrightarrow H_1(X \times S^n) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{(i^*, j^*)_0} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{\partial_0} 0$$

$\ker(i^*, j^*)_0 = 0 \Rightarrow \operatorname{Im} \partial_1 = 0 \Rightarrow H_1(X \times S^n) = 0$

since ∂_1 injective by exactness

So
$$\begin{cases} H_0(X \times S^n) = \mathbb{Z} \\ H_1(X \times S^n) = 0 \\ H_i(X \times S^n) \cong H_i(X \times S^{n-1}) & i > 1. \end{cases}$$

(2.) C_1, C_2 disjoint circles in $\mathbb{R}^3$, $A = S^1 \times [0,1]$ cylinder

Let $X = \mathbb{R}^3 \amalg A$ via gluing ∂A to C_1 & C_2 by homeomorphisms:

<u>Compute $\pi_1(X)$.</u>

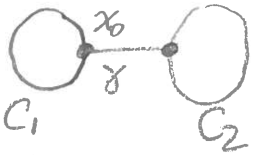
Consider curve γ connecting C_1 & C_2

Now let $A_1 = A \cup C_1 \cup C_2 \cup \gamma$ and let $A_2 = \mathbb{R}^3$

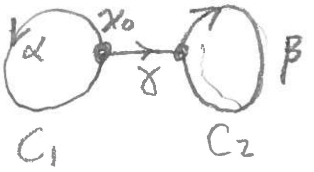
Then $A_1 \cup A_2 = X$ and $A_1 \cap A_2 = C_1 \cup C_2 \cup \gamma$, which is connected, hence we may apply <u>Seifert – Van Kampen</u>:

$$\pi_1(X) \cong \pi_1(A_1) * \pi_1(A_2) \big/ \langle i_{12}(\omega)\, i_{21}(\omega)^{-1} \rangle$$

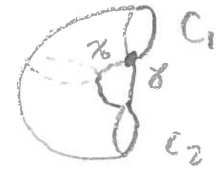
where $i_{12} : \pi_1(A_1 \cap A_2) \rightarrow \pi_1(A_1)$
$i_{21} : \pi_1(A_1 \cap A_2) \rightarrow \pi_1(A_2)$ are induced by inclusion

See that $A_1 \cap A_2 = C_1 \cup C_2 \cup \gamma =$ , hence $\pi_1(A_1 \cap A_2) = \mathbb{Z} * \mathbb{Z}$

with generators the loops α, $\gamma\beta\overline{\gamma}$ (based at x_0):



So we need only consider $i_{12}(\alpha)$, $i_{12}(\gamma\beta\overline{\gamma})$, $i_{21}(\alpha)$, $i_{21}(\gamma\beta\overline{\gamma})$

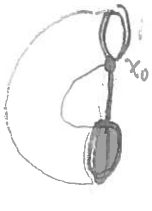
see that $A_1 =$  $\simeq$ , hence $\pi_1(A_1) \cong \mathbb{Z} * \mathbb{Z}$
$= \langle \alpha, \gamma \rangle$

and $A_2 = \mathbb{R}^3$, hence $\pi_1(A_2) = 0$.

So now: $i_{12} : \pi_1(A_1 \cap A_2) \longrightarrow \pi_1(A_1)$

$\parallel$

$\longmapsto$



$i_{12}(\alpha) = a$

$i_{12}(\gamma\beta\bar{\gamma}) = bab^{-1}$

$\langle \alpha, \gamma\beta\bar{\gamma} \rangle$ $\qquad$ $\langle \alpha, \gamma \rangle$

$i_{21} : \pi_1(A_1 \cap A_2) \longrightarrow \pi_1(A_2) = 1$,

hence $i_{21}(\alpha) = 1$
$i_{21}(\gamma\beta\bar{\gamma}) = 1$

Then $\pi_1(X) = \pi_1(A_1) * \pi_1(A_2) \Big/ \langle i_{12}(\alpha)\, i_{21}(\alpha)^{-1}, \ i_{12}(\gamma\beta\bar{\gamma})\, i_{21}(\gamma\beta\bar{\gamma})^{-1} \rangle$

$= \overset{(a)}{\mathbb{Z}} * \overset{(b)}{\mathbb{Z}}) * 1 \Big/ \langle a, bab^{-1} \rangle$

$= \langle a, b : a = 1, bab^{-1} = 1 \rangle$

$= \langle b \rangle = \boxed{\mathbb{Z}}$

③ Consider the determinant function $\det : M_n(\mathbb{R}) \to \mathbb{R}$

Compute the tangent map of det at I_n

We want to find $T_I \det : \underset{\text{\rotatebox{90}{$\cong$}}}{T_I M_n(\mathbb{R})} \longrightarrow \underset{\text{\rotatebox{90}{$\cong$}}}{T_1 \mathbb{R}}$

$$\Rightarrow T_I \det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$$

Consider $A \in T_I M_n(\mathbb{R})$, i.e. $\exists \alpha : (-\varepsilon, \varepsilon) \to M_n(\mathbb{R})$ s.t. $\alpha(0) = I_n$, $\alpha'(0) = A$.

Now let $p_t(x)$ = min poly of $\alpha(t)$ $= \det(\alpha(t) - x I_n)$

$$= (x - \lambda_1(t)) \cdots (x - \lambda_n(t))$$

where $\lambda_i(t)$ are the eigenvalues of $\alpha(t)$; by their definition we can see they are continuous, and $\lambda_i(0) = 1$, $\lambda_i'(0) = \mu_i$, where μ_i e.values of A

Now: $T_I \det(A) = (\det \circ \alpha)'(0)$

So see that $(\det \circ \alpha)(t) = \det(\alpha(t)) = \prod_{i=1}^{n} \lambda_i(t)$

and then:

$$(\det \circ \alpha)' = \frac{d}{dt}\left(\prod_{i=1}^{n} \lambda_i\right) = \lambda_1'(\lambda_2 \cdots \lambda_n) + \lambda_1(\lambda_2 \cdots \lambda_n)'$$

$$= \lambda_1'(\lambda_2 \cdots \lambda_n) + \lambda_1 \lambda_2'(\lambda_3 \cdots \lambda_n) + \lambda_1 \lambda_2 (\lambda_3 \cdots \lambda_n)'$$

$$= \lambda_1'(\lambda_2 \cdots \lambda_n) + \lambda_2'(\lambda_1 \lambda_3 \cdots \lambda_n) + \lambda_3'(\lambda_1 \lambda_2 \lambda_4 \cdots \lambda_n) + \lambda_1 \lambda_2 \lambda_3 (\lambda_4 \cdots \lambda_n)'$$

$$= \sum_{i=1}^{n} \lambda_i' \left(\prod_{j \neq i} \lambda_j\right)$$

hence $T_I \det(A) = (\det \circ \alpha)'(0) = \sum_{i=1}^{n} \lambda_i'(0) \cancelto{1}{\left(\prod_{j \neq i} \lambda_j(0)\right)}$ since $\lambda_j(0) = 1$ for all j

$$= \sum_{i=1}^{n} \mu_i = tr(A)$$

④ M cpt, orntble, $\dim M = n$, $\partial M \approx S^{n-1}$ via $f: \partial M \to S^{n-1}$.
Let $F: M \to \mathbb{R}^n$ continuous s.t. $F|_{\partial M} = f$

Show that $F(M)$ contains $0 \in \mathbb{R}^n$ (the center of S^{n-1})

Suppose that $0 \notin F(M)$. Then $F(M) \subseteq \mathbb{R}^n \setminus \{0\}$ and we have:

$$\begin{tikzcd}
\partial M \arrow[r, hook, "i"] & M \arrow[r, "F"] & \mathbb{R}^n \setminus \{0\} \arrow[r, "r"] & S^{n-1} \arrow[r, "f^{-1}"] & \partial M \\
p \arrow[r, mapsto] & p \arrow[r, mapsto] & f(p) \in S^{n-1} \arrow[r, mapsto] & f(p) \arrow[r, mapsto] & p
\end{tikzcd}$$

where r is the deformation retraction $r(x) = \frac{x}{\|x\|}$

Hence $f^{-1} \circ r \circ F \circ i = id_{\partial M}$; now consider the induced homomorphisms on homology:

$$\begin{tikzcd}
H_{n-1}(\partial M) \arrow[r, "i^*"'] \arrow[d, phantom, "\cong" rotate=90] & H_{n-1}(M) \arrow[r, "(f^{-1}\circ r \circ F)^*"'] & H_{n-1}(\partial M) \arrow[d, phantom, "\cong" rotate=90] \\
H_{n-1}(S^{n-1}) \cong \mathbb{Z}\{\alpha\} \arrow[d, phantom, "\|"] & & H_{n-1}(S^{n-1}) \cong \mathbb{Z}\{\alpha\} \\
{} & &
\end{tikzcd}$$

Now, see that $H_{n-1}(\partial M)$ is generated by a single $(n-1)$-cell that is the boundary of M, hence $i(\alpha)$ boundary in M, i.e. $i^*(\alpha) = 0$ in $H_{n-1}(M)$. Then:

$(f^{-1}\circ r\circ F\circ i)^*(\alpha) = (f^{-1}\circ r\circ F)^*\circ(i)^*(\alpha) = (f^{-1}\circ r\circ F)^*(0) = 0$, a contradiction since we showed $(f^{-1}\circ r\circ F\circ i)^* = id^*$

Therefore, $0 \in F(M)$

⑤ $\Omega = \{(x,y) : 1 < x^2+y^2 < 10\} \subseteq \mathbb{R}^2$, $\omega = \frac{x\,dy - y\,dx}{4x^2+y^2}$

(a) Show ω is closed in Ω

(b) Show ω is not exact in Ω.

(a) $d(\omega) = d\left(\frac{x}{4x^2+y^2}\right) \wedge dy - d\left(\frac{y}{4x^2+y^2}\right) \wedge dx$

Now see:

$$d\left(\frac{x}{4x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{4x^2+y^2}\right)dx + \frac{\partial}{\partial y}\left(\frac{x}{4x^2+y^2}\right)dy$$
$$= \left(\frac{y^2-4x^2}{(4x^2+y^2)^2}\right)dx + \left(\frac{-2xy}{(4x^2+y^2)^2}\right)dy$$

and:

$$d\left(\frac{y}{4x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{y}{4x^2+y^2}\right)dx + \frac{\partial}{\partial y}\left(\frac{y}{4x^2+y^2}\right)dy$$
$$= \left(\frac{-8xy}{(4x^2+y^2)^2}\right)dx + \left(\frac{4x^2-y^2}{(4x^2+y^2)^2}\right)dy$$

So then

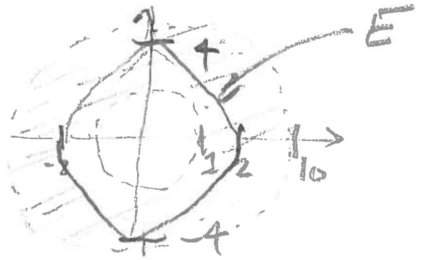
$$d(\omega) = \left[\frac{y^2-4x^2}{(4x^2+y^2)^2}dx + \frac{-2xy}{(4x^2+y^2)^2}dy\right] \wedge dy$$
$$- \left[\frac{-8xy}{(4x^2+y^2)^2}dx + \frac{4x^2-y^2}{(4x^2+y^2)^2}dy\right] \wedge dx$$

$$= \frac{y^2-4x^2}{(4x^2+y^2)^2}dx \wedge dy - \frac{4x^2-y^2}{(4x^2+y^2)^2}dy \wedge dx$$

(since $dx \wedge dx = -dx \wedge dx$ $\Rightarrow dx \wedge dx = 0$)

$$= \frac{y^2-4x^2}{(4x^2+y^2)^2}dx \wedge dy + \frac{4x^2-y^2}{(4x^2+y^2)^2}dx \wedge dy = 0$$

hence $d\omega = 0$, i.e. ω closed.

(b) See that: $\Omega =$ 

Let E be the ellipse $4x^2+y^2=16$, i.e. $E \subseteq \Omega$

Recall that the E is parametrized by:

$x(t) = 2\cos t$, $y(t) = 4\sin t$, $0 \le t \le 2\pi$

Now integrate:

$$\int_E \omega = \int_{\partial E} \frac{x\,dy - y\,dx}{4x^2+y^2}$$

$$= \frac{1}{16}\int_{\partial E} x\,dy - y\,dx$$

$$= \frac{1}{16}\int_0^{2\pi} (x(t)\,y'(t) - y(t)\,x'(t))\,dt$$

$$= \frac{1}{16}\int_0^{2\pi} ((2\cos t)(4\cos t) - (4\sin t)(-2\sin t))\,dt$$

$$= \frac{1}{16}\int_0^{2\pi} 8(\cos^2 t + \sin^2 t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} dt = \frac{2\pi}{2} = \pi \neq 0,$$

But, if ω were exact, by Stokes we get:

$$\int_E \omega = \int_E d\alpha = \int_{\partial \tilde{E}} d\alpha = \int_{\tilde{E}} d^2\alpha = \int_{\tilde{E}} 0 = 0$$

(where $\tilde{E}$ is ellipsoid bdd by E), which is a contradiction

Hence ω is not <u>exact</u>

⑥ Consider $\varphi : \mathbb{R}^2 \to \mathbb{RP}^2$ with $\varphi(x,y) = [x,y,1]$. If $C = \{(x,y) \in \mathbb{R}^2 : y^2 = x^3 - x\}$, show that $\overline{\varphi(C)}$ is a differentiable submanifold of $\mathbb{RP}^2$:

See that $\varphi(C) = \{[x,y,1] : y^2 = x^3 - x\}$

$$= \{[\tfrac{x}{z}, \tfrac{y}{z}, 1] : (\tfrac{y}{z})^2 = (\tfrac{x}{z})^3 - (\tfrac{x}{z}),\ z \neq 0\}$$

$$= \{[x,y,z] : zy^2 = x^3 - z^2x,\ z \neq 0\}$$

(recall $[0,0,0] \notin \mathbb{RP}^2$)

Now, if $z=0$, then $x^3 = 0$, hence $z \neq 0$ only excludes the point $[0,1,0]$, i.e. $\varphi(C) \cup \{[0,1,0]\} = \{[x,y,z] : zy^2 = x^3 - z^2x\}$

Now, note that $\varphi(C) \subseteq \varphi(C) \cup \{[0,1,0]\} = \{[x,y,z] : zy^2 = x^3 - z^2x\}$, but this set is closed since for $f : \mathbb{RP}^2 \to \mathbb{R}$

$$[x,y,z] \mapsto zy^2 + z^2x - x^3$$

we have $f^{-1}(0) = \varphi(C) \cup \{[0,1,0]\}$, hence $\varphi(C) \cup \{[0,1,0]\}$ is closed since it is the inverse image of a closed set ($\{0\}$) in $\mathbb{R}$ under continuous map f. Therefore $\varphi(C) \cup \{[0,1,0]\}$ is the smallest closed set containing $\varphi(C)$, hence $\overline{\varphi(C)} = \varphi(C) \cup \{[0,1,0]\} = \{[x,y,z] : zy^2 = x^3 - z^2x\}$

Now we may proceed by the Regular Value Theorem, i.e. since $\overline{\varphi(C)} = f^{-1}(0)$, we must show T_pf surj. $\forall p \in f^{-1}(0)$:

$$T_pf : T_p\mathbb{RP}^2 \to T_p\mathbb{R}$$

$$T_pf = [z^2 - 3x^2,\ 2zy,\ 2zx + y^2]\big|_{p=[x,y,z]}.$$

Since $p \in \mathbb{RP}^2$, all of x, y, z cannot be zero, and since codom$(T_pf) = \mathbb{R}$ and linear, must only show one elt nonzero and get surjectivity.

- $\underline{x \neq 0} \Rightarrow z^2 - 3x^2 \neq 0$ (done) $\underline{or}$ $z^2 \neq 0 \Rightarrow z \neq 0 \Rightarrow 2zx + y^2 \neq 0$ (done) or $y^2 \neq 0 \Rightarrow y \neq 0 \Rightarrow 2zy \neq 0$ (done).
- $\underline{y \neq 0} \Rightarrow 2zy \underset{(\text{done})}{\neq 0}$ or $z = 0 \Rightarrow 2zx + y^2 = y^2 \neq 0$ (done) $\Rightarrow$ $\underline{T_pf \text{ surjective}}$
- $\underline{z \neq 0} \Rightarrow z^2 - 3x^2 \neq 0$ (done) or $x \neq 0 \Rightarrow 2zx + y^2 \neq 0$ (done) or $y^2 \neq 0 \Rightarrow 2zy \neq 0$ (done)

⑦ M, N cpt, dim n, $f: M \to N$ with $\deg f \neq 0$
Show $f_*(\pi_1(M, x_0)) \leq \pi_1(N, f(x_0))$ has finite index

N cpt, connected mfld $\Rightarrow$ conn, path conn, semiloc s-conn.
$\Rightarrow$ apply classification of connected covers.

Let $p: \tilde{N} \to N$ be a cover corresponding to $f_*(\pi_1(M, x_0)) \leq \pi_1(N, f(x_0))$
Then $f_*(\pi_1(M, x_0)) = p_*(\pi_1(\tilde{N}, \tilde{x}_0))$ by construction, hence by the lifting criterion $\exists \tilde{f}$ such that $f = p \circ \tilde{f}$ $\rightarrow$

$$\left[\begin{tikzcd} & \tilde{N} \arrow[d, "p"] \\ M \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & N \end{tikzcd}\right]$$

So then by construction:

$$\#\text{sheets of } p = [p_*(\pi_1(\tilde{N}, \tilde{x}_0)) : \pi_1(N, f(x_0))]$$
$$= [f_*(\pi_1(M, x_0)) : \pi_1(N, f(x_0))]$$

Hence we're left to show that p finite-sheeted cover. (i.e. $\deg p < \infty$)
Suppose not; i.e. p an infinite sheeted covering space, hence $\tilde{N}$ is not compact.

Recall $\tilde{f}: M \to \tilde{N}$, M compact, $\tilde{N}$ non-compact $\Rightarrow \deg \tilde{f} = 0$

This a contradiction since
$0 \neq \deg f = \deg p \circ \tilde{f} = \deg p \cdot \deg \tilde{f}$,
therefore $\tilde{N}$ is compact; a compact is cover of a compact space is finite-sheeted
hence p is finite-sheeted, hence $]<\infty$

$\rightarrow$ If $\tilde{f}$ surjective, then $\tilde{N} = f(M)$ is compact, a contradiction
So $\tilde{f}$ not surjective, hence $\deg \tilde{f} = 0$

$$[f_*(\pi_1(M, x_0)) : \pi_1(N, f(x_0))] = [p_*(\pi_1(\tilde{N}, \tilde{x}_0)) : \pi_1(N, f(x_0))] < \infty$$

Geometry/Topology Qualifying Exam

Incomplete

2(b).

Spring 2007

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

✓ 1. (15 pts) Let $M_n(\mathbb{R})$ be the space of all $n \times n$ matrices with real entries. (This is, of course, a differentiable manifold.) For $A \in M_n(\mathbb{R})$, define a tangent vector to $M_n(\mathbb{R})$ at the identity matrix I to be the class of the curve $I + tA$, $-\epsilon < t < \epsilon$. Denote this tangent vector by $\bar{A}$.

(a) For any $X \in M_n(\mathbb{R})$, let $R_X : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be defined by $R_X(B) = B \cdot X$. Prove that R_X is differentiable.

(b) For any $\bar{A} \in T_I M_n(\mathbb{R})$, define a vector field $\xi_{\bar{A}}$ on $M_n(\mathbb{R})$ so that $\xi_{\bar{A}}(X) = (R_X)_*(I)(\bar{A})$ (Here $(R_X)_*(I)$ is the derivative of R_X at I.) Compute the Lie bracket $[\xi_{\bar{A}}, \xi_{\bar{B}}]$.

– ○ 2. (15 pts) Let C be the subset of $\mathbb{C}^2$ with coordinates z, w, defined by the equation $w^2 = P(z)$, where $P(z)$ is a polynomial of degree 3.

✓ (a) Prove that if P has no repeated roots, then C is a submanifold of $\mathbb{C}^2$. (Remark: C is a complex submanifold, and hence is also a real submanifold.)

(b) Suppose that P has no repeated roots. Compute the fundamental group of $C - \{(z,w) | w = 0\}$. (Hint: Think of covering spaces.)

✓ 3. (10 pts) Prove that the tangent bundle TM of a smooth manifold M has the structure of a smooth orientable manifold. (Do not assume that M itself is orientable.)

✓ 4. (10 pts) Consider the differential 1-form $\omega = dz - y\,dx$ on $\mathbb{R}^3$ with coordinates (x, y, z). Prove that $f\omega$ is not closed for any nowhere zero function $f : \mathbb{R}^3 \to \mathbb{R}$.

✓ 5. (10 pts) Define the notion of a *deformation retraction* of a space X onto a subset $A \subset X$. Prove that if A is the knot in the solid torus $X = S^1 \times D^2$ as drawn in the picture below, then there is no deformation retraction *of* X onto A.

FIGURE 1

– ✓ 6. (10 pts) Construct a topological space X such that $H_0(X;\mathbb{Z}) = \mathbb{Z}$, $H_1(X;\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}$, $H_2(X;\mathbb{Z}) = \mathbb{Z}$, and all other homology groups are zero.

1.

2.

3. Jacobian of coordinate changes

4.

5. If A is a def. ret. of X, then i_* is an isomorphism on π_1.

6.

Geo/Top – Spr'07 :

① Define tangent vector $A \in T_I(M_n(\mathbb{R}))$ by curve $\alpha : (-\varepsilon, \varepsilon) \to M_n(\mathbb{R})$, $\alpha(t) = I_n + tA$

(a) For $X \in M_n(\mathbb{R})$, let $R_X : M_n(\mathbb{R}) \to M_n(\mathbb{R})$
$\xi \longmapsto \xi X$

Show R_X is differentiable :

Let $\varphi : M_n(\mathbb{R}) \longrightarrow \mathbb{R}^{n^2}$
$(a_{ij}) \longmapsto (a_{11}, a_{12}, \cdots a_{nn})$; then $(M_n(\mathbb{R}), \varphi)$ is an atlas for $M_n(\mathbb{R})$.

Now we just need that $\varphi \circ R_X \circ \varphi^{-1}$ differentiable:

$$\varphi \circ R_X \circ \varphi^{-1}(a_{11}, a_{12}, \ldots, a_{nn}) = \varphi \circ R_X\, [a_{ij}]$$
$$= \varphi([a_{ij}]X) = \varphi([p_{ij}(a_{11}, \ldots, a_{nn})])$$
$$= (p_{11}(a_{11}, \ldots, a_{nn}), \ldots, p_{nn}(a_{11}, \ldots, a_{nn}))$$

where p_{ij} are polynomial the a_{ij} with coefficients the entries of X.

Therefore $\varphi \circ R_X \circ \varphi^{-1}$ has polynomial component functions, hence differentiable in $\mathbb{R}^{n^2}$, hence R_X differentiable.

(b) $A \in T_I(M_n(\mathbb{R}))$, $\xi_A(X) = T_I(R_X)(A)$ vector field.

Compute $[\xi_A, \xi_B]$:

$A \in T_I(M_n(\mathbb{R}))$, $\alpha(t) = I_n + tA$, hence $\alpha(0) = I_n$, $\alpha'(0) = A$;
then $T_I(R_X)(A) = (R_X \circ \alpha)'(0)$:

$$(R_X \circ \alpha)(t) = R_X(\alpha(t)) = R_X(I_n - tA) = X + tAX$$

$\Rightarrow (R_X \circ \alpha)'(t) = AX \Rightarrow T_I(R_X)(A) = (R_X \circ \alpha)'(0) = AX$,
hence $\xi_A(X) = AX$. Then $[\xi_A, \xi_B](X) = \xi_A \circ \xi_B(X) - \xi_B \circ \xi_A(X)$
$= \xi_A(BX) - \xi_B(AX) = ABX - BAX = (AB - BA)X$.

②. $C = \{(z,w) \in \mathbb{C}^2 : w^2 = P(z)\} \subseteq \mathbb{C}^2$ where p polynomial of degree 3

(a) If p has no repeated roots, show $C \subseteq \mathbb{C}^2$ submanifold

---

Consider the map $f: \mathbb{C}^2 \to \mathbb{C}$

$$(z,w) \longmapsto P(z) - w^2$$

Then $f^{-1}(0) = C$, so we may apply the Regular val. thm:

Consider the tangent map:

$$T_pf : T_p\mathbb{C}^2 \to T_{f(p)}\mathbb{C} \quad , \quad p = (z,w) \in f^{-1}(0) = C$$

$$T_pf = [\,P'(z) \quad -2w\,]$$

Now, since P is over $\mathbb{C}$ (alg. closed) and has no repeated roots, it has the form $P(x) = (x-\alpha)(x-\beta)(x-\gamma)$, i.e. it is a separable polynomial, hence we know that $\gcd(P, P') = 1$, hence P & P' do not share any roots, hence for $p = (z,w) \in f^{-1}(0)$ such that $w^2 = 0$, we have $P'(z) \neq 0$.

On the other hand, if $w^2 \neq 0$, then $-2w \neq 0$.

Hence in either case $T_pf \neq 0$, hence surjective since linear map onto 1-dim'l codomain.

$\longrightarrow$ Thus $f^{-1}(0) = C$ a submfld by reg. val. thm.

(b) If p has no repeated roots, find $\pi_1(C \setminus \{w=0\})$.

③ M smooth mfld; show TM is orientable.

Recall that $TM = \coprod_{p\in M} T_pM = \{(p,v) : p\in M,\ v\in T_pM\}$

Since M a smooth mfld it has an atlas $\{(U,\varphi)\}$ where $\varphi: U \to \mathbb{R}^n$.

Now we can define an atlas $\{(\tilde{U},\tilde{\varphi})\}$ on TM via the following:

Let (U,φ) be a chart on M. Then let:

$$\tilde{U} = \coprod_{p\in U} T_pM \subseteq TM \quad \text{and} \quad \tilde{\varphi}(p,v) = (\varphi(p), T_p\varphi(v)) \in \mathbb{R}^{2n}$$

Now let $(\tilde{U},\tilde{\varphi}), (\tilde{V},\tilde{\psi})$ be charts defined as above, and consider the transition $\varphi\circ\tilde{\psi}^{-1} : \tilde{\psi}(\tilde{U}\cap\tilde{V}) \to \tilde{\varphi}(\tilde{U}\cap\tilde{V})$.

Let $(x_1,\dots,x_{2n}) \in \tilde{\psi}(\tilde{U}\cap\tilde{V}) \subseteq \mathbb{R}^{2n}$, and then:

$$\tilde{\varphi}\circ\tilde{\psi}^{-1}(x_1,\dots,x_{2n}) = \tilde{\varphi}(\psi^{-1}(x_1,\dots,x_n), T_{(x_1,\dots,x_n)}(\psi^{-1})(x_{n+1},\dots,x_{2n}))$$

$$= (\varphi\circ\psi^{-1}(x_1,\dots,x_n), T_{\psi^{-1}(x_1,\dots,x_n)}\circ T_{(x_1,\dots,x_n)}(\psi^{-1})(x_{n+1},\dots,x_{2n}))$$

Chain rule →

$$= (\varphi\circ\psi^{-1}(x_1,\dots,x_n), T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})(x_{n+1},\dots,x_{2n}))$$

" which is a diffeomorphism since $\varphi\circ\psi^{-1}$ is a diffeomorphism by virtue of it being a transition from a known atlas.

Now to show that $\{(\tilde{U},\tilde{\varphi})\}$ is orientable atlas, we must show that $\det(T_p(\tilde{\varphi}\circ\tilde{\psi}^{-1})) > 0$ for all p; now see:

$$T_{(x_1,\dots,x_{2n})}(\tilde{\varphi}\circ\tilde{\psi}^{-1})(v_1,\dots,v_{2n})$$

$$= \left(T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})(v_1,\dots,v_n),\ T_{(x_{n+1},\dots,x_{2n})}\left(T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})\right)(v_{n+1},\dots,v_{2n})\right)$$

Since $T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})$ is a linear map on $T_{(x_1,\dots,x_n)}\mathbb{R}^n \cong \mathbb{R}^n$, its tangent map induces the same linear map on $T_{(x_{n+1},\dots,x_{2n})}(T_{(x_1,\dots,x_n)}\mathbb{R}^n) \cong \mathbb{R}^n$:

→ { Let $F: \mathbb{R}^m \to \mathbb{R}^n$ be linear, $\alpha(t) = p + tv$; then:

$(F\circ\alpha)(t) = F(\alpha(t)) = F(p+tv) = F(p) + tF(v)$

$\Rightarrow T_pF(v) = (F\circ\alpha)'(0) = F(v)$

Therefore,

$$T_{(x_1,\dots,x_{2n})}(\tilde{\varphi}\circ\tilde{\psi}^{-1})(v_1,\dots,v_{2n})$$

$$= \left(T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})(v_1,\dots,v_n),\ T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})(v_{n+1},\dots,v_{2n})\right)$$

Now let w_i be the eigenvectors of $T_{(x_1,\dots,x_n)}(\varphi\circ\psi^{-1})$, hence $\{(w_i,0),(0,w_i)\}$ are the eigenvectors of $T_{(x_1,\dots,x_{2n})}(\tilde{\varphi}\circ\tilde{\psi}^{-1})$.

Now let λ_i be the eigenvalues corresponding to the w_i and see that:

$$T_x(\tilde{\varphi}\circ\tilde{\psi}^{-1})(w_i,0) = (T_x(\varphi\circ\psi^{-1})(v_i),0) = (\lambda_i v_i, 0) = \lambda_i(v_i,0)$$

$$T_x(\tilde{\varphi}\circ\tilde{\psi}^{-1})(0,w_i) = (0, T_x(\varphi\circ\psi^{-1})(v_i)) = (0,\lambda_i v_i) = \lambda_i(0,v_i),$$

hence the eigenvalues of $T_x(\tilde{\varphi}\circ\tilde{\psi}^{-1})$ are $\{\lambda_1,\lambda_1,\lambda_2,\lambda_2,\dots\}$, hence

$$\underline{\det(T_x(\tilde{\varphi}\circ\tilde{\psi}^{-1})) = \prod_i \lambda_i^2 > 0}$$

since $\lambda_i \neq 0$ for all i since $\varphi\circ\psi^{-1}$ homeomorphism

$\Rightarrow T_x(\varphi\circ\psi^{-1})$ invertible.

④ Consider $\omega = dz - y\,dx \in \Omega^1(\mathbb{R}^3)$

Show $f\omega$ is not closed for any nonzero $f: \mathbb{R}^3 \to \mathbb{R}$

First compute the exterior derivative:

$$d(f\omega) = d(f \wedge \omega) = df \wedge \omega + f \wedge d\omega :$$

- $d\omega = d(dz - y\,dx) = \cancel{d^2 z}^{\to 0} - d(y\,dx) = -dy \wedge dx = \underline{dx \wedge dy}$
- $df = f_x dx + f_y dy + f_z dz$
- $df \wedge \omega = (f_x dx + f_y dy + f_z dz) \wedge (dz - y\,dx)$

$$= f_x dx \wedge (dz - y\,dx) + f_y dy \wedge (dz - y\,dx) + f_z dz \wedge (dz - y\,dx)$$

$$= f_x dx \wedge dz - f_x \cancel{dx \wedge y\,dx}^{\to 0} + f_y dy \wedge dz - f_y y\, dz \wedge dx + f_z \cancel{dz \wedge dz}^{\to 0} - f_z dz \wedge y\,dx$$

$$= f_x dx \wedge dz + f_y dy \wedge dz - f_y y\, dy \wedge dx - f_z y\, dz \wedge dx$$

$$= f_x dx \wedge dz + f_z y\, dx \wedge dz + f_y dy \wedge dz + f_y y\, dx \wedge dy$$

$$= (f_x + f_z y)\, dx \wedge dz + f_y y\, dx \wedge dy + f_y dy \wedge dz$$

- $f \wedge d\omega = f\, dx \wedge dy.$

So $d(f\omega) = (f_x + f_z y)\, dx \wedge dz + (f_y y)\, dx \wedge dy + (f_y)\, dy \wedge dz + f\, dx \wedge dy$

$$= (f_x + f_z y)\, dx \wedge dz + (f_y y + f)\, dx \wedge dy + (f_y)\, dy \wedge dz$$

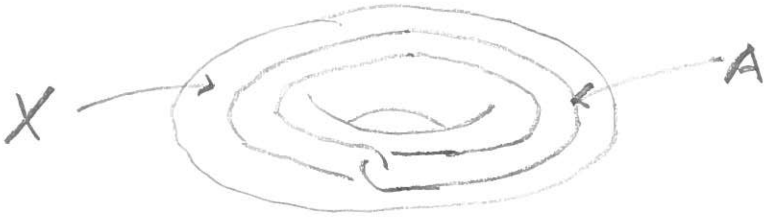
Suppose now that $f\omega$ is closed, i.e. $d(f\omega) = 0$.

Then: $\left.\begin{aligned} f_x + f_z y &= 0 \\ f_y y + f &= 0 \\ f_y &= 0 \end{aligned}\right\} \Rightarrow (0)y + f = 0 \Rightarrow f = 0$, a contradiction since $f \neq 0$ by hypothesis.

<u>So $f\omega$ not closed</u>

(5.) A the pictured knot, $X = S^1 \times D^2$ solid torus.

Show $\nexists$ deformation retract of X onto A:



Recall that if A is a deformation retract of X, then the induced homomorphism on fund. grp of the inclusion $i: A \hookrightarrow X$ is an isomorphism

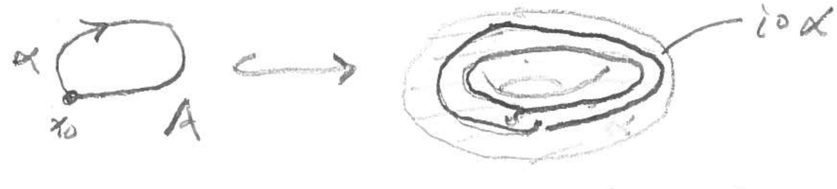
$$i_*: \pi_1(A) \xrightarrow{\cong} \pi_1(X)$$

So suppose that A def. retract of X. Then

$i_*: \pi_1(A) \to \pi_1(X)$ isomorphism

But now let the loop α be a generator for $\pi_1(A, x_0) \cong \pi_1(S^1, x_0) \cong \mathbb{Z}\langle\alpha\rangle$. Then:

$$i_*: \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0)$$



And see that $i \circ \alpha$ is a nullhomotopic path in $X = S^1 \times D^2$ since solid, i.e. $i_*(\alpha) = 1$, hence i_* must be trivial since α was the generator of $\pi_1(A)$, hence i_* not an isomorphism, a contradiction; hence, A not def. ret. of X.

⑥ Construct top. space X s.t. $H_0(X)=\mathbb{Z}$, $H_3(X)=\mathbb{Z}/5\mathbb{Z}$, $H_5(X)=\mathbb{Z}$, and all others are 0

Our space will consist of 1 0-cell v, 1 3-cell α, 1 4-cell β, and 1 5-cell γ, with the following chain complex:

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}\{\gamma\} \arrow[r, "\partial_5"] & \mathbb{Z}\{\beta\} \arrow[r, "\partial_4"] & \mathbb{Z}\{\alpha\} \arrow[r, "\partial_3"] & 0 \arrow[r, "\partial_2"] & 0 \arrow[r, "\partial_1"] & \mathbb{Z}\{v\} \arrow[r, "\partial_0"] & 0 \\ & \gamma \arrow[r, mapsto] & 0 & & & & & \\ & & \beta \arrow[r, mapsto] & 5\alpha & & & & \end{tikzcd}$$

So we obtain our space by starting with a 0-cell v. Now attach α to v along its boundary.

Then attach β to α by wrapping the boundary of β around α 5 times.

Finally, attach γ so that β is the boundary of γ, i.e. $\partial e^5_\gamma = e^4_\beta$ (i.e. identify the boundary of γ with β).

<u>check</u>:

$$H_3(X) = \ker\partial_3 / \operatorname{im}\partial_4 = \mathbb{Z}\{\alpha\} / \mathbb{Z}\{5\alpha\} \cong \mathbb{Z}/5\mathbb{Z}$$

$$H_4(X) = \ker\partial_4 / \operatorname{im}\partial_5 = 0/0 = 0$$

$$H_5(X) = \ker\partial_5 / \operatorname{im}\partial_6 = \ker\partial_5 = \mathbb{Z}\{\gamma\} \cong \mathbb{Z}$$

Geometry/Topology Qualifying Exam

September 2006

Solve all SEVEN *problems. Partial credit will be given to partial solutions.*

1. Let M, N be compact oriented manifolds of dimension n (without boundary), and let $f : M \to N$ be a differentiable map. Prove that, if the induced homomorphism $f^* : H^n_{dR}(N;\mathbb{R}) \to H^n_{dR}(M;\mathbb{R})$ between de Rham cohomology groups is surjective, then f is surjective.

2. Let D^2 be the closed unit disk in the complex plane $\mathbb{C}$, bounded by the unit circle S^1. Consider the 2-dimensional torus $T^2 = S^1 \times S^1$ and two copies D_1 and D_2 of D^2. For two integers p, q, let X_{pq} be the quotient space of the disjoint union
$$T^2 \sqcup D_1 \sqcup D_2$$
by the equivalence relation that identifies each point $e^{i\theta}$ in the boundary of D_1 to $(e^{ip\theta}, 1) \in S^1 \times S^1$, and identifies each point $e^{i\varphi}$ in the boundary of D_2 to $(1, e^{iq\varphi}) \in S^1 \times S^1$. Compute the fundamental group of X_{pq}.

3. Prove that any two continuous maps $f, g : X \to S^1$ from a simply-connected space X to the circle S^1 are homotopic.

4. Calculate the relative homology groups $H_*(S^1 \times D^2, S^1 \times \partial D^2)$, where D^2 denotes the 2-dimensional closed disk and S^1 is the circle.

check

5. Let M be a compact oriented n-manifold with $H^1_{dR}(M;\mathbb{R}) = 0$ and let $f : M \to T^n$ be a smooth map. Show that the degree of f is equal to 0. (Possible hint: Write $T^n = S^1 \times \cdots \times S^1$; if θ_i is the angular coordinate for the i th factor S^1, then $d\theta_1 \wedge \cdots \wedge d\theta_n$ is a volume form for T^n.)

check

6. Recall that the *rank* of a matrix is the dimension of the span of its row vectors. Show that the space of all 2×3 matrices of rank 1 forms a smooth manifold.

7. Consider the group SO(3) of orientation-preserving isometries of the 2-dimensional sphere S^2. Namely, SO(3) consists of all rotations of $\mathbb{R}^3$ whose axis passes through the origin or, equivalently of all 3×3 matrices A such that $AA^t = \mathrm{Id}$ and $\det(A) = 1$. Prove that, if ω is a 1-form (not necessarily closed) on S^2 such that $\phi^*(\omega) = \omega$ for every $\phi \in \mathrm{SO}(3)$, then $\omega = 0$.

1. If f were not surjective, $\deg f = 0$.
2. Van kampen
3. Lifting criterion and Covering spaces
4. Long Exact Sequence for relative homology
5. Use pullbacks and $H^1_{dR}(M) = 0$
6. Regular Value Theorem, use cross product.
7. $SO(3)$ acts transitively on the tangent bundle of S^2.

Geo/Top - Fall 06 :

① M, N cpt, ornt, dim. n, $f: M \to N$ differentiable.
Suppose $f^*: H^n_{dR}(N) \to H^n_{dR}(M)$ is surjective.
Then prove f is surjective:

Suppose that f is not surjective. Then $\exists y \in N$ such that $y \notin f(M)$, hence the fiber $f^{-1}(y)$ is empty, hence the geom. degree of f is zero here, hence $\deg f = 0$. since y is trivially a regular value and deg f must agree with geometric degree at all regular values.

Recall that since M, N compact we have top-dim de Rham isomorphisms $I: H^n(M) \to \mathbb{R}$, $I: H^n(N) \to \mathbb{R}$, and then the map $I \circ f^* \circ I^{-1} \equiv 0$ since $\deg f = 0$, hence $I \circ f^* \circ I^{-1}$ not surjective, hence f^* not surjective since we already know the I maps are isomorphisms.

This is a contradiction since we assumed f^* was surjective; hence f is surjective.

② $D^2 \subseteq \mathbb{C}$ closed unit disk.

Consider torus $T^2 = S^1 \times S^1$ and two copies D_1, D_2 of D^2.

For integers p, q, let $X_{p,q} = T^2 \amalg D_1 \amalg D_2 / \sim$

where $e^{i\theta} \in \partial D_1 \sim (e^{ip\theta}, 1) \in S^1_a \times S^1_b$

$e^{i\theta} \in \partial D_2 \sim (1, e^{iq\theta}) \in S^1_a \times S^1_b$

Compute $X_{p,q}$:

We'll apply Seifert–van Kampen twice.

Step 1: Let $X_p := T^2 \amalg D_1 / \sim$ where $e^{i\theta} \in \partial D_1 \sim (e^{ip\theta}, 1) \in T^2$

Then let $A \simeq T^2$, $B = D_1$, $A \cap B = \partial D_1 \simeq S^1$, $A \cup B = X_p$:

See that $\pi_1(A) \cong \pi_1(T^2) \cong \mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\}$ (see fig.)

and: $\pi_1(A \cap B) \cong \pi_1(S^1) \cong \mathbb{Z}\{\alpha\}$.

Now consider the inclusion maps:

$i_1^*: \pi_1(A \cap B) \cong \langle \alpha \rangle \longrightarrow \pi_1(A) \cong \langle a \rangle \oplus \langle b \rangle$.

∂D_1

$\alpha \longmapsto a^p$

and $i_2^*: \pi_1(A \cap B) \longrightarrow \pi_1(B) \cong \pi_1(D^2) \cong 1$, hence $i_2^*(\alpha) = 1$.

So by Seifert–van Kampen:

$$\pi_1(X_p) = \pi_1(A) * \pi_1(B) \Big/ \langle i_1^*(\alpha) i_2^*(\alpha)^{-1} \rangle = (\mathbb{Z}\{a\} \oplus \mathbb{Z}\{b\}) * 1 \Big/ \langle a^p \rangle$$

$$\cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}$$

2 cont'd :

Step 2 : Now see that $X_{p,q} = T^2 \sqcup D_1 \sqcup D_2/\sim \; = X_p \sqcup D_2/\sim$

Let $A = X_p$, $B = D_2$, $A \cap B = \partial D_2 \simeq S^1$.

Now see that $\pi_1(X_p) = \mathbb{Z}\{a\}/\langle a^p \rangle \oplus \mathbb{Z}\{b\}$

$$\text{and } \pi_1(A \cap B) \cong \pi_1(\partial D_2) \simeq \pi_1(S^1) \cong \mathbb{Z}\{\beta\}$$

and consider the inclusions:

$$i_1^*: \pi_1(A\cap B) \cong \langle \beta \rangle \longrightarrow \pi_1(A) \cong \langle a : a^p \rangle \oplus \langle b \rangle$$

$\underline{\underline{\partial D_2}}$:

$\underline{i_1^*(\beta) = b^q}$

$$\beta \longmapsto b^q$$

and $i_2^*: \pi_1(A \cap B) \longrightarrow \pi_1(B) \cong 1$, hence $\underline{i_2^*(\beta) = 1}$.

So by Seifert-vanKampen:

$$\pi_1(X_{p,q}) = \pi_1(A) * \pi_1(B) / \langle i_1^*(\beta)\, i_2^*(\beta)^{-1} \rangle$$

$$\cong \left(\mathbb{Z}\{a\}/\langle a^p \rangle \oplus \mathbb{Z}\{b\} \right) * 1 / \langle b^q \rangle$$

$$\cong \mathbb{Z}\{a\}/\langle a^p \rangle \oplus \mathbb{Z}\{b\}/\langle b^q \rangle \cong \underline{\underline{\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/q\mathbb{Z}}}$$

③ Any two continuous maps $f, g : X \to S^1$ where $\pi_1(X) = 1$ are homotopic :

Consider the covering map $p : \mathbb{R} \to S^1$.
Since $\pi_1(X)$ is trivial, we get that $f_*, g_* : \pi_1(X) \to \pi_1(S^1)$ are both trivial, hence $f_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}))$

$g_*(\pi_1(X)) \subseteq p_*(\pi_1(\mathbb{R}))$,

i.e. the lifting criterion is satisfied, hence we obtain:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & S^1 \end{tikzcd} \quad \& \quad \begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ X \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & S^1 \end{tikzcd}$$

Now, $\mathbb{R}$ is contractible, hence $\exists$ homotopy $h_t : \mathbb{R} \to \mathbb{R}$ such that $h_0 = id_{\mathbb{R}}$ and $h_1 = const$; then see that $\tilde{f} \circ h_0 = \tilde{f}$, $\tilde{g} \circ h_0 = \tilde{g}$, and $\tilde{f} \circ h_1 = const.$, $\tilde{g} \circ h_1 = const.$, therefore $\tilde{f}_t = \tilde{f} \circ h_t$ and $\tilde{g}_t = \tilde{g} \circ h_t$ give homotopies from $\tilde{f}, \tilde{g}$ to const., i.e. $\tilde{f}$ and $\tilde{g}$ are homotopic.

Thus let F_t be the homotopy such that $F_0 = \tilde{f}$ and $F_1 = \tilde{g}$ and consider the new homotopy $G_t := p \circ F_t$. Then:

$$G_0 = p \circ F_0 = p \circ \tilde{f} = f$$
$$G_1 = p \circ F_1 = p \circ \tilde{g} = g \quad , \quad \text{hence } f \simeq g.$$

④ Calculate the relative homology groups $H_n(S^1\times D^2; S^1\times\partial D^2)$ where D^2 = closed disk, S^1 = circle.

Recall the long exact sequence for relative homology:

$$\cdots \to H_n(A) \xrightarrow[i_*]{} H_n(X) \xrightarrow[j_*]{} H_n(X,A) \xrightarrow[\partial]{} H_{n-1}(A) \to H_{n-1}(X) \to \cdots \to H_0(X,A) \to 0$$

Let $X = S^1\times D^2$ = solid torus $\simeq S^1$

$A = S^1\times\partial D^2 \cong S^1\times S^1 \cong T^2$ torus.

Then $H_n(X) \cong H_n(S^1) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} & n=1 \\ 0 & n>1 \end{cases}$, $H_n(A) \cong H_n(T^2) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z} & n=1 \\ \mathbb{Z} & n=2 \\ 0 & n>2 \end{cases}$

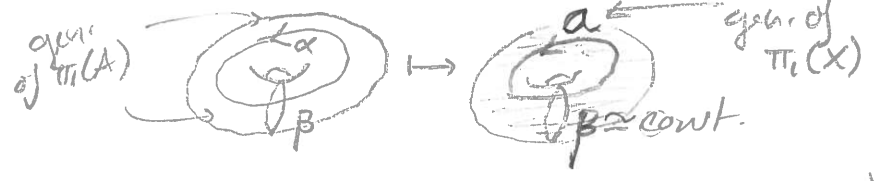
and then we get the sequence:

$$\underbrace{0 \to H_3(X,A) \to \mathbb{Z} \to 0}_{} \to H_2(X,A) \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to H_1(X,A) \to \mathbb{Z} \to \mathbb{Z} \to H_0(X,A) \to 0$$

↳ By exactness, $\underline{H_3(X,A) \cong \mathbb{Z}}$

- Consider the part $0 \to H_2(X,A) \to \underset{\langle\alpha\rangle\oplus\langle\beta\rangle}{\overset{\shortparallel}{H_1(A)}} \xrightarrow[i_*]{} \underset{\langle a\rangle}{\overset{\shortparallel}{H_1(X)}}$ and the inclusion:

$i: A \hookrightarrow X$



Then clearly the generator α is sent to generator a and the generator β is sent to a contractible loop, hence $i_*(\alpha)=a$ and $i_*(\beta)=0$, i.e. $\ker i_* = \mathbb{Z}$ & $\operatorname{im} i_* = \mathbb{Z}$.

By exactness, $H_2(X,A) \xrightarrow{\partial} H_1(A)$ is injective, hence $H_2(X,A) \cong \ker i_* = \mathbb{Z}$. $\Rightarrow$ $\underline{H_2(X,A) \cong \mathbb{Z}}$

- Now consider the part $H_1(X) \xrightarrow{j_*} H_1(X,A) \xrightarrow{\partial} H_0(A) \xrightarrow{i_*} H_0(X) \to$ -

$A\subseteq X$ and both path connected, hence this i_* is injective, i.e. $\underline{0 = \ker i_* = \operatorname{im}\partial}$; but we saw $\mathbb{Z} = \operatorname{im}(i_*: H_1(A)\to H_1(X)) = \ker j_*$, hence $\operatorname{im} j_* = 0$ since $H_1(X)\cong\mathbb{Z}$, i.e. $\underline{0 = \operatorname{im} j_* = \ker\partial}$, hence $\underline{H_1(X,A) \cong 0}$

<u>4 cont'd</u> :

Lastly, consider the part $\overset{\mathbb{Z}}{\overset{\cong}{H_0(A)}} \xrightarrow{i_*} \overset{\mathbb{Z}}{\overset{\cong}{H_0(X)}} \xrightarrow{j_*} H_0(X,A) \longrightarrow 0$.

j_* is surjective by exactness, hence $H_0(X,A) \cong \operatorname{im} j_*$; we already showed that i_* is injective, hence $\mathbb{Z} \cong \operatorname{im} i_* \cong \ker j_*$, hence $\operatorname{im} j_* \cong 0$, hence $\underline{H_0(X,A) \cong 0}$

Thus we've obtained: $H_n(X,A) = \begin{cases} \mathbb{Z} & n=3 \\ \mathbb{Z} & n=2 \\ 0 & n=1 \\ 0 & n=1 \end{cases}$

⑤ M cpt, ont, dim n, $H^1_{dR}(M)=0$, $f: M \to T^n$ smooth. Show $\deg f = 0$.

$T^n = S^1 \times \cdots \times S^1$ is an orientable manifold, hence it will have a volume form. Consider the coordinate system on T^n given by $\theta_i : S^1 \to \mathbb{R}$, hence we may write our volume form on T^n as $d\theta_1 \wedge \cdots \wedge d\theta_n \in \Omega^n(T^n)$

Consider the pullback map $f^*: \Omega^k(T^n) \to \Omega^k(M)$; then:

$$f^*(d\theta_1 \wedge \cdots \wedge d\theta_n) = f^*(d\theta_1) \wedge \cdots \wedge f^*(d\theta_n) \in \Omega^n(M)$$

where the $d\theta_i \in \Omega^1(T^n)$; clearly the $d\theta_i$ are closed, hence $[d\theta_i] \in H^1(T^n)$ well-defined, hence $f^*\underline{(d\theta_i) \in H^1(M) = 0}$, hence $[f^*(d\theta_i)] = 0$ for all i, and then:

$$f^*(d\theta_1 \wedge \cdots \wedge d\theta_n) = f^*(d\theta_1) \wedge \cdots \wedge f^*(d\theta_n) = 0 \in H^n(M)$$

Since the class of the volume form $[d\theta_1 \wedge \cdots \wedge d\theta_n]$ generates $H^n(T^n)$ (by top-dual de Rham, T^n cpt),

$$\begin{cases} f^*: H^n(T^n) \to H^n(M) \\ [\alpha] \longmapsto (\deg f)\alpha \end{cases}$$ is thus the zero map,

hence $\underline{\deg f = 0}$

⑥ Show $M = \{A \in M_{2,3}(\mathbb{R}) : \mathrm{rk} A = 1\}$ smooth manifold

See that $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \in M_{2,3}(\mathbb{R})$ is rank 1 if $a \doteq (a_1, a_2, a_3)$ & $b \doteq (b_1, b_2, b_3)$ are linearly dependent, i.e. colinear in $\mathbb{R}^3$, i.e. $a \times b = 0$, so define

$$f : M_{2,3}(\mathbb{R}) \cong \mathbb{R}^6 \longrightarrow \mathbb{R}^3$$
$$A = (a,b) \longmapsto a \times b$$

hence $M = f^{-1}(0)$; now apply <u>regular value theorem</u>:

For $P = (a,b) \in f^{-1}(0)$, consider $T_p f : T_p \mathbb{R}^6 \longrightarrow T_0 \mathbb{R}^3$

Then choose $V = (V_1, V_2) \in T_p\mathbb{R}^6$ and representative curve $\alpha(t) = tV + P$; then $T_pf(V) = (f \circ \alpha)'(0)$:

$$(f\circ\alpha)(t) = f(tV+P) = f(tv_1 + a, tv_2 + b)$$
$$= (tv_1 + a) \times (tv_2 + b)$$

$$\Rightarrow (f\circ\alpha)'(t) = \frac{d}{dt}\left[(tv_1+a)\times(tv_2+b)\right] = \left(\frac{d}{dt}(tv_1+a)\right)\times(tv_2+b)$$
$$+ (tv_1+a)\times\left(\frac{d}{dt}(tv_2+b)\right)$$
$$= (v_1)\times(tv_2+b) + (tv_1+a)\times(v_2)$$

$$\Rightarrow T_pf(V) = T_{(a,b)}f(V_1,V_2) = (f\circ\alpha)'(0) = V_1 \times b + a \times V_2$$

Now recall that $P = (a,b) \in f^{-1}(0)$, hence $a \times b = 0$, hence $b = \lambda a$, so we have:

$$T_pf(V_1,V_2) = V_1 \times \lambda a + (a \times V_2)$$
$$= -\lambda(a \times V_1) + (a \times V_2) = a \times (V_2 - \lambda V_1)$$

Now let
$$\left.\begin{aligned} u &= (1,1,1,1+\lambda,\lambda,\lambda) \\ v &= (1,1,1,\lambda,1+\lambda,\lambda) \\ w &= (1,1,1,\lambda,\lambda,1+\lambda) \end{aligned}\right\} \in T_p\mathbb{R}^6 \Rightarrow \begin{aligned} T_pf(u) &= a \times i \\ T_pf(v) &= a \times j \\ T_pf(v) &= a \times k \end{aligned}$$

which are linearly independent in $\mathbb{R}^3$, hence T_pf surjective.
(supposing a is multiple of a canonical basis vector; then replace corresponding u, v, w with $x = (1,1,1,1+\lambda,1+\lambda,1+\lambda)$, i.e. $T_pf(x) = a \times (1,1,1)$ & still get basis)

⑦ $SO_3 = \{A \in M_3(\mathbb{R}) : \text{def } A = 1,\ AA^T = I\}$

Prove that if $\omega \in \Omega^1(S^2)$ s.t. $\phi^*(\omega) = \omega$ for all $\phi \in SO_3$, then $\omega \equiv 0$.

SO_3 acts on TS^2 via $\phi(p,v) = (\phi(p), T_p\phi(v))$ and the action is transitive, i.e. for each $(p,v), (q,w) \in TS^2$, $\exists\ \phi \in SO_3$ such that $\phi(p,v) = (q,w)$.

Now, $\phi^*(\omega) = \omega$ for all $\phi \in SO_3$, hence for $(p,v) \in TS^2$, (and since ω 1-form):

$$\omega_p(v) = \phi^*(\omega)_p(v) = \omega_{\phi(p)}(T_p\phi(v))$$

But recall that the action is transitive, hence we can choose $\phi_{p,v} \in SO_3$ s.t. $\phi_{p,v}(p,v) = (p,0)$, and then since $\phi^*(\omega) = \omega$ $\forall \phi \in SO_3$, it is true for $\phi_{p,v}$ in particular, hence for arbitrary $(p,v) \in TS^2$,

$$\omega_p(v) = \phi_{p,v}^*(\omega)_p(v) = \omega_{\phi_{p,v}(p)}(T_p\phi_{p,v}(v))$$

$$= \omega_p(0) = 0$$

Hence $\omega_p(v) = 0$ for arbitrary (p,v), hence $\omega \equiv 0$.

Geometry/Topology Qualifying, Spring 2006

Partial credit for partial solutions

1. Let (x, y, z, w) be Cartesian coordinates on $\mathbf{R}^4$. Is the set defined by the equation $x^2 + xy^3 + yz^4 - w^5 = -1$ a smooth manifold of $\mathbf{R}^4$? Prove your assertion.

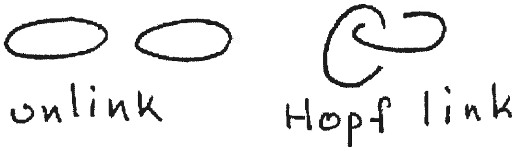
2. a) State the definition of the ith de Rham cohomology group $H^i_{dR}(M)$ of a smooth manifold M.
b) Compute the ith de Rham cohomology groups of the real line $\mathbf{R}$ directly from the definition for all $i \geq 0$.

3. Let X be the quotient space obtained from the n-dimensional sphere S^n by identifying three distinct points to a single common point $p \in X$. In other words, let $q, r, s \in S^3$ be pairwise distinct points, let $X = S^n/\sim$ where $x \sim y$ if $x = y$ or if $x, y \in \{q, r, s\}$, and let $p \in X$ denote the equivalence class $\{q, r, s\}$. Calculate $\pi_1(X, p)$.

4. Let $S^3 = \{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\} \subset \mathbf{R}^4$ and let $\omega = w\, dx \wedge dy \wedge dz$. Compute $\int_{S^3} \omega$.

5. Recall that the *genus* of a closed orientable surface Σ is defined to be $\frac{1}{2} \dim_{\mathbf{R}} H^1_{dR}(\Sigma)$. Let S and T be closed orientable surfaces of respective genera $g(S)$ and $g(T)$. Assume $g(S) < g(T)$. Show that the degree of any smooth map $h : S \to T$ equals zero. [You may use the fact that on a closed orientable surface Σ, the wedge product of one-forms induces a skew-symmetric non-degenerate bilinear pairing $H^1_{dR}(\Sigma) \otimes H^1_{dR}(\Sigma) \to H^2_{dR}(\Sigma) \approx \mathbf{R}$, where $H^i_{dR}(F)$ denotes the ith de Rham cohomology group of Σ.]

6. Define the *unlink* to be the union of two unknotted circles in the three-dimensional sphere S^3, where there are two disjoint three-dimensional balls in S^3 containing the circles. Define the *Hopf link* to be the union of two unknotted disjoint circles in S^3, where each circle meets a disk bounding the other circle in a single point. These links are illustrated in the figure below drawn in $\mathbf{R}^3 = S^3 - \{\text{the point at infinity}\}$. Let U be the complement in S^3 to the unlink and let H be the complement in S^3 to the Hopf link. Calculate the homology groups of U and H.



7. Let X denote a bouquet of $n + 1$ circles, i.e., X is the quotient of the disjoint union of $n + 1$ circles with base points obtained by identifying all the base points to a single point p in the quotient.
a) Prove that $\pi_1(X, p)$ is a free group F_{n+1} on $n + 1$ generators.
b) Let H be a subgroup of F_{n+1} of index k. Show that H is a free group with $kn + 1$ generators.

1. Regular Value Theorem

2. a) Cohomology groups of the cochain complex $0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \to \dots$

   b) Given any smooth f, $g = \int_0^x f(t)dt$ satisfies $d_0 g = f dx$, so $H^0_{dR}(\mathbb{R}) = \mathbb{R}$, $H^1_{dR}(\mathbb{R}) = 0$.

3. $X \overset{h.e.}{\simeq} S^n \vee S^1 \vee S^1$

4. Stokes

5. A dimension argument

$$\begin{tikzcd}
H_1^{dR}(T) \otimes H_1^{dR}(T) \arrow[r, "h^* \otimes h^*"] \arrow[d, "\wedge"] & H_1^{dR}(S) \otimes H_1^{dR}(S) \\
H_2^{dR}(T) &
\end{tikzcd}$$

Write down a basis for $H_1^{dR}(T) \otimes H_1^{dR}(T)$.

6. Think of S^3 as $\mathbb{R}^3 - \{\infty\}$, let one circle pass through ∞ (z-axis). So $U \overset{h.e.}{\simeq}$ Torus. Use "Inverse" Mayer Vietoris on H.

7. Covering Spaces and Van Kampen

Geo./Top - spr '06

① $M = \{(x,y,z,w) : x^2+xy^3+yz^4-w^5 = -1\} \subseteq \mathbb{R}^4$ smooth mfld?

Regular value theorem: Let $f : \mathbb{R}^4 \to \mathbb{R}$

$$(x,y,z,w) \longmapsto x^2+xy^3+yz^4-w^5+1.$$

Now we have $M = f^{-1}(0)$, hence must only show 0 reg. value.
Choose $p \in f^{-1}(0)$;

$$T_p f : T_p\mathbb{R}^4 \to T_0\mathbb{R} \cong \mathbb{R}$$

$$T_p f = [\,2x+y^3 \quad 3xy^2+z^4 \quad 4yz^3 \quad -5w^4\,]\big|_{p=(x,y,z,w)}$$

This is a linear map into $\mathbb{R}$, hence to be surjective we only need that it be nonzero, ie. one entry of T_pf must be nonzero.

Recall that $p=(x,y,z,w) \in f^{-1}(0)$, hence $x^2+xy^3+yz^4-w^5=-1$, hence not all of x,y,z,w may be zero.

Case $x \neq 0$: Then $2x+y^3 \neq 0$ (done) or $y \neq 0$. If $y \neq 0$, then $3xy^2+z^4 \neq 0$ (done) or $z \neq 0$. If $z \neq 0$, then $4yz^3 \neq 0$ (done)

Case $y \neq 0$: Then $2x+y^3 \neq 0$ (done) or $x \neq 0$. If $x \neq 0$, then $3xy^2+z^4 \neq 0$ (done) or $z \neq 0$. If $z \neq 0$ then $4yz^3 \neq 0$ (done).

Case $z \neq 0$: then $4yz^3 \neq 0$ (done) or $y=0$. If $y=0$ then $3xy^2+z^4 = z^4 \neq 0$ (done).

Case $w \neq 0$: $-5w^4 \neq 0$ (done).

Thus T_pf is surjective, hence $f^{-1}(0) = M$ is a submanifold.

② (a) Def'n of $H^i_{dR}(M)$

(b) Compute de Rham cohomology of $\mathbb{R}$ from def'n.

(a) $H^i_{dR}(M)$ are the cohomology groups of the cochain complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \longrightarrow \cdots$$

(b) Consider the cochain complex:

$$0 \longrightarrow \Omega^0(\mathbb{R}) \xrightarrow{d_0} \Omega^1(\mathbb{R}) \xrightarrow{d_1} 0$$

Then we have $H^1(\mathbb{R}) := \ker d_1 / \operatorname{im} d_0$ & $H^0(\mathbb{R}) := \ker d_0$

A form in $\Omega^1(\mathbb{R})$ is of the form $f\,dx$ where $f \in C^\infty(\mathbb{R})$

See that we have $g(x) = \int_0^x f(t)\,dt \in \Omega^0(\mathbb{R})$ such that $d_0 g = f\,dx$ by the Fundamental Thm of Calculus, hence d_0 is surjective, i.e. $\operatorname{im} d_0 = \Omega^1(\mathbb{R})$.

On the other hand, $\ker d_0 = \{ f \in \Omega^0(\mathbb{R}) : d_0 f = 0 \}$

$= \{$ Constant functions on $\mathbb{R} \}$

$\cong \mathbb{R}$

So now: $H^1(\mathbb{R}) = \Omega^1(\mathbb{R}) / \Omega^1(\mathbb{R}) \cong 0$

$H^0(\mathbb{R}) = \mathbb{R}$

③ $X := S^n/_{\{q,r,s\}}$ = S^n with 3 pts identified

Let $p = [q] \in X$ and compute $\pi_1(X, p)$

$X =$ 

$\simeq$ 

$\simeq S^n \vee S^1 \vee S^1$

by Seifert-van Kampen

Hence $\pi_1(X,p) \cong \pi_1(S^n \vee S^1 \vee S^1, p) \cong \pi_1(S^n) * \pi_1(S^1) * \pi_1(S^1)$

$$= \begin{cases} \mathbb{Z}*\mathbb{Z}*\mathbb{Z} & n=1 \\ \mathbb{Z}*\mathbb{Z} & n>1 \end{cases}$$

---

④ $S^3 = \{(x,y,z,w) : x^2+y^2+z^2+w^2=1\} \in \mathbb{R}^4$

Let $\omega = w\,dx \wedge dy \wedge dz$ and compute $\int_{S^3} \omega$.

See that $d\omega = d(w\,dx\wedge dy \wedge dz) = dw \wedge dx \wedge dy \wedge dz$ and apply Stokes.

$$\int_{S^3} \omega = \int_{\partial B^4} \omega = \int_{B^4} d\omega = \int_{B^4} dw \wedge dx \wedge dy \wedge dz = \underline{\mathrm{vol}(B^4)}$$

(5.) Σ closed, orientable surface

$g(\Sigma) = \frac{1}{2}\dim_{\mathbb{R}} H^1_{dR}(\Sigma)$ (genus)

Let S, T be closed, ornt surfaces of genus $g(S), g(T)$ s.t. $g(S) < g(T)$ show that for smooth map $h: S \to T$, $\deg h = 0$.

(Given: for closed, ornt Σ, wedge yields skew-symm bilinear function $H^1(\Sigma) \otimes H^1(\Sigma) \to H^2(\Sigma) \cong \mathbb{R}$)

Recall S, T surfaces, hence degree given by the map

$$h^*: H^2(T) \to H^2(S)$$
$$[\alpha] \longmapsto \deg h\,[\alpha]$$

Now consider the following diagram:

$$\begin{tikzcd}
H^1(T) \otimes H^1(T) \arrow[r, "h^* \otimes h^*"] \arrow[d, "\wedge"'] & H^1(S) \otimes H^1(S) \arrow[d, "\wedge"] \\
H^2(T) \arrow[r, "h^*"] & H^2(S) \\
{[\alpha]} \arrow[r, mapsto] & (\deg h)[\alpha]
\end{tikzcd}$$

Recall that $g(S) < g(T)$, hence:

$$\dim H^1(S) = 2g(S) < 2g(T) = \dim H^1(T),$$

hence $\dim(H^1(T) \otimes H^1(T)) > \dim(H^1(S) \otimes H^1(S))$, hence the map $h^* \otimes h^*$ must have non-trivial kernel by rank-nullity; let $\alpha \otimes b \in \ker(h^* \otimes h^*)$.

Then $\alpha \wedge \beta \in H^2(T)$ and:

$$h^*(\alpha \wedge \beta) = h^*\alpha \wedge h^*\beta = \wedge\left(h^* \otimes h^*(\alpha \otimes \beta)\right)$$
$$= \wedge(0) = 0$$

hence $\ker h^* \neq 0$, hence $\deg h = 0$

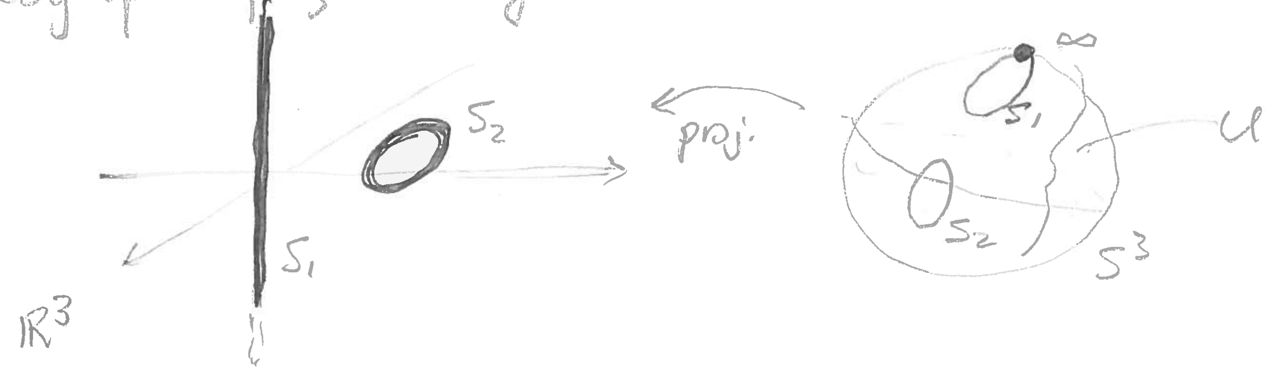
⑥ $U =$  $\subseteq S^3$

$H =$  $\subseteq S^3$ (drawn in $\mathbb{R}^3 = S^3\setminus\{\infty\}$ here)

Find $H_i(S^3\setminus U)$ and $H_i(S^3\setminus H)$:

---

unlink: Consider S^3 as $\mathbb{R}^3 \cup \{\infty\}$ and translate one of the circles in U so that it passes through infinity; then the stereographic projection of $U \subseteq S^3$ onto $\mathbb{R}^3$ looks like:



Therefore, we now have $S^3\setminus U \subseteq S^3\setminus\{\infty\} \cong \mathbb{R}^3$ viewable as $\mathbb{R}^3 - \{\text{circle \& axis}\}$.

Let $A = S^3\setminus S_1 \simeq S^3\setminus S^1 \simeq S^1$

$B = S^3\setminus S_2 \simeq S^3\setminus S^1 \simeq S^1$

and then $A\cap B = S^3\setminus(S_1\cup S_2) = S^3\setminus U$ & $A\cup B = S^3$

Hence: $H_i(A) = H_i(B) = H_i(S^1) = \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & i>1 \end{cases}$

$H_i(A\cup B) = H_i(S^3) = \begin{cases} \mathbb{Z} & i=0,3 \\ 0 & i=1,2,\ i>3 \end{cases}$

Now apply Mayer-Vietoris:

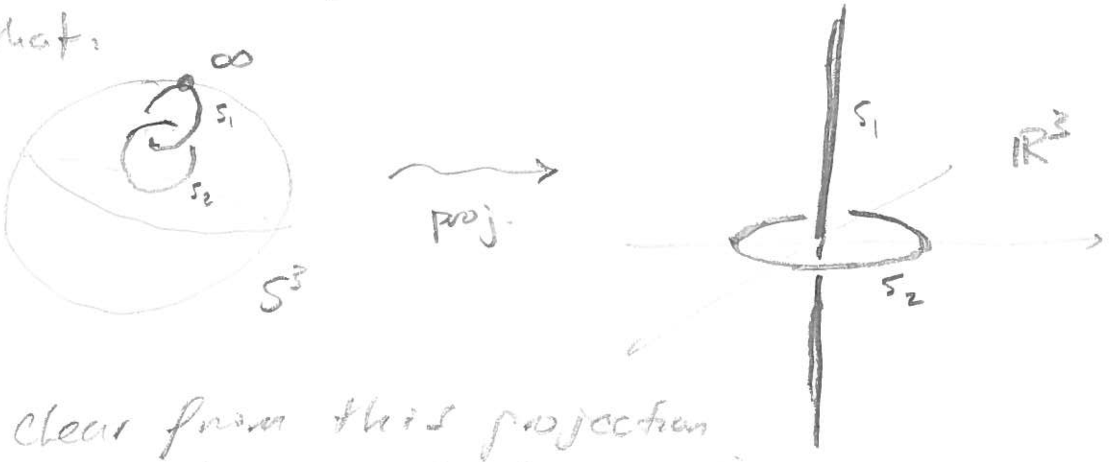
$\to H_3(A\cap B) \to H_3(A)\oplus H_3(B) \to H_3(A\cup B) \to H_2(A\cap B) \to H_2(A)\oplus H_2(B)$
$\to H_2(A\cup B) \to H_1(A\cap B) \to H_1(A)\oplus H_1(B) \to H_1(A\cup B) \to H_0(A\cap B)$
$\to H_0(A)\oplus H_0(B) \to H_0(A\cup B) \to 0$

$\Rightarrow\ 0 \to H_3(S^3\setminus U) \to 0 \to \mathbb{Z} \to H_2(S^3\setminus U) \to 0 \to 0 \to H_1(S^3\setminus U) \to \mathbb{Z}\oplus\mathbb{Z}$
$\to 0 \to H_0(S^3\setminus U) \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$:

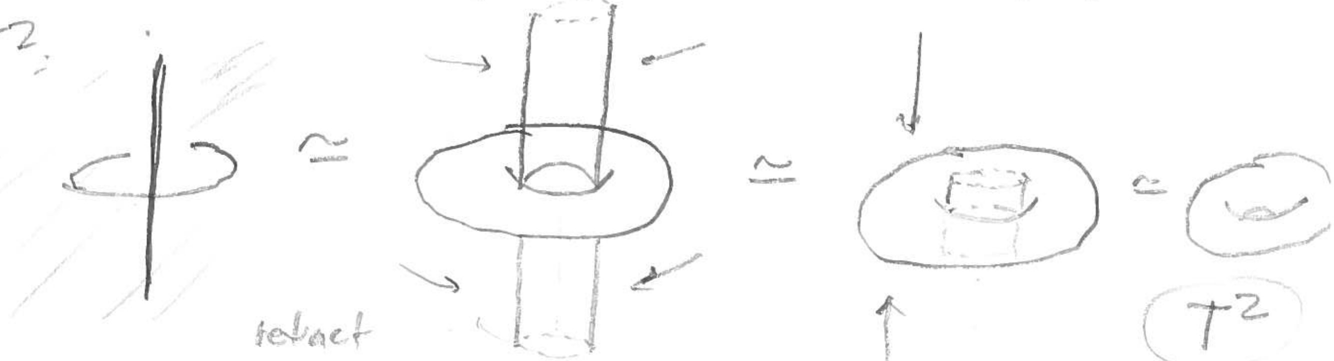
Clearly $H_3(S^3\setminus U)\cong 0$, $H_2(S^2\setminus U)\cong\mathbb{Z}$, $H_1(S^2\setminus U)\cong\mathbb{Z}\oplus\mathbb{Z}$ by exactness, and $H_0(S^2\setminus U)\cong\mathbb{Z}$ by path-connectedness.

So we have:
$$H_i(S^3\setminus U)=\begin{cases}\mathbb{Z} & i=0\\ \mathbb{Z}\oplus\mathbb{Z} & i=1\\ \mathbb{Z} & i=2\\ 0 & i>2\end{cases}$$

Hopf link: Again, translate one of the circles so that it passes thru infinity in $S^3=\mathbb{R}^3\cup\{\infty\}$. Then project and see that:



Now its clear from this projection that $S^3\setminus H\subseteq S^3\setminus\{\infty\}\cong\mathbb{R}^3$, hence we may view the projection as $S^3\setminus H$ itself, and see that the projection is $\simeq T^2$:



Hence
$$H_i(S^3\setminus H)\cong H_i(T^2)=\begin{cases}\mathbb{Z} & i=0\\ \mathbb{Z}\oplus\mathbb{Z} & i=1\\ \mathbb{Z} & i=2\\ 0 & i>2\end{cases}$$

⑦ X = bouquet of $n+1$ circles $= \bigvee^{n+1} S^1$

F_n = free group on n generators

(a) Prove that $\pi_1(X, p) \cong F_{n+1}$

(b) $H \leq F_{n+1}$, $[F_{n+1} : H] = k$. Show that $H = F_{kn+1}$

(a) Let $X_2 = S^1 \vee S^1$. Then let $A =$  $\simeq S^1$

$B =$  $\simeq S^1$

Then $A \cap B = \times \simeq pt.$ and $A \cup B = X_2$

Now apply Seifert-van Kampen; note that both $i_1^* : \pi_1(A \cap B) \to \pi_1(A)$ and $i_2^* : \pi_1(A \cap B) \to \pi_1(B)$ are trivial since $\pi_1(A \cap B) = \pi_1(pt)$ is trivial. Hence:

$$\pi_1(X_2) = \pi_1(A) * \pi_1(B) \Big/ \langle i_1(\omega) i_2(\omega)^{-1} \rangle$$

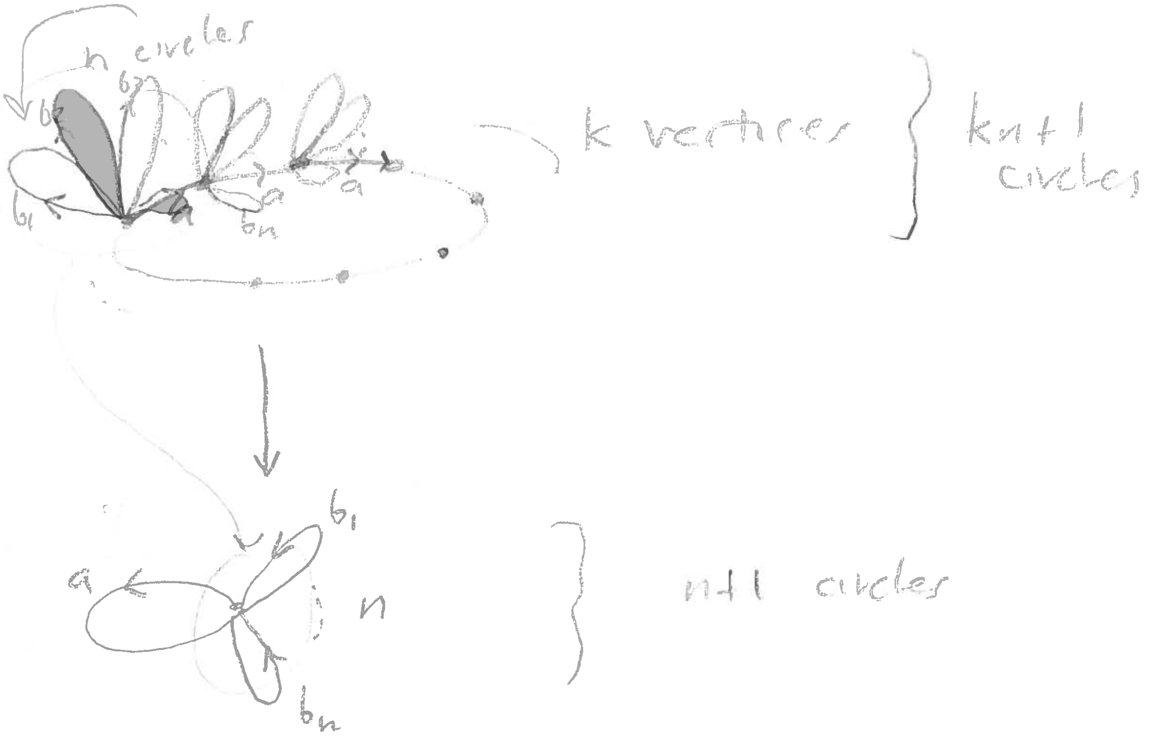
$$\cong \pi_1(S^1) * \pi_1(S^1) \cong \mathbb{Z} * \mathbb{Z} = F_2$$

Now let $X_n = \bigvee^n S^1$ and suppose for induction that $\pi_1(X_n) = F_n$. and let $A \simeq X_n$ and $B \simeq S^1$ so that $A \cup B = X_{n+1}$ and $A \cap B \simeq pt.$ Both inclusions again have trivial induced homomorphisms since $\pi_1(A \cap B) = \pi_1(pt)$ is trivial, hence:

$$\pi_1(X_{n+1}) = \pi_1(A) * \pi_1(B) = \pi_1(X_n) * \pi_1(S^1) = F_n * \mathbb{Z} = \underline{\underline{F_{n+1}}}$$

(β): Let $H \leq F_{n+1}$ with $[F_{n+1} : H] = k$, i.e. $H \leq \pi_1(V^{n+1}S^1)$, hence let $p: \tilde{X} \to V^{n+1}S^1$ be the covering space corresponding to H, i.e. the covering space with $\pi_1(\tilde{X}) \cong p_*(\pi_1(\tilde{X})) = H$. By the choice, $\tilde{X}$ will be a k-sheeted covering space of $V^{n+1}(S^1)$; now we must only show that such a cover is a wedge of $kn+1$ copies of S^1:

See the following; consider a circle S^1 with k vertices chosen, and attach n circles at each vertex



This is the k-sheeted covering of $V^{n+1}S^1$, hence $\pi_1(\tilde{X}) = F_{kn+1}$, i.e. $H = F_{kn+1}$

Incomplete:
6.

Qualifying Exam in Geometry/Topology Fall 2005

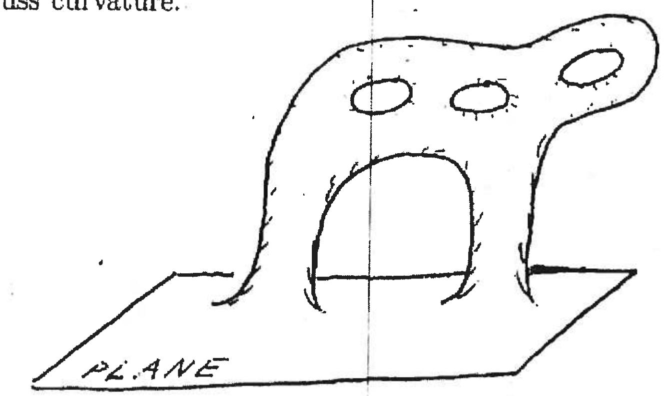
Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. Show that the complement of a finite set of points in R^n is simply connected if $n \geq 3$.

2. Fix a space X and say that two covers $p_i : \tilde{X}_i \to X$, for $i = 1, 2$, are *equivalent* if there is a homeomorphism $f : \tilde{X}_1 \to \tilde{X}_2$ so that $p_1 = p_2 \circ f$. Recall that real projective 2-space RP^2 has its fundamental group isomorphic to the integers mod two, and describe the equivalence classes of connected covers of $RP^2 \times RP^2$.

3. Let α be a closed 2-form on $S^4 = \{(x_1, \ldots, x_5) \in R^5 : x_1^2 + \cdots + x_5^2 = 1\}$. Show that $\alpha \wedge \alpha = 0$ at some point $p \in S^4$.

4. Consider the surface $M \subset R^3$ pictured below. Compute the integral $\int_M K dA$, where K is the Gauss curvature.



(picture of the surface M)

5. Show that for any space X, we have $H_i(X \times S^1) \approx H_i(X) \oplus H_{i-1}(X)$, where S^1 denotes the circle.

6. Given a smooth manifold M, define the *cotangent bundle* $T^*(M)$ to be the set of all pairs (p, q), where $p \in M$ and q lies in the dual vector space to the tangent space $T_p(M)$ of M at p. Show that $T^*(M)$ has the structure of a smooth orientable manifold. (Do not assume that M itself is orientable.)

7. Let M be a smooth manifold. Let $\Omega_c^i(M) \subset \Omega^i(M)$ be the set of smooth i-forms with compact support, i.e., $\omega \in \Omega_c^i(M)$ is zero outside a compact set. Then there is a chain complex

$$0 \to \Omega_c^0(M) \xrightarrow{d_0} \Omega_c^1(M) \xrightarrow{d_1} \Omega_c^2(M) \xrightarrow{d_2} \cdots,$$

where d is the exterior derivative restricted to forms with compact support. Define the ith de Rham cohomology of M *with compact support* to be $ker(d_i)/im(d_{i-1})$. Compute the ith de Rham cohomology of the real line R with compact support for all $i \geq 0$. (Your answer will differ from the usual de Rham cohomology of R.)

1. $\mathbb{R}^n - \{M \text{ points}\} \overset{h.e.}{\simeq} \bigvee_{i=1}^{M} S^{n-1}$

2. $\pi_1(RP^2 \times RP^2) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \Rightarrow \exists$ 4 covers, ~~2 subgroups are con~~

3. Stokes, volume forms have nonzero integral.

4. $M \overset{h.e.}{\simeq}$ surface of genus $4 \setminus pt \Rightarrow \chi(M) = 2-2\cdot 4-1 = -7 \Rightarrow \int_M k\, dA = -14\pi$.

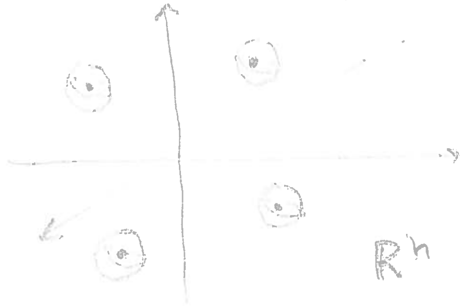
5. Mayer Vietoris / Short Exact Sequence

6. Identify dual of T_pM w/ itself, use tangent bundle trick, ie, tangent maps of coordinate changes are block matrices.

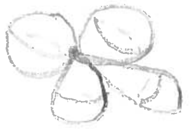
7. $\ker d_0 = 0$ because the only constant functions w/ compact support is 0. $\operatorname{Im} d_0 = \ker d_1$.

GeoTop – Fall 05

① The complement of a finite set of pts in $\mathbb{R}^n$ is simply-connected if $n \geq 3$



$\simeq$



$\simeq \vee^k S^{n-1}$

See that $\mathbb{R}^n \setminus \{p_1, \dots, p_k\} \simeq \vee^k S^{n-1}$, a bouquet of k $(n-1)$-spheres.

Then $\pi_1(\mathbb{R}^n \setminus \{p_1, \dots, p_k\}) \cong \pi_1(\vee^k S^{n-1}) \boxed{\cong} *^k \pi_1(S^{n-1})$

↳ by Seifert – van Kampen.

But for $n \geq 3 \Rightarrow (n-1) \geq 2$, hence $\pi_1(S^{n-1}) = 1$,

hence $\pi_1(\mathbb{R}^n \setminus \{p_1, \dots, p_k\}) \cong *^k 1 \cong 1 \Rightarrow$ simply connected

② Recall that $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$; now describe the equivalence classes of connected covers of $\mathbb{RP}^2 \times \mathbb{RP}^2$.

Recall equiv. classes of connected covers of $\mathbb{RP}^2 \times \mathbb{RP}^2$ correspond to subgroups of $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^3) \cong \pi_1(\mathbb{RP}^2) \times \pi_1(\mathbb{RP}^2)$

$$\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

This group has subgroups 1, $\mathbb{Z}_2 \oplus 1$, $1 \oplus \mathbb{Z}_2$, and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

- $1 \oplus 1$ corresponds to universal cover $S^2 \times S^2$
- $1 \oplus \mathbb{Z}_2$, $\mathbb{Z}_2 \oplus 1$ correspond to the universal covers of the coordinate spaces: $S^2 \times \mathbb{RP}^2$ & $\mathbb{RP}^2 \times S^2$ respectively.
- $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ corresponds to the trivial cover $\mathbb{RP}^2 \times \mathbb{RP}^2$.

③ $\alpha \in \Omega^2(S^4)$ closed. Show that $\alpha \wedge \alpha = 0$ at some $p \in S^4$.

See that $\alpha\wedge\alpha \in \Omega^4(S^4)$ and
$$d(\alpha\wedge\alpha) = d\alpha\wedge\alpha + \alpha\wedge d\alpha = 0\wedge\alpha + \alpha\wedge 0 = 0$$

hence $\alpha\wedge\alpha$ is closed.

Suppose that $\alpha\wedge\alpha \neq 0$ everywhere on S^4, hence a volume form. Now apply Stokes:

$$\int_{S^4} \alpha\wedge\alpha = \int_{B^5} d(\alpha\wedge\alpha) = \int_{B^5} 0 = 0$$

which is a contradiction since volume forms have non-zero integral, hence $\alpha\wedge\alpha = 0$ somewhere on S^4.

---

④ $M =$

$\subseteq \mathbb{R}^3$

Compute $\int_M K\,dA$ where K = Gaussian curvature

See that $M \simeq$

$\simeq$

$\simeq$

$\simeq M_4 \setminus \{pt\}$.

removing a point adds a 1-cell!

Then $\chi(M) = \chi(M_4 \setminus pt) = (2-2g)-1 = (2-8)-1 = -7$

and then by Gauß-Bonnet:

$$\int_M K\,dA = 2\pi\chi(M) = 2\pi(-7) = -14\pi$$

⑤ For any X, $H_i(X\times S^1) \cong H_i(X)\oplus H_{i-1}(X)$

$X\times S^1 = X\times \bigcirc$, so Let $A = X\times \mathsf{C}$, $B = X\times \mathsf{Ↄ}$

$A\cap B = X\times \overset{\frown}{\smile} = X \amalg X$

Now apply Mayer-Vietoris:

$$\cdots \to H_{i+1}(A)\oplus H_{i+1}(B) \longrightarrow H_{i+1}(X\times S^1) \to H_i(A\cap B) \to H_i(A)\oplus H_i(B)$$

$$\longrightarrow H_i(X\times S^1) \to H_{i-1}(A\cap B) \longrightarrow \cdots$$

Now consider the map (for any i)

$$H_i(A\cap B) \cong H_i(X\times \overset{\frown}{\smile}) \xrightarrow{(i^*,j^*)} H_i(A)\oplus H_i(B)$$
$$\cong H_i(X\times \mathsf{C})\oplus H_i(X\times \mathsf{Ↄ})$$

induced from $i: A\cap B \hookrightarrow A$, $j: A\cap B \hookrightarrow B$.

So:

$i^*: H_i(A\cap B) \longrightarrow H_i(A)$ $\qquad$ $j^*: H_i(A\cap B) \longrightarrow H_i(B)$

$X\times \overset{\frown}{\smile} \underset{i}{\hookrightarrow} X\times \mathsf{C}$ $\qquad$ $X\times \overset{\frown}{\smile} \underset{j}{\hookrightarrow} X\times \mathsf{Ↄ}$

Clearly an i-chain in $X\times\overset{\frown}{\smile}$ is the same in $X\times \mathsf{C}$ & $X\times \mathsf{Ↄ}$ (and cannot, say, become a boundary). Hence both are identity, hence

$$(i^*,j^*): \overset{\cong H_i(X)\oplus H_i(X)}{H_i(A\cap B)} \longrightarrow \overset{\cong H_i(X)\oplus H_i(X)}{H_i(A)\oplus H_i(B)} \quad \text{for all } i,$$
$$\alpha \longmapsto (\alpha,\alpha)$$

i.e. $\operatorname{Im}(i^*,j^*)_i = \langle(\alpha,\alpha)\rangle = H_i(X)$

Therefore, we now have the sequence:

$$\longrightarrow H_i(A\cap B) \xrightarrow{(i^*,j^*)_i} H_i(A)\oplus H_i(B) \longrightarrow H_i(X\times S^1) \xrightarrow{\partial_i} H_{i-1}(A\cap B)$$

$$\xrightarrow{(i^*,j^*)_{i-1}} H_{i-1}(A)\oplus H_{i-1}(B) \longrightarrow H_{i-1}(X\times S^1) \longrightarrow -$$

$$\Longrightarrow \dashrightarrow H_i(X)\oplus H_i(X) \xrightarrow{(i^*,j^*)_i} H_i(X)\oplus H_i(X) \xrightarrow{\phi} H_i(X\times S^1) \xrightarrow{\partial_i} H_{i-1}(X)\oplus H_{i-1}(X$$

$$\xrightarrow{(i^*,j^*)_{i-1}} H_{i-1}(X)\oplus H_{i-1}(X) \longrightarrow -$$

Now, $\operatorname{im}(i^*,j^*)_i = H_i(X)$ for all i, hence:

$$H_i(X) = \operatorname{im}(i^*,j^*)_i = \ker\phi \Rightarrow \operatorname{im}\phi = H_i(X)$$

$$\Rightarrow \underline{\ker\partial_i = H_i(X)}.$$

and $\operatorname{im}(i^*,j^*)_{i-1} = H_{i-1}(X) \Rightarrow \ker(i^*,j^*)_{i-1} = H_{i-1}(X)$

$\Rightarrow \underline{\operatorname{im}\partial_i = H_{i-1}(X)}$

Hence $\underline{H_i(X\times S^1) \cong H_{i-1}(X)\oplus H_i(X)}$

⑦ Find $H^1_c(\mathbb{R})$, the de Rham cohomology w/ compact support, from the definition!

We have the chain complex

$$0 \longrightarrow \Omega^0_c(\mathbb{R}) \xrightarrow{d_0} \Omega^1_c(\mathbb{R}) \xrightarrow{d_1} 0$$

and we know $H^0_c(\mathbb{R}) = \ker d_0$, $H^1_c(\mathbb{R}) = \ker d_1 / \operatorname{im} d_0$.

$\underline{H^0_c(\mathbb{R})}$: Recall that $\Omega^0_c(\mathbb{R}) = C^\infty_c(\mathbb{R}) \subseteq C^\infty(\mathbb{R}) = \Omega^0(\mathbb{R})$, and recall that $\ker(d_0 : \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R})) = \{f \in \Omega^0(\mathbb{R}) : d_0 f = 0\}$

$= \{\text{constant functions}\}$

But the only constant function with compact support is 0, hence $\ker(d_0 : \Omega^0_c(\mathbb{R}) \to \Omega^1_c(\mathbb{R})) \cong \{0\}$, hence $\underline{H^0_c(\mathbb{R}) \cong 0}$.

$\underline{H^1_c(\mathbb{R})}$: For $f \in \Omega^0_c(\mathbb{R})$, i.e. smooth fn. with compact support, we know $\exists M > 0$ s.t. $|x| > M \Rightarrow f(x) = 0$. Therefore,

$$\int_{\mathbb{R}} df = \int_{\mathbb{R}} f' dx = \lim_{x\to\infty} f(x) - \lim_{x\to-\infty} f(x) = 0,$$ hence

$$\operatorname{im} d_0 \subseteq \{\omega = g(x)dx : g \text{ cpt support } \& \int_{\mathbb{R}} g = 0\} \subseteq \Omega^1_c(\mathbb{R}).$$

Choose g s.t. g has cpt supp. & $\int_{\mathbb{R}} g = 0$. Now consider the function $h(x) = \int_{-\infty}^{x} g(t)dt$; clearly $dh = g(x)dx$, hence if h has compact support then $\operatorname{im} d_0 = \{g(x)dx : g \text{ cpt supp}, \int_{\mathbb{R}} g = 0\}$

Since g has compact support, $\exists M$ s.t. $|x| > M \Rightarrow g(x) = 0$.

So then $\int_{-\infty}^{M} g(t)dt = 0$ & $\int_{M}^{\infty} g(t)dt = 0$

$\Rightarrow h(x) = \overset{\to 0}{\int_{-\infty}^{-M} g(t)dt} + \int_{-M}^{x} g(t)dt$; hence h nonzero only in $[-M, M]$, i.e. h has compact support

Now consider the linear map $I : \Omega^1_c(\mathbb{R}) \to \mathbb{R}$

$$\omega \longmapsto \int_{\mathbb{R}} \omega$$

Clearly $\ker I = \{ \omega = g\,dx : g \text{ cpt support } \& \int_{\mathbb{R}} g = 0 \}$

$= \operatorname{im} d_0$

Hence by isomorphism theorem:

$$\Omega^1_c(\mathbb{R}) / \operatorname{im} d_0 \cong \mathbb{R}, \text{ i.e.:}$$

$$H^1_c(\mathbb{R}) = \ker d_1 / \operatorname{im} d_0 = \Omega^1_c(\mathbb{R}) / \operatorname{im} d_0 \cong \mathbb{R}.$$

$$\Rightarrow \underline{\underline{H^1_c(\mathbb{R}) \cong \mathbb{R}}}$$

Incomplete:
3, 5

Geometry/Topology Qualifying Exam

February 2005

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

1. For each $n > 0$ and every $m \in \mathbf{Z}$, show that there exists a smooth map $f : S^n \to S^n$ of degree m.

2. Let $T^2 = \mathbf{R}^2/\mathbf{Z}^2$ be a 2-dimensional torus with standard Euclidean coordinates (x, y) inherited from $\mathbf{R}^2$.
   (a) Prove that for any 2-form ω_2 on T^2 there is a 1-form ω_1 on T^2 and a real number a such that
$$\omega_2 = a dx \wedge dy + d\omega_1.$$
   (b) Prove that for any closed 1-form ω_1 on T^2 there is a smooth function f on T^2 and real numbers a, b so that
$$\omega_1 = a dx + b dy + df.$$

3. Let M be a nonorientable smooth manifold and $i : M \to \mathbf{R}^m$ be an immersion. Define the *normal bundle* $\nu \to M$ to be the set of points (x, v) where $x \in M$ and $v \in \mathbf{R}^m$ is orthogonal to $i_*(T_xM)$ (with respect to the standard Euclidean metric on $\mathbf{R}^m$). Here i_* is the induced map $T_xM \to T_{i(x)}\mathbf{R}^m$ between tangent spaces and we are identifying $T_{i(x)}\mathbf{R}^m$ with $\mathbf{R}^m$.
   (a) Prove that ν can be given the structure of a smooth manifold.
   (b) Is ν an orientable manifold?

4. Let A be a nonsingular symmetric $n \times n$ matrix and c a nonzero real number. (A matrix is *nonsingular* if $\det A \neq 0$ and *symmetric* if $A^T = A$.) Show that
$$\{x \in \mathbf{R}^n \mid \langle x, Ax \rangle = c\}$$
is a submanifold of $\mathbf{R}^n$. Here $\langle , \rangle$ is the standard inner product on $\mathbf{R}^n$. What is the dimension of the submanifold?

5. Compute the second homotopy group $\pi_2(S^2 \vee S^1)$ of the wedge sum of S^2 and S^1.

6. Let Σ be an embedded compact surface without boundary in $\mathbf{R}^3$. Then prove that there is a point $x \in \Sigma$ where the Gaussian curvature $K(x)$ is positive. Here the Gaussian curvature is computed with respect to the metric induced from $\mathbf{R}^3$.

Continued on the next page.

⑦ Let X be the complement of the knot K in the solid torus $S^1 \times D^2$ as in Figure 1. Compute the homology groups $H_i(X; \mathbf{Z})$.

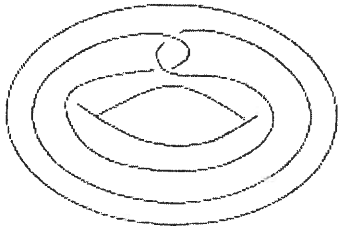


FIGURE 1

1. $S^n \xrightarrow{g} \vee_m S^n \xrightarrow{h} S^n$, g collapses the complement of m B^n's to a point. h maps each S^n to S^n. For all but 1 $y \in S^n$, $f^{-1}(y)$ consists of m points.

2. a) Let $a = \int_{T^2} \omega_2$. Use $I: H^2_{dR}(T^2) \to \mathbb{R}$ isomorphism. ; Both are questions of dimension of $H^0_{dR}(T^2)$

b)

3.

4. Regular Value Theorem

5. If $p: \tilde{X} \to X$ is a covering space, $p^*: \pi_n(\tilde{X}) \to \pi_n(X)$ is an isomorphism for all $n \geq 2$.

6. Use definition of K, enclose Σ is smallest sphere, look at point where Σ touches sphere.

7. Inverse Mayer Vietoris

Geo/Top. Spr. 05:

① For each $n>0$, $m \in \mathbb{Z}$, show $\exists\, f: S^n \to S^n$ of degree m

Consider m disjoint copies of B_ε^n contained in S^n. Call them $B_1, \dots, B_m$.

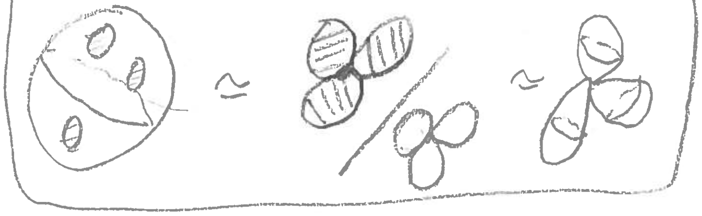
Now see that

$$S^n/(B_1\cup \dots \cup B_m)^c \simeq B_1 \vee \dots \vee B_m / \partial B_1 \vee \dots \vee \partial B_m$$

$$\simeq B_1/\partial B_1 \vee \dots \vee B_m/\partial B_m$$

$$\simeq S^n \vee \dots \vee S^n \quad (m \text{ times})$$

Picture:



So define $g: S^n \to S^n/(B_1 \cup \dots \cup B_m)^c \simeq \vee^m S^n$

as the map collapsing $(B_1 \cup \dots \cup B_m)^c$ to a point

Now define $h: \vee^m S^n \to S^n$ to be the map sending each copy of S^n to S^n via the identity.

Now $h \circ g: S^n \to \vee^m S^n \to S^n$

Now, $h^{-1}(x)$ consists of m points, one in each copy of S^n, and then $g^{-1}(h^{-1}(x))$ is also m distinct points since each point on the distinct S^n's in $\vee^m S^n$ has preimage a point in one of the distinct B_i's. Therefore $\deg(h \circ g) = m$

(for negative m, let h map via antipode)

② $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x,y) inherited

(a) For any $\omega_2 \in \Omega^2(T^2)$, $\exists\, \omega_1 \in \Omega^1(T^2)$ & $a \in \mathbb{R}$ s.t. $\omega_2 = a\,dx \wedge dy + d\omega_1$

Since $\Omega^3(T^2) = 0$, $d\omega_2 = 0$, hence $[\omega_2] \in H^2(T^2)$ well-def.

Since T^2 cpt, orient, the map $I : H^2(T^2) \to \mathbb{R}$, $\omega \mapsto \int_{T^2} \omega$ is iso.

$H^2(T^2)$ is generated by the volume form $dx \wedge dy$,

hence $[\omega] = \left(\int_{T^2} \omega\right)[dx \wedge dy]$; let $\int_{T^2} \omega = a$; then

$[\omega_2] = [a\,dx \wedge dy]$

Now, $[\omega_2 - a\,dx \wedge dy] = 0$, hence exact, hence $\exists$

$\exists\, \omega_1 \in \Omega^1(T^2)$ s.t. $d\omega_1 = \omega_2 - a\,dx \wedge dy \Rightarrow \underline{\omega_2 = d\omega_1 + a\,dx \wedge dy}$

(b) Prove for any closed $\omega_1 \in \Omega^1(T^2)$, $\exists$ smooth $f : T^2 \to \mathbb{R}$ and $a, b \in \mathbb{R}$ s.t. $\omega_1 = a\,dx + b\,dy + df$

ω_1 closed $\Rightarrow [\omega_1] \in H^1(T^2)$ well-def.

Now, the classes of the coordinate covector fields dx, dy form an $\mathbb{R}$-basis for $H^1(T^2)$, hence $\exists\, a, b$ s.t.

$[\omega_1] = [a\,dx + b\,dy]$

$\Rightarrow [\omega_1 - a\,dx - b\,dy] = 0 \Rightarrow \omega_1 - a\,dx - b\,dy$ exact

$\Rightarrow \exists f$ s.t. $df = \omega_1 - a\,dx - b\,dy$

$\Rightarrow \underline{\omega_1 = df + a\,dx + b\,dy}$

④ $A \in M_n(\mathbb{R})$, $\det A \neq 0$, $A^T = A$, $c \in \mathbb{R}$ nonzero

Show that $M = \{x \in \mathbb{R}^n : \langle x, Ax \rangle = c\}$ is a submfld

---

We'll apply the reg. val. thm.

Define $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\quad$ Then $f^{-1}(c) = M \subseteq \mathbb{R}^n$

$x \longmapsto \langle x, Ax \rangle$

Now must only show $T_p f$ surjective $\forall p \in f^{-1}(c)$.

Consider $v \in T_p\mathbb{R}^n$; then we have representative curve $\alpha(t) = tv + p$. Then $T_p f(v) = (f \circ \alpha)'(0)$

So: $(f\circ\alpha)(t) = f(\alpha(t)) = f(tv+p) = \langle tv+p, A(tv+p) \rangle$

$= \langle tv+p, tAv + Ap \rangle = \langle tv, tAv + Ap \rangle + \langle p, tAv + Ap \rangle$

$= \langle tv, tAv \rangle + \langle tv, Ap \rangle + \langle p, tAv \rangle + \langle p, Ap \rangle$

$= t^2 \langle v, Av \rangle + t\langle v, Ap \rangle + t\langle p, Av \rangle + c$

$\Rightarrow (f\circ\alpha)'(t) = 2t\langle v, Av \rangle + \langle v, Ap \rangle + \langle p, Av \rangle$

$\Rightarrow T_p f(v) = (f\circ\alpha)'(0) = \langle v, Ap \rangle + \langle p, Av \rangle$.

Therefore, $T_p f(p) = \langle p, Ap \rangle + \langle p, Ap \rangle = 2c \neq 0$,

hence $T_p f$ is surjective since codom $= \mathbb{R}$ & $\neq 0$.

⑤ Compute $\pi_2(S^2 \vee S^1)$.

Cf. Ex. 4.26 Hatcher

Recall that if $p: \tilde{X} \to X$ covering space, then $p^*: \pi_n(\tilde{X}) \to \pi_n(X)$ iso. for $n \geq 2$.

So consider $X = S^2 \vee S^1$.

S^2 has universal cover itself & S^1 has universal cover $\mathbb{R}$, hence a universal cover of X is:

e.g., a countable wedge of S^2's.

Therefore we have $\pi_2(S^2 \vee S^1) \cong \pi_2(\vee^N S^2)$

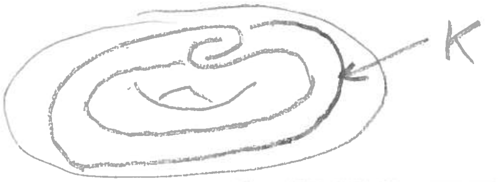
Ⓖ $\Sigma \subseteq \mathbb{R}^3$ embedded cpt surface.

Then prove $\exists x \in \Sigma$ st. $K(x) > 0$

Since Σ is compact, we may cover it with a large enough closed ball B_r^3

Let r be minimal, hence at least one point of Σ will touch ∂B_r^3, hence that point will be in $\partial B_r^3 \simeq S_r^2$; but every point on S^2 has positive curvature, hence this pt in Σ will have positive curvature!

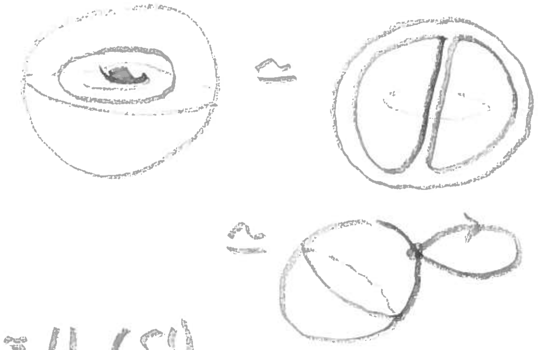
⑦ $X=(S^1\times D^2)\setminus K$ where K pictured:



Find $H_i(X)$

Let $A=\mathbb{R}^3\setminus K$, $B=S^1\times D^2$, $A\cap B=X$, $A\cup B=\mathbb{R}^3$

See that $A=\mathbb{R}^3\setminus K\simeq S^1\vee S^2$:



and B = solid torus $\simeq S^1$!

So we have:

$$H_i(A)\cong H_i(S^1\vee S^2)\cong\begin{cases}\mathbb{Z} & i=0\\ \mathbb{Z} & i=1\\ \mathbb{Z} & i=2\\ 0 & i>2\end{cases}\qquad H_i(B)\cong H_i(S^1)=\begin{cases}\mathbb{Z} & i=0,1\\ 0 & i>1\end{cases}$$

$$H_i(A\cup B)=H_i(pt)\cong\begin{cases}\mathbb{Z} & i=0\\ 0 & i>0\end{cases}$$

Now apply Mayer-Vietoris:

$$0\to H_3(X)\to H_3(A)\oplus H_3(B)\to H_3(\overset{pt}{A\cup B})\to H_2(X)\to H_2(A)\oplus H_2(B)$$
$$\to H_2(\overset{pt}{A\cup B})\to H_1(X)\to H_1(A)\oplus H_1(B)\to H_1(\overset{pt}{A\cup B})\to H_0(X)\to$$
$$H_0(A)\oplus H_0(B)\to H_0(A\cup B)\to 0$$

$$\Rightarrow\ 0\to H_3(X)\to 0\to 0\to H_2(X)\to\mathbb{Z}\to 0\to H_1(X)\to\mathbb{Z}\oplus\mathbb{Z}$$
$$\to 0\to H_0(X)\to\mathbb{Z}\oplus\mathbb{Z}\to\mathbb{Z}\to 0$$

- X path connected, hence $\underline{H_0(X)\cong\mathbb{Z}}$
- $H_3(X)=0$ and $H_2(X)=\mathbb{Z}$ and $H_1(X)\cong\mathbb{Z}\oplus\mathbb{Z}$ by exactness.

$$\Rightarrow\ \boxed{H_i(X)\cong\begin{cases}\mathbb{Z} & i=0\\ \mathbb{Z}\oplus\mathbb{Z} & i=1\\ \mathbb{Z} & i=2\\ 0 & i>2\end{cases}}$$

Geometry/Topology Qualifying Exam

September 2004

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

✓ 1. Prove that a k-form ω on a k-dimensional torus T^k is exact if and only if $\int_{T^k} \omega = 0$.

✓ → 2. Consider the following $(n-1)$-form ω on $\mathbf{R}^n$ with coordinates $(x_1, \ldots, x_n)$:

$$\omega = \frac{\sum_{i=1}^n (-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \cdots \wedge dx_n}{(\sum_{i=1}^n x_i^2)^{n/2}},$$

where $\widehat{dx_i}$ means the dx_i term is omitted.
(a) Show that the form ω is closed on $\mathbf{R}^n - \{0\}$. — $dx \wedge dy = -dy \wedge dx$
(b) Compute $\int_E \omega$, where E is the ellipsoid

$$E = \left\{ \frac{x_1^2}{9} + \sum_{i=2}^n x_i^2 = 2004 \right\}, \quad \simeq S^{n-1}$$

and the orientation of E is the outward orientation (induced from the compact region of $\mathbf{R}^n$ bounded by E). You may leave your answer in terms of the volume $vol(B^n)$ of the n-dimensional unit ball B^n.

✓ → 3. Let X be the topological space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ on the torus. [Here a *Möbius band* is obtained from $[0,1] \times [0,1]$ by identifying $(x, 0) \sim (1-x, 1)$ for all $x \in [0,1]$.]
(a) Compute its fundamental group $\pi_1(X)$.
(b) Compute its homology groups $H_n(X; \mathbf{Z})$ for all $n \geq 0$.

20 ✓ 4. Carefully state the Gauss-Bonnet Theorem and use it compute the total Gaussian curvature $\int_\Sigma \kappa$, where Σ is a compact oriented surface of genus 2004 which is embedded in $\mathbf{R}^3$.

✓ 5. Let X be the topological space obtained from $\mathbf{R}^3$ (with standard coordinates (x, y, z)) by removing two subsets $A_1 = \{x = y = 0\}$ (the z-axis) and $A_2 = \{x^2+y^2 = 1, z = 0\}$ (the boundary of the unit disk in $\mathbf{R}^2 \subset \mathbf{R}^3$). Calculate the fundamental group of X.

✓ 6. Show that there exists no smooth (C^∞-differentiable) surjective map from S^2 to S^3.

Continued on the next page.

7. Let f be a homogeneous polynomial in k (real) variables. Homogeneity means that there is some positive integer m for which

$$f(tx_1, \dots, tx_k) = t^m f(x_1, \dots, x_k),$$

for all $t \in \mathbf{R}$ and $x_1, \dots, x_k \in \mathbf{R}$. Prove that the set of points $x \in \mathbf{R}^k$ for which $f(x) = a$ is a $(k-1)$-dimensional submanifold of $\mathbf{R}^k$, provided $a \neq 0$. [Hint: Use Euler's identity for homogeneous polynomials, which states that $\sum_{i=1}^{k} x_i \frac{\partial f}{\partial x_i} = m \cdot f$.]

1. Stokes, $[\omega] \mapsto \int_M \omega$ is an isomorphism $H^n_{dR}(M^n) \to \mathbb{R}$ if M^n is connected.

2. $\int_E \omega = \int_{S^2} \omega$ because $d\omega = 0$ in between E and S^2.

3. Van Kampen $\langle a, b \mid a b^2 a^{-1} b^{-2} \rangle$

4. $\int_M k \, dA = 2\pi \chi(M)$

5. $X \overset{h.e}{\cong} S^1 \times S^1$

6. $f: S^2 \to S^3$, $\operatorname{im} f$ has no regular values $\Rightarrow$ $\operatorname{im} f$ has measure 0.

7. Suppose $f(x) = a$. Then $T_x f(x) = a \cdot m \neq 0$ $\Rightarrow$ Regular Value Thm.

Geo/Top – Fall 07:

① $\omega \in \Omega^k(T^k)$ exact $\Leftrightarrow \int_{T^k} \omega = 0$

($\Rightarrow$) Suppose ω exact. Then $\exists\, \alpha \in \Omega^{k-1}(T^k)$ s.t. $d\alpha = \omega$.

Now apply Stokes:

$$\int_{T^k} \omega = \int_{\partial(\text{Solid } T^k)} \omega = \int_{\text{Solid } T^k} d\omega = \int_{\text{Solid } T^k} d(d\alpha) = \int_{\text{Solid } T^k} 0 = 0$$

($\Leftarrow$) Suppose $\int_{T^k} \omega = 0$. Since $\dim T^k = k$, we have $\Omega^{k+1}(T^k) = 0$, hence $d\omega = 0$, hence $[\omega] \in H^k(T^k)$ well-defined.

Now, since T^k oper, connected, cmt, consider the top dim'l de Rham iso. $I : H^k(T^k) \to \mathbb{R}$

$$\omega \mapsto \int_{T^k} \omega$$

Since $\int_{T^k} \omega = 0$, $[\omega] = 0 \Rightarrow$ <u>ω exact</u>

② $\omega \in \Omega^{n-1}(\mathbb{R}^n)$ given by $\omega = \dfrac{\sum_{i=1}^n (-1)^{i+1} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{\left(\sum_{i=1}^n x_i^2\right)^{n/2}}$

(a) Show ω closed in $\mathbb{R}^n \setminus \{0\}$

(b) Compute $\int_E \omega$ where $E = \{ \frac{x_1^2}{9} + \sum_{i=2}^n x_i^2 = 2007 \}$

(a) $d\omega = \sum_{i=1}^n (-1)^{i+1}\, d\left(\dfrac{x_i}{\left(\sum_{j=1}^n x_j^2\right)^{n/2}}\right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

and: $d\left(\dfrac{x_i}{(\sum x_j^2)^{n/2}}\right) = \sum_{k=1}^n \dfrac{\partial}{\partial x_k}\left[x_i \left(\sum_{j=1}^n x_j^2\right)^{-n/2}\right] dx_k$

see that: $\dfrac{\partial}{\partial x_k}\left(x_i (\sum x_j^2)^{-n/2}\right) = 2 x_k x_i \dfrac{-n}{2} (\sum x_j^2)^{\frac{-n-2}{2}} = \underline{\dfrac{-x_k x_i n}{(\sum x_j^2)^{\frac{n+2}{2}}}}$ $k \neq i$

$\dfrac{\partial}{\partial x_i}\left(x_i (\sum x_j^2)^{-n/2}\right) = (\sum x_j^2)^{-n/2} + x_i \left(\dfrac{-n}{2}\right) 2 x_i (\sum x_j^2)^{\frac{-n-2}{2}}$

$$= \underline{\underline{\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{\frac{n+2}{2}}}}}$$

So:

$$d\left(\frac{x_i}{(\sum x_j^2)^{n/2}}\right) \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n = \cancelto{0}{\left(\sum_{k \neq i} \frac{-x_k x_i n}{(\sum x_j^2)^{\frac{n+2}{2}}} dx_k\right)} \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$+ \left(\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{\frac{n+2}{2}}}\right) dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

contains $dx_k \wedge dx_k$ $\forall k \neq i$

hence $d\omega = \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{\frac{n+2}{2}}} \right) \underline{dx_i} \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

$= \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{\frac{n+2}{2}}} \right) \boxed{(-1)^{i-1}} dx_1 \wedge \cdots \wedge \underline{dx_i} \wedge \cdots \wedge dx_n$

$= \sum_{i=1}^{n} (-1)^{2i} \left(\frac{1}{(\sum x_j^2)^{n/2}} + \frac{-x_i^2 n}{(\sum x_j^2)^{\frac{n}{2}+1}} \right) dx_1 \wedge \cdots \wedge dx_n$

$= \sum_{i=1}^{n} \left(\frac{(\sum x_j^2) + -x_i^2 n}{(\sum x_j^2)^{n/2+1}} \right) dx_1 \wedge \cdots \wedge dx_n$

$= \frac{1}{(\sum x_j^2)^{n/2+1}} \cdot \left(\sum_{i=1}^{n} (\sum x_j^2) - \sum_{i=1}^{n} x_i^2 n \right) dx_1 \wedge \cdots \wedge dx_n$

$= \frac{1}{(\sum x_j^2)^{\frac{n+2}{2}}} \cdot \left(\cancelto{0}{n \sum_{j=1}^{n} x_j^2 - n \sum_{i=1}^{n} x_i^2} \right) dx_1 \wedge \cdots \wedge dx_n = 0$

(b) $\underline{E = \{ \frac{x_1^2}{9} + \sum_{i=2}^{n} x_i^2 = 2024 \} \text{; find } \int_E \omega}$.

ω is closed on all of $\mathbb{R}^n \setminus \{0\}$ and both $E, S^{n-1} \subseteq \mathbb{R}^n \setminus \{0\}$. Hence ω is closed on both, hence $[\omega]$ well-defined in $H^{n-1}(E) \overset{?}{=} H^{n-1}(S^{n-1})$; but $E \simeq S^{n-1}$ (homotopy equiv), hence $H^{n-1}(E) \cong H^{n-1}(S^{n-1})$, hence $\int_E \omega = \int_{S^{n-1}} \omega$ by hop-inv. de Rham iso.

So $\int_E \omega = \int_{S^{n-1}} \omega = \int_{S^{n-1}} \frac{\sum_{i=1}^{n} (-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{(\sum_{i=1}^{n} x_i^2)^{n/2}}$

$= \int_{S^{n-1}} \sum_{i=1}^{n} (-1)^{i+1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ → $\underline{\text{apply Stokes}}$

$= \int_{B^n} \sum_{i=1}^{n} (-1)^{i+1} dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

$= \int_{B^n} \sum_{i=1}^{n} (-1)^{i+1} (-1)^{i-1} dx_1 \wedge \cdots \wedge dx_n = \int_{B^n} \sum_{i=1}^{n} dx_1 \wedge \cdots \wedge dx_n$

$= n \int_{B^n} dx_1 \wedge \cdots \wedge dx_n = \underline{n \operatorname{vol}(B^n)}$

③ $X = S^1 \times S^1 \amalg M \big/ \partial M \sim (S^1 \times \{x_0\})$

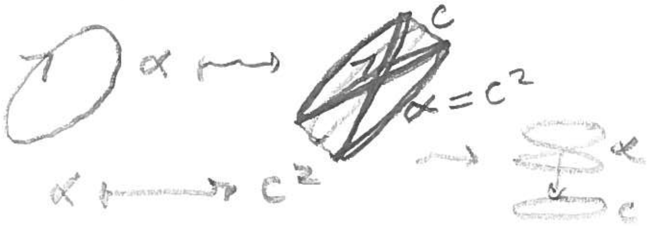
(a) $\pi_1(X)$

(b) $H_i(X)$

(a) Let $A = M$, $B = T^2$, $A \cap B = \partial M \cong S^1$, $A \cup B = X$

and consider the inclusions $i_1 : A\cap B \hookrightarrow A$, $i_2 : A \cap B \hookrightarrow B$

$$\begin{tikzcd} i_1^* : \pi_1(A\cap B) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \pi_1(A) \arrow[d, phantom, sloped, "\cong"] \\ \mathbb{Z}\langle \alpha \rangle & \mathbb{Z}\langle c \rangle \end{tikzcd} \qquad \begin{tikzcd} i_2 : \pi_1(A\cap B) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \pi_1(B) \arrow[d, phantom, sloped, "\cong"] \\ \mathbb{Z}\langle \alpha \rangle & \mathbb{Z}\langle a \rangle \oplus \mathbb{Z}\langle b \rangle \end{tikzcd}$$



$\alpha \longmapsto c^2$

$\alpha \longmapsto a$

→ clearly the boundary circle $A\cap B \sim \partial M$ wraps around the generating circle c twice

→ the circle $A \cap B \sim S^1 \times \{x_0\}$ clearly corresponds to the generating circle a.

Now apply Seifert-van Kampen:

$$\pi_1(X) = \pi_1(A) * \pi_1(B) \big/ \langle i_1^*(\alpha)\, i_2^*(\alpha)^{-1} \rangle$$

$$= (\mathbb{Z}\langle c \rangle) * (\mathbb{Z}\langle a \rangle \oplus \mathbb{Z}\langle b \rangle) \big/ \langle c^2 a^{-1} \rangle$$

$$= \langle a, b, c : aba^{-1}b^{-1}, c^2a^{-1} \rangle \quad \rightsquigarrow a = c^2$$

$$= \langle b, c : c^2bc^{-2}b^{-1} \rangle$$

(b) $H_i(X)$:

Let $A = M$, $B = T^2$, $A \cap B = S^1$, $A \cup B = X$ as before

Now we have :

$$H_i(A) \cong H_i(M) = \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & i>1 \end{cases} \qquad H_i(A \cap B) \cong H_i(S^1) = \begin{cases} \mathbb{Z} & i=0,1 \\ 0 & i>1 \end{cases}$$

$$H_i(B) \cong H_i(T^2) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}$$

Now apply Mayer-Vietoris :

$$0 \to H_2(A)\oplus H_2(B) \to H_2(X) \to H_1(A\cap B) \to H_1(A)\oplus H_1(B) \to H_1(X)$$
$$\to H_0(A\cap B) \to H_0(A)\oplus H_0(B) \to H_0(X) \to 0$$

$$\Rightarrow \quad 0 \to \mathbb{Z} \to H_2(X) \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \to H_1(X) \longrightarrow \mathbb{Z} \longrightarrow$$
$$\mathbb{Z}\oplus\mathbb{Z} \longrightarrow H_0(X) \longrightarrow 0 .$$

Now, X still path-connected, hence $\underline{H_0(X) \cong \mathbb{Z}}$

and $H_1(X) = \pi_1(X) / \pi_1(X)' = \langle \gamma, \beta : \gamma^2\beta\gamma^{-2}\beta^{-1} \rangle / \langle \gamma\beta\gamma^{-1}\beta^{-1} \rangle$

$$= \langle \gamma, \beta : \gamma\beta\gamma^{-1}\beta^{-1} \rangle = \underline{\underline{\mathbb{Z}\oplus\mathbb{Z} = H_1(X)}}$$

So we have the sequence:

$$0 \to \mathbb{Z} \to H_2(X) \xrightarrow{\varphi} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\delta} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}\oplus\mathbb{Z}$$
$$\xrightarrow{\alpha} \mathbb{Z} \to 0 .$$

α surjective $\Rightarrow \ker\alpha = \mathbb{Z} \Rightarrow \operatorname{im}\beta = \mathbb{Z} \Rightarrow \ker\beta = 0 \Rightarrow \operatorname{im}\gamma = 0$

$\Rightarrow \ker\gamma = \mathbb{Z}\oplus\mathbb{Z} \Rightarrow \operatorname{im}\delta = \mathbb{Z}\oplus\mathbb{Z} \Rightarrow \ker\delta = \mathbb{Z} \Rightarrow \operatorname{im}\varepsilon = \mathbb{Z} \Rightarrow \ker\varepsilon = 0$

$\Rightarrow \operatorname{im}\varphi = 0 \Rightarrow \ker\varphi = H_2(X)$.

But $0 \to \mathbb{Z} \to H_2(X)$ implies injectivity by exactness, hence

$\ker\varphi = \mathbb{Z}$, hence $\underline{\underline{H_2(X) \cong \mathbb{Z}}}$

④ Find $\int_\Sigma K\,dA$ where Σ cpt, ornt, genus 2004, $\subseteq \mathbb{R}^3$

Since Σ cpt in $\mathbb{R}^3$, it is closed & bdd, hence no boundary, and we may apply Gauss-Bonnet:

$$\int_M K\,dA = 2\pi\chi(M)$$

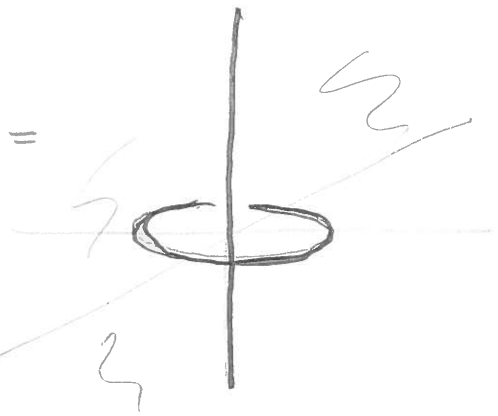
So then:

$$\begin{aligned}\int_\Sigma K\,dA = 2\pi\chi(\Sigma) &= 2\pi(2-2g)\\ &= 2\pi(2-2(2004))\\ &= 2\pi(2-4008)\\ &= -2\pi(4006)\\ &= \boxed{-8012\pi}\end{aligned}$$

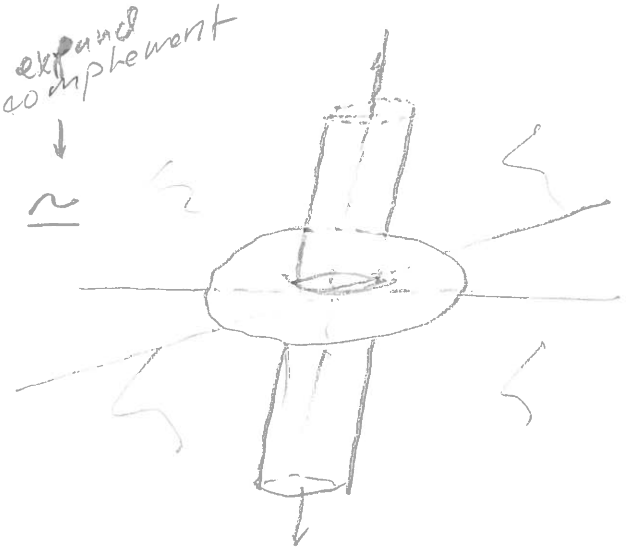
⑤ $X = \mathbb{R}^3 \setminus (A_1 \cup A_2)$ where $A_1 = \{x=y=0\} = z$ axis
$A_2 = \{x^2+y^2=1, z=0\} \cong S^1$

$\pi_1(X)$?

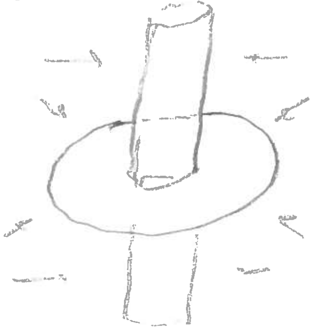
See that $X =$



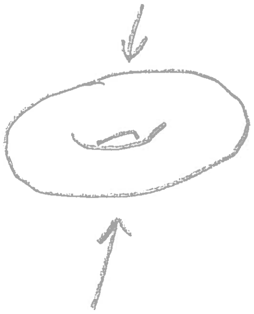
expand complement
↓
$\simeq$



contract space to surface
↓
$\simeq$



$\simeq$
↑
contract cylinder to circle



$= T^2$

Hence $\pi_1(X) \cong \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$.

⑥ $\nexists$ smooth $f: S^2 \to S^3$ surjective

Suppose such a map exists.
then consider $T_pf: T_pS^2 \to T_pS^3 \cong T_pf: \mathbb{R}^2 \to \mathbb{R}^3$
which is never surjective since linear and
$\dim(\operatorname{dom} f) < \dim(\operatorname{codom} f)$, hence f has no
regular pts, hence no reg. values, hence
by Sard, $\lambda(\operatorname{im} f) = 0$

Thus $\operatorname{im} f \neq S^3$ since $\lambda(S^3) > 0$.

Hence f not surjective.

⑦ f homogeneous polynomial in k real variables,

i.e. $f(tx_1, \dots, tx_k) = t^m f(x_1, \dots, x_k)$. for all $t \in \mathbb{R}$.

Prove that $M = \{f(X) = a\}$ is a $(k-1)$-dim'l submfld of $\mathbb{R}^k$ provided $a \neq 0$.

Hint: Euler identity: $\sum_{i=1}^{k} x_i \frac{\partial f}{\partial x_i} = m \cdot f$

Clearly $M = f^{-1}(a)$, hence apply regular value thm.

$p \in f^{-1}(a)$, $T_p f : T_p\mathbb{R}^k \longrightarrow T_p\mathbb{R}$

$$T_p f = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}\right]\Big|_p$$

So then, letting $p = (\phi_1, \dots, \phi_k)$ (coord. rep'n):

$$T_p f(p) = \sum \phi_i \frac{\partial f}{\partial x_i}(p) = m \cdot f(p) = ma \neq 0$$

since $a \neq 0$,

hence $T_p f \neq 0$, hence surjective since linear & codomain $= \mathbb{R}$.

Hence $f^{-1}(a) = M$ a submanifold of dim $\dim \mathbb{R}^k - \dim \mathbb{R} = \underline{k-1}$.

Geometry/Topology Qualifying Exam

February 2004

Partial credit will be given to partial solutions.

1. Let M be a compact orientable manifold of dimension n (without boundary). Let $\omega \in \Omega^n(M)$ be an n-form on M and X a vector field on M. Prove that $\mathcal{L}_X\omega = 0$ at some point $p \in M$. (Here $\mathcal{L}_X\omega$ is the Lie derivative of ω in the direction X.)

2. Let
$$\omega = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$
be a 2-form defined on $\mathbf{R}^3 - \{0\}$. If $i : S^2 = \{x^2 + y^2 + z^2 = 1\} \to \mathbf{R}^3$ is the inclusion, then compute $\int_{S^2} i^*\omega$. Also compute $\int_{S^2} j^*\omega$, where $j : S^2 \to \mathbf{R}^3$ maps $(x, y, z) \to (3x, 2y, 8z)$.

3. Consider the set $X \subset \mathbf{R}^4$ defined by the simultaneous equations $x^2 + y^2 - z^2 - w^2 = 1$ and $xz + yw = 1$. Is X a smooth submanifold of $\mathbf{R}^4$?

4. Show that any smooth function $g : \mathbf{RP}^{2n} \to \mathbf{RP}^{2n}$ has a fixed point. Here $\mathbf{RP}^k$ is the *real projective space*, defined as the quotient of the k-dimensional sphere $S^k = \{|x| = 1\} \subset \mathbf{R}^{k+1}$ by the equivalence relation $x \sim -x$.

5. Let $S^1 = \{x^2 + y^2 = 1, z = 0\}$ denote the boundary of the unit disk in $\mathbf{R}^2 \subset \mathbf{R}^3$ (where $\mathbf{R}^3$ has standard coordinates (x, y, z)). Calculate the fundamental group of $\mathbf{R}^3 - S^1$.

6. Let X be a connected covering space of the 2-dimensional torus $T^2 = S^1 \times S^1$. List all the possible homeomorphism types of X.

7. For a topological space X, its *suspension* ΣX is the quotient $(X \times [0, 1])/\sim$ of $X \times [0, 1]$ obtained by collapsing $X \times \{0\}$ to one point and $X \times \{1\}$ to another point. (More precisely, the equivalence relation $\sim$ is given by:
$$\forall x, x' \in X \ (x, 0) \sim (x', 0) \text{ and } (x, 1) \sim (x', 1).)$$
For any $p \geq 2$, prove that $H_p(\Sigma X, \mathbf{Z})$ is isomorphic to $H_{p-1}(X, \mathbf{Z})$, where $\mathbf{Z}$ is the set of integers. What happens when $p = 0, 1$?

1. Stokes, $\mathcal{L}_X \omega = d \circ i_X \omega + i_X \circ d\omega = d \circ i_X \omega$

2. Consider the Ellipse $9x^2 + 4y^2 + 64z^2 \leq 1$ minus S^2.

3. Regular Value Theorem

4. If $f: S^k \to S^k$ fixes no point, $\deg f = (-1)^{k+1}$. If $f(x) \neq -x \ \forall x \in S^k$, then $f \simeq id \Rightarrow \deg f = 1$. Lifting Criterion.

5. $\mathbb{R}^3 - S^1 \overset{h.e.}{\simeq} S^1 \vee S^2$

6. Cylinder, Torus, $\mathbb{R}^2$

7. Mayer Vietoris

(1.) M cpt, ornt, dim n, $\partial M = \emptyset$, $\omega \in \Omega^n(M)$, $X \in \mathfrak{X}(M)$.

Prove that $\mathcal{L}_X \omega = 0$ at some point $p \in M$.

Since $\dim M = n$, $\Omega^{n+1}(M) = 0$, hence $d\omega = 0$.

Now recall that

$$\mathcal{L}_X \omega = d \circ i_X(\omega) + \overset{=0}{\cancel{i_X \circ d(\omega)}} = d(i_X(\omega))$$

We know $i_X(\omega) \in \Omega^{n-1}(M) \Rightarrow d(i_X(\omega)) \in \Omega^n(M)$,

So :

$$\int_M \mathcal{L}_X \omega = \int_M d(i_X(\omega)) \overset{\text{Stokes}}{=} \int_{\partial M} i_X(\omega) = \int_{\emptyset} i_X(\omega) = 0$$

Hence $\int_M \mathcal{L}_X \omega = 0$, hence $(\mathcal{L}_X \omega)_p = 0$ for some $p \in M$ by Intermediate Value Theorem.

② $$\omega = \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2(\mathbb{R}^3\setminus\{0\})$$

$$i: S^2 \hookrightarrow \mathbb{R}^3, \quad j: S^2 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longmapsto (3x, 2y, 8z)$$

Compute $\int_{S^2} i^*\omega$ & $\int_{S^2} j^*\omega$:

- $$\int_{S^2} i^*\omega = \int_{S^2} \frac{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}{(x^2+y^2+z^2)^{3/2}}$$

$$= \int_{S^2} x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$$

defined at zero, so may apply Stokes.

$$= \int_{B^3} d(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$= \int_{B^3} dx\wedge dy\wedge dz + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy$$

$$= \int_{B^3} 3\,dx\wedge dy\wedge dz = \underline{3\,\mathrm{vol}(B^3)}.$$

- $$\int j^*w = \frac{(x\circ j)(d(y\circ j)\wedge d(z\circ j)) + (y\circ j)\,d(z\circ j)\wedge d(x\circ j) + (z\circ j)\,d(x\circ j)\wedge d(y\circ j)}{((x\circ j)^2+(y\circ j)^2+(z\circ j)^2)^{3/2}}$$

$$= \frac{(3x)\,d(2y)\wedge d(8z) + (2y)\,d(8z)\wedge d(3x) + (8z)\,d(3x)\wedge d(2y)}{[(3x)^2+(2y)^2+(8z)^2]^{3/2}}$$

$$= \frac{48\,(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)}{[9x^2+4y^2+64z^2]^{3/2}}$$

Now, $j^*\omega \in \Omega^2(S^2)$ and closed since $\Omega^3(S^2)=0$, hence $[j^*\omega] \in H^2(S^2)$ well-defined. Now consider the ellipsoid $E = \{9x^2+4y^2+64z^2 \leq 1\}$; clearly $\partial E \simeq S^2$ (homotopy equiv.), hence $H^2(\partial E) \cong H^2(S^2)$, but recall top dim'l de Rham iso gives iso. $H^2(S^2) \cong \mathbb{R}$ by $\omega \mapsto \int_{S^2}\omega$, hence $\underline{\int_{S^2} j^*\omega = \int_{\partial E} j^*\omega}$

2 cont'd : So:

$$\int_{S^2} j^*\omega = \int_{\partial E} j^*\omega = \int_{\partial E} \frac{48\,(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)}{(9x^2+4y^2+64z^2)^{3/2}}$$

$$= \int_{\partial E} 48(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy) \rightarrow$$ defined at 0, so apply Stokes

$$= 48 \int_E d(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)$$

$$= 48 \int_E dx\wedge dy\wedge dz + dy\wedge dz\wedge dx + dz\wedge dx\wedge dy$$

$$= 48 \int_E 3\,dx\wedge dy\wedge dz = \underline{144\,vol(E)}$$

③ $X = \{ x^2+y^2-z^2-w^2=1 , xz+yw=1 \} \subset \mathbb{R}^4$

Is X smooth mfld?

We'll apply regular value theorem twice.

Let $X_1 = \{x^2+y^2-z^2-w^2=1\}$, hence for $f_1: \mathbb{R}^4 \longrightarrow \mathbb{R}$, $(x,y,z,w) \longmapsto x^2+y^2-z^2-w^2-1$,

$X_1 = f_1^{-1}(0)$; So choose $p \in f_1^{-1}(0)$ and consider:

$$T_pf_1 : T_p\mathbb{R}^4 \longrightarrow T_0\mathbb{R}$$

$$T_pf_1 = [2x \;\; 2y \; -2z \; -2w]\Big|_p$$

But $p=(x,y,z,w)$ satisfies $x^2+y^2-z^2-w^2=1$, hence not all coordinates are zero, hence T_pf_1 must have one nonzero entry, i.e. $T_pf_1 \neq 0$, hence surjective.

So X_1 is a smooth mfld.

Now let $X = \{(x,y,z,w) \in X_1 : xz+yw=1\} \subseteq X_1 \subseteq \mathbb{R}^4$.
and consider the map $f: X_1 \longrightarrow \mathbb{R}$, $(x,y,z,w) \longmapsto xz+yw-1$; again, $f^{-1}(0) = X$.

So choose $q \in f^{-1}(0)$ and consider $T_qf : T_qX_1 \rightarrow T_0\mathbb{R}$

$$T_qf = [z \;\; w \;\; x \;\; y]\Big|_q$$

But recall that $q \in X_1$, hence again $q=(x,y,z,w)$ satisfies $x^2+y^2-z^2-w^2=1$, hence not all 0, hence $T_qf \neq 0$, hence surjective.

So X is a submfld of submfld X_1, hence a submfld of $\mathbb{R}^4$.

④ Show any smooth fn. $g: \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.
$\left(\mathbb{R}P^{2n} = S^{2n}/_{\alpha(x)\sim x}\right)$

3,

Suppose $g: \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has $g(x) \neq x$ for all $x \in \mathbb{R}P^{2n}$.
Now recall the quotient map $\pi: S^{2n} \to \mathbb{R}P^{2n}$
and consider the map $g \circ \pi : S^{2n} \to \mathbb{R}P^{2n}$.

Then $(g\circ\pi)^*(\pi_1(S^{2n})) = (g\circ\pi)^*(1) = 1 \leq \pi^*(\pi_1(S^{2n})) = 1$,
hence lifting criterion is satisfied and we get $\tilde{g}$!

$$\begin{tikzcd} & S^{2n} \arrow[d, "\pi"] \\ g\circ\pi : S^{2n} \arrow[ur, dashed, "\tilde{g}"] \arrow[r] & \mathbb{R}P^{2n} \end{tikzcd}$$

i.e. $\pi \circ \tilde{g} = g \circ \pi$

$\Rightarrow \pi\circ\tilde{g}(p) = g\circ\pi(p)$ $\quad$ g has no fixed pts by hypothesis

$\Rightarrow [\tilde{g}(p)] = g([p]) \neq [p]$

hence $\tilde{g}(p) \neq p$ or $-p$ for all $p \in S^{2n}$,

but then $\tilde{g}(p) \neq p \;\forall_p \Rightarrow \deg\tilde{g} = \deg\alpha = (-1)^{2n+1} = \boxed{-1}$!
$\tilde{g}(p) \neq -p \;\forall_p \Rightarrow \deg\tilde{g} = \deg id = \boxed{1}$,

a contradiction, hence g must have had a fixed point

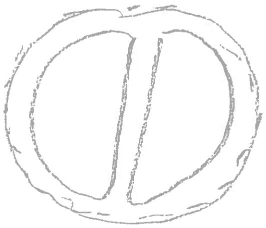
⑤ $\underline{\pi_1(\mathbb{R}^3\setminus S^1)}$:

See the following:

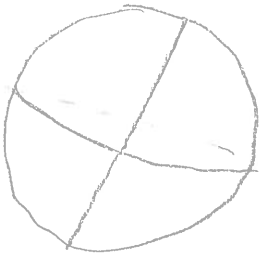
$\mathbb{R}^3\setminus S^1 =$

$\simeq$

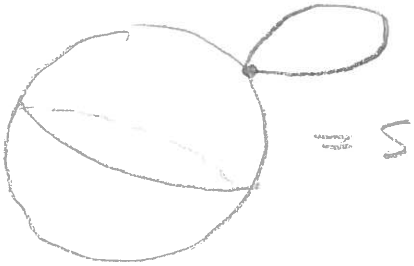
$\simeq$



$\simeq$



$\simeq$



$\simeq S^1 \vee S^2$

Hence $\pi_1(\mathbb{R}^3\setminus S^1) \cong \pi_1(S^1\vee S^2) \cong \pi_1(S^1) * \pi_1(S^2) = \mathbb{Z} * 0$

$= \boxed{\mathbb{Z}}$

Seifert-van Kampen

⑥ List the homeomorphism classes of connected covers of $T^2 = S^1 \times S^1$:

Recall the classification of connected covers gives 1-1 Galois correspondence between subgroups of $\pi_1(T^2)$ and the connected covering spaces.

See that $\pi_1(T^2) \cong \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Now we have:

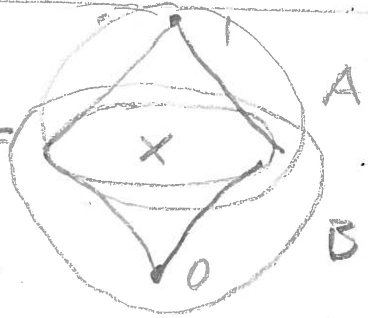
<u>Subgroup</u>	<u>Cover</u>
$0 \oplus 0$	$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ (universal cover)
$0 \oplus \mathbb{Z}$	$\mathbb{R} \times S^1 \simeq$ cylinder
$\mathbb{Z} \oplus 0$	$S^1 \times \mathbb{R} \simeq$ cylinder
$\mathbb{Z} \oplus \mathbb{Z}$	T^2 (trivial cover)

<u>Note</u>: if $\mathbb{Z} = \langle x \rangle$, then subgroup $\langle x^2 \rangle \cong \mathbb{Z}$, hence subgroups like $0 \oplus \langle x^2 \rangle$ still correspond to covering <u>space</u> $\mathbb{R} \times S^1$, but w/ different # of sheets (2 here).

⑦ X top space, $\Sigma X = (X \times [0,1]) \Big/ \begin{smallmatrix} (x,0)\sim(x',0) \\ (x,1)\sim(x',1) \end{smallmatrix} \; \forall x,x'$

For $p \geq 2$, prove $H_p(\Sigma X) \cong H_{p-1}(X)$. What about $p=0,1$?

See that $\Sigma X =$



Let A, B be as pictured; then $A \simeq pt$, $B \simeq pt$, $A \cap B \simeq X$, and $A \cup B = \Sigma X$. Now apply Mayer-Vietoris ($p \geq 2$)

$$\dashrightarrow H_p(A) \oplus H_p(B) \to H_p(\Sigma X) \to H_{p-1}(X) \longrightarrow H_{p-1}(A) \oplus H_{p-1}(B) \to$$

$$\Rightarrow \quad 0 \to H_p(\Sigma X) \to H_{p-1}(X) \to 0 \quad \Rightarrow \quad \underline{H_p(\Sigma X) \cong H_{p-1}(X)}$$

for $p \geq 2$ by exactness

For $p=0$, we have $\underline{H_0(\Sigma X) \cong \mathbb{Z}}$ since ΣX path-connected by construction.

For $p=1$, consider the end of the Mayer-Vietoris sequence:

$$\begin{tikzcd} & & & & \mathbb{Z} \arrow[d, equal] & \\ 0 \arrow[r] & H_1(\Sigma X) \arrow[r, "\gamma"] & H_0(X) \arrow[r, "\beta"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\alpha"] & H_0(\Sigma X) \arrow[r] & 0 \end{tikzcd}$$

By exactness, α is surjective, hence $\ker \alpha = \mathbb{Z} \Rightarrow \operatorname{im} \beta = \mathbb{Z}$

& by exactness, γ is injective; have $H_1(\Sigma X) = \operatorname{im} \gamma = \ker \beta$,

i.e. $H_0(X) \cong H_1(\Sigma X) \oplus \mathbb{Z}$, i.e. $\underline{H_1(\Sigma X) \cong \tilde{H}_0(X)}$