Geometry/Topology Qualifying Exam
Fall 2003

1. Let T^n be the n-dimensional torus $S^1 \times S^1 \times \cdots \times S^1$. Construct a differentiable embedding of T^n in $\mathbf{R}^{n+1}$.

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2. Let S^n denote the n-dimensional sphere, and consider a differentiable map $f : S^n \to \mathbf{R}^n$ such that $f(S^n)$ has non-empty interior in $\mathbf{R}^n$.
   a) Warm-up: Show there there is at least one point $x \in S^n$ where f is a local diffeomorphism, namely such that there exists an open neighborhood $U \subset M$ of x such that restriction $f_{|U} : U \to f(U)$ is a diffeomorphism.
   b) Show that there exists at least two points x, $y \in S^n$ such that f is a local diffeomorphism at x and y, f is orientation-preserving at x, and f is orientation-reversing at y.

3. Let M be a manifold with fundamental group isomorphic to $(\mathbb{Z}/2) \times (\mathbb{Z}/3) \times (\mathbb{Z}/5)$. Up to isomorphism, how many 3-fold covers does it have? Recall that a 3-fold cover is a covering map $p : \widetilde{M} \to M$ such that each $p^{-1}(x)$ consists of 3 points, and that two such covers $p : \widetilde{M} \to M$ and $p' : \widetilde{M}' \to M$ are isomorphic if there exists a homeomorphism $\varphi : \widetilde{M} \to \widetilde{M}'$ such that $p' \circ \varphi = p$.

4. Let M be a manifold of dimension n, and let ω be a differential form of degree $n-1$ on M. Suppose that $\int_N \omega = 0$ for every $(n-1)$-dimensional submanifold N of M. Show that $d\omega = 0$. (hint: look at small spheres.)

5. Let S^n denote the n-dimensional sphere and define $X = S^1 \times S^2$. Also, choose a point $p_n \in S^n$, for $n = 1, 2, 3$, and take the quotient Y of the disjoint union of S^1, S^2, S^3 by the equivalence relation identifying p_1, p_2, p_3 to a single point $p \in Y$.
   a) Calculate the homology groups of X and of Y.
   b) Calculate the fundamental groups as well.
   c) Are these spaces homeomorphic?

6. Let $T = S^1 \times S^1$ denote the 2-dimensional torus. Identify the circle S^1 to $\{z \in \mathbb{C}; |z| = 1\}$, and the 2-dimensional disk B^2 to $\{z \in \mathbb{C}; |z| \leq 1\}$ in the complex plane $\mathbb{C}$. Adjoin to T two copies D_1 and D_2 of B^2, where the boundary $\partial D_1 = \partial B^2$ of the disk D_1 is glued to $S^1 \times \{1\}$ by the map $z \mapsto z^3$ and where the boundary ∂D_2 of D_1 is glued to $\{1\} \times S^1$ by the map $z \mapsto z^5$. Calculate the fundamental group of X.

1. Use polar coordinates and induction

2. a) Sard, b) $\deg f = 0$ because f is not surj (S^n compact, $\mathbb{R}^n$ is not), Sard and definition of degree.

3. Classification of n-sheeted covering spaces

4. Stokes, definition of integration using charts.

5. a) For X, Mayer Vietoris, For Y $\tilde{H}_n(\vee_\alpha X_\alpha) \cong \oplus_\alpha \tilde{H}_n(X_\alpha)$ provided (X_α, x_α) are good pairs.
   b) $\pi_1(A \times B) = \pi_1(A) \times \pi_1(B)$ if A and B are path connected.
   c) Higher homotopy groups.

6. Van Kampen

(2) $f: S^n \to \mathbb{R}^n$ s.t. $f(S^n)^\circ \neq \emptyset$.

(a) Show $\exists\, p \in S^n$ s.t. f is a local diffeomorphism.

S^n cpt, hence $f(S^n)$ cpt $\Rightarrow$ closed and bdd, and $f(S^n)^\circ \neq \emptyset$, hence $\lambda(f(S^n)) > 0$.
Now by Sard's theorem, the regular values will have full measure in $f(S^n)$, hence we may select a regular value $x \in f(S^n)$, i.e. we may select a regular point $p \in f^{-1}(x) \neq \emptyset$.

Since p regular, $T_pf: T_pS^n \to T_p\mathbb{R}^n$ is surjective, i.e. $T_pf: \mathbb{R}^n \to \mathbb{R}^n$ is surjective, hence invertible by equality of dimension + linearity.

By the Inverse Fn. Thm, $\exists$ neighborhood U of p s.t. $f|_U: U \to f(U)$ is a diffeomorphism (i.e. f local diffeo. at p).

(b) Show $\exists\, x, y \in S^n$ s.t. f local diffeo at x & y, and f is orientation preserving @ x & reversing @ y.

Since $f: S^n \to \mathbb{R}^n$ is a map from cpt to non-cpt spaces, $\deg f = 0$ (deg well-def since same dim, both orientable)
Therefore, for any regular value p, the geometric degree is 0.

$$0 = \deg_p f = \sum_{q \in f^{-1}(p)} \deg_q f \quad \text{where} \quad \deg_q f = \begin{cases} +1 & \text{if } f \text{ orient-pres at } q \\ -1 & \text{if } f \text{ orient-rev at } q. \end{cases}$$
$$= \operatorname{sign}(\det(T_q f))$$

Since we've shown that $f^{-1}(p) \neq \emptyset$,
we must have some $x \in f^{-1}(p)$ with $\deg_x f = +1$
and some $y \in f^{-1}(p)$ with $\deg_y f = -1$ to make the
sum $\deg_p f = 0$.

These are both regular points, hence the maps
$T_x f : \mathbb{R}^n \to \mathbb{R}^n$ and $T_y f : \mathbb{R}^n \to \mathbb{R}^n$ are both surjective,
hence invertible, and $\det T_x f > 0$ & $\det T_y f < 0$

So f is a local diffeomorphism at x that is orientation preserving

& f is a local diffeomorphism at y that is orientation-reversing

(3.) M mfld with $\pi_1(M) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$.
Up to iso, how many 3-fold covers of M?

Recall that 3-sheeted covering spaces of M are in correspondence with the homomorphisms $f: \pi_1(M) \to S_3$, so consider these, i.e.

$$f: \mathbb{Z}_2 \oplus \mathbb{Z}/_3 \oplus \mathbb{Z}/_5 \longrightarrow S_3$$

Since the domain is abelian f may only map to commuting permutations; f is determined by its action on the generators, so let $\langle\alpha\rangle = \mathbb{Z}_2$, $\langle\beta\rangle = \mathbb{Z}_3$, $\langle\gamma\rangle = \mathbb{Z}_5$

Then $f(\alpha)$ is order 2
$f(\beta)$ is order 3
$f(\gamma)$ is order 5

Recall $S_3 = \{id, (123), (123)^2 = (132), (12), (23), (13)\}$

- $5 \nmid 3! = 6 = |S_3|$, hence there are no order 5 elements, hence $f(\gamma) = id$ for all hom.'s f.
- $f(\alpha)$ is an order 2 permutation; there are $id, (12), (23), (13)$
- $f(\beta)$ is an order 3 permutation, there are $id, (123), (132)$

Now all of $f(\alpha), f(\beta), f(\gamma)$ must commute, but no nontrivial transposition will commute with (123) or (132), and vice-versa.

So we get

$f(\alpha)$	$f(\beta)$	$f(\gamma)$
id	id	id
(12)	id	id
(23)	id	id
(13)	id	id
id	(123)	id
id	(132)	id

hence there are 6 total.

(4.) M mfld dim n, $\omega \in \Omega^{n-1}(M)$, $\int_N \omega = 0$ for every $(n-1)$-dimensional submfld $N \subseteq M$. 4.

Show $d\omega \equiv 0$:

Consider the point $p \in M$ and coordinate neighborhood (U, φ) of p. Then consider the point $\varphi(p) \in \mathbb{R}^n$ and consider the ε-ball centered there, $B^n_\varepsilon(\varphi(p))$, and then consider the p-nbhd $\varphi^{-1}(B^n_\varepsilon(\varphi(p)))$.

$$\int_{\varphi^{-1}(B^n_\varepsilon(\varphi(p)))} d\omega = \int_{B^n_\varepsilon(\varphi(p))} (\varphi^{-1})^*(d\omega) = \int_{B^n_\varepsilon(\varphi(p))} d(\varphi^{-1})^*(\omega) \underset{\text{Stokes}}{=} \int_{S^{n-1}_\varepsilon(\varphi(p))} (\varphi^{-1})^*(\omega)$$

$$= \int_{\varphi^{-1}(S^{n-1}_\varepsilon(\varphi(p)))} d\omega = 0$$ since φ homeom. onto image, $\varphi^{-1}(S^{n-1}_\varepsilon(\varphi(p)))$ is $(n-1)$-submfld of M.

This is true for any $\varepsilon > 0$, hence $d\omega_p = 0$; but p was arbitrary, so $d\omega \equiv 0$

(5.) $X = S^1 \times S^2$, $Y = S^1 \vee S^2 \vee S^3$ 5.

(a) $H_i(X)$, $H_i(Y)$

(b) $\pi_1(X)$, $\pi_1(Y)$

(c) $X \simeq Y$? :

(a) <u>Homology of X</u>: Let $A = S^2 \times ($, $B = S^2 \times)$

$A \cap B = S^2 \amalg S^2$, $A \cup B = S^2 \times S^1 = X$

X is still path connected, hence $H_0(X) \cong \mathbb{Z}$; now apply Mayer-Vietoris

$$H_3(S^2 \amalg S^2) \to H_3(S^2) \oplus H_3(S^2) \to H_3(X) \to H_2(S^2 \amalg S^2) \to H_2(S^2) \oplus H_2(S^2)$$
$$\to H_2(X) \to H_1(S^2 \amalg S^2) \to H_1(S^2) \oplus H_1(S^2) \to H_1(X) \to H_0(S^2 \amalg S^2)$$
$$\to H_0(S^2) \oplus H_0(S^2) \to H_0(X) \to 0$$

$$\Rightarrow 0 \to \overbrace{H_3(X) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_2(X)}^{(*)} \to 0 \to 0 \to H_1(X)$$
$$\to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to H_0(X) \cong \mathbb{Z} \to 0$$

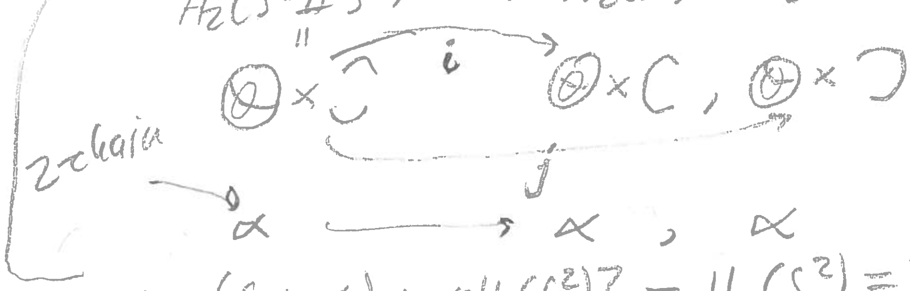
Consider portion $(*)$:

$$0 \to H_3(X) \xrightarrow{\partial_3} H_2(S^2 \amalg S^2) \xrightarrow{(i^*, j^*)} H_2(S^2) \oplus H_2(S^2) \xrightarrow{q} H_2(X) \to 0$$

$H_2(S^2 \amalg S^2) \cong H_2(S^2) \oplus H_2(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$, $\quad H_2(S^2) \oplus H_2(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$

Now consider the map (i^*, j^*): ; it sends $(i^*, j^*)(\alpha) = (\alpha, \alpha)$

$$H_2(S^2 \amalg S^2) \xrightarrow{(i^*, j^*)} H_2(S^2) \oplus H_2(S^2)$$



so $\operatorname{im}(i^*, j^*) = \{(\alpha, \alpha) : \alpha \in H_2(S^2)\} = H_2(S^2) = \mathbb{Z}$,

hence $\ker(i^*, j^*) = \mathbb{Z}$.

Now, by exactness, ∂_3 is injective and q is surjective, so:

$$\underline{H_3(X) = \operatorname{im}\partial_3 = \ker(i^*, j^*) = \mathbb{Z}}$$

and

$$\mathbb{Z} = \operatorname{im}(i^*, j^*) = \ker q \Rightarrow \operatorname{im} q = \mathbb{Z} \quad \text{since domain} = \mathbb{Z}^2$$

but $\underline{H_2(X) = \operatorname{im} q = \mathbb{Z}}$

Finally, for the rest of the sequence, we have:

$$0 \to H_1(X) \xrightarrow{\alpha} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \to 0$$

Then by exactness α is injective and γ surjective.

So $H_1(X) = \operatorname{im}\alpha = \ker\beta$

But $\operatorname{im}\gamma = \mathbb{Z} \Rightarrow \ker\gamma = \mathbb{Z} \Rightarrow \operatorname{im}\beta = \mathbb{Z} \Rightarrow \ker\beta = \mathbb{Z}$,

hence $\underline{H_1(X) \cong \mathbb{Z}}$.

So: $H_i(X) \cong \begin{cases} \mathbb{Z} & i = 0,1,2,3 \\ 0 & i > 3 \end{cases}$

Homology of Y. Each sphere is a good pair with the basepoint of the wedge, hence:

$$\tilde{H}_i(S^1 \vee S^2 \vee S^3) \cong \tilde{H}_i(S^1) \oplus \tilde{H}_i(S^2) \oplus \tilde{H}_i(S^3) \cong \begin{cases} 0 & i = 0 \\ \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 2 \\ \mathbb{Z} & i = 3 \end{cases}$$

Recall $H_i(S^1 \vee S^2 \vee S^3) \cong \tilde{H}_i(S^1 \vee S^2 \vee S^3)$ for $i > 0$

& $H_0(S^1 \vee S^2 \vee S^3) \cong \tilde{H}_0(S^1 \vee S^2 \vee S^3) \oplus \mathbb{Z} \cong \mathbb{Z}$

hence $H_i(S^1 \vee S^2 \vee S^3) \cong \begin{cases} \mathbb{Z} & i = 0,1,2,3 \\ 0 & i > 3 \end{cases}$

(b). $\pi_1(X), \pi_1(Y)$?

$\pi_1(X) = \pi_1(S^1 \times S^2) \cong \pi_1(S^1) \times \pi_1(S^2) = \mathbb{Z} \times 1 = \mathbb{Z}$

$\pi_1(Y) = \pi_1(S^1 \vee S^2 \vee S^3) \cong \pi_1(S^1) * \cancel{\pi_1(S^2)}^{0} * \pi_1(S^3) = \mathbb{Z}$

Seifert van Kampen

(c) Are X & Y homeomorphic ?

#1 No. Consider the higher homotopy groups. First:

$\pi_2(X) = \pi_2(S^1 \times S^2) \cong \pi_2(S^1) \times \pi_2(S^2) = 0 \times \mathbb{Z} = \mathbb{Z}$

Now consider the covering space for $S^1 \vee S^2 \vee S^3$.

It has infinitely many copies of S^2, hence $\pi_2(S^1 \vee S^2 \vee S^3)$ will be infinitely generated,

hence $\pi_2(X) \not\cong \pi_2(Y)$.

#2 No. If you remove a pt. from $S^1 \times S^2$ it is still connected ($S^1 \times S^2$ is manifold), but remove wedge point of $S^1 \vee S^2 \vee S^3$ will yield a disconnected space (i.e. wedge is not mfld), hence not homeomorphic!

⑥ $T^2 = S^1 \times S^1$, $D_1 = D_2 = B^2$

Let $X = T^2 \amalg D_1 \amalg D_2 \Big/ \begin{smallmatrix} \partial D_1 \sim_1 S^1 \times \{1\} \\ \partial D_2 \sim_2 \{1\} \times S^1 \end{smallmatrix}$ where $\begin{smallmatrix} \partial D_1 \longrightarrow S^1 \\ z \longmapsto z^3 \\ \partial D_2 \longrightarrow S^1 \\ z \longmapsto z^5 \end{smallmatrix}$

Calculate $\pi_1(X)$:

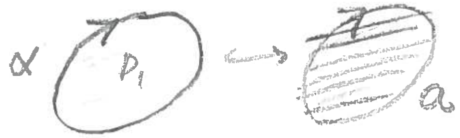
Let $A_1 = D_1$, $B_1 = T^2$

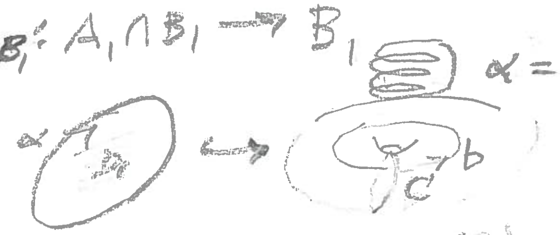
Then $A_1 \cap B_1 = \partial D_1 \cong S^1$

We'll consider $X_1 = T^2 \amalg D_1 /\sim_1$:

Now consider the inclusions :

$i_{A_1}: A_1 \cap B_1 \longrightarrow A_1$, $i_{B_1}: A_1 \cap B_1 \longrightarrow B_1$ $\quad \alpha = b^3$





$i_{A_1}^*: \pi_1(A_1 \cap B_1) \longrightarrow \pi_1(A_1)$; $\pi_1(A_1 \cap B_1) \cong \mathbb{Z}\langle\alpha\rangle$, $\pi_1(A_1) \cong 1$,

$i_{B_1}^*: \pi_1(A_1 \cap B_1) \longrightarrow \pi_1(B_1) \cong \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle$; $\pi_1(A_1 \cap B_1) \cong \mathbb{Z}\langle\alpha\rangle$, $\alpha \longmapsto b^3$

So $i_{A_1}^*(\alpha) = 1$ & $i_{B_1}^*(\alpha) = b^3$

Then $\pi_1(X_1) \cong \pi_1(A) * \pi_1(B_1) \Big/ \langle i_{A_1}^*(\alpha)\, i_{B_1}^*(\alpha)^{-1}\rangle \cong \langle b\rangle \oplus \langle c\rangle \Big/ \langle b^{-3}\rangle \cong \mathbb{Z}_3 \oplus \mathbb{Z}\langle c\rangle$

Now we'll attach D_2 to X_1 , i.e.

Let $A_2 = D_2$ and $B_2 = X_1$; again $A_2 \cap B_2 = \partial D_2 \cong S^1$.

Then consider the inclusions :

$i_{B_2}: A_1 \cap B_1 \hookrightarrow B_1$ $\leadsto$ $i_{B_2}^*: \pi_1(A_1 \cap B_1) \longrightarrow \pi_1(B_1)$; $\pi_1(A_1 \cap B_1) = \mathbb{Z}\langle\alpha\rangle$, $\pi_1(B_1) \cong \mathbb{Z}\langle b\rangle \oplus \mathbb{Z}\langle c\rangle$

$\longrightarrow$ $i_{B_2}^*(\alpha) = c^5$

Now, $A_2 = D_2$ is contractible, so $i_{A_2}^*(\alpha) = 1$,

So we have now:

$$\pi_1(X) \cong \pi_1(X_1) * \pi_1(A_2) / \langle c^5 \rangle$$

$$\cong (\mathbb{Z}_3 \oplus \mathbb{Z}\langle c \rangle * 1) / \langle c^5 \rangle \cong \boxed{\mathbb{Z}_3 \oplus \mathbb{Z}_5}$$

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Geometry/Topology Qualifying Exam

February 2003

Partial credit will be given to partial solutions.

1. Let M be a compact orientable manifold M of dimension $2n$ (without boundary), and let ω be a *symplectic form* on M, namely a closed differential form of degree 2 whose n-th exterior power $\omega \wedge \omega \wedge \cdots \wedge \omega$ does not vanish at any point. Prove that the second de Rham cohomology $H^2_{dR}(M; \mathbf{R}) \neq 0$ by showing that ω is not exact.

2. Show that the set $Sl(n, \mathbf{R})$ of $n \times n$ matrices A with entries in the real numbers and which satisfy $\det(A) = 1$ is a manifold. What is its dimension?

3. On $\mathbf{R}^4$ with coordinates x_1, y_1, x_2, y_2, consider the 2-form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Given a smooth function f on $\mathbf{R}^4$, let X be the vector field
$$X = \frac{\partial f}{\partial y_1}\frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial f}{\partial y_2}\frac{\partial}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial}{\partial y_2}.$$
Then compute $\mathcal{L}_X\omega$, the Lie derivative of ω in the direction X.

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4. Let M be a compact oriented n-dimensional manifold (without boundary), where $n > 1$. Show that there exists a differentiable map $f : M \to S^n$ of degree 1.

5. Recall that two coverings $p : \widetilde{X} \to X$ and $p' : \widetilde{X}' \to X$ are *equivalent* if there exists a homeomorphism $\varphi : \widetilde{X} \to \widetilde{X}'$ such that $p' \circ \varphi = p$. When X is the 2-dimensional torus $S^1 \times S^1$, determine the number of equivalence classes of all coverings $p : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 3 points (for an arbitrary x_0).

6. Compute the homology groups $H_n(X; \mathbf{Z})$ of the complement $X = \mathbf{R}^5 - A$ of a subset $A \subset \mathbf{R}^5$ consisting of 4 points.

7. Let B^n be the closed unit ball in $\mathbf{R}^n$, and let S^{n-1} be its boundary, namely the $(n-1)$-dimensional sphere. If $f : B^n \to \mathbf{R}^n$ is a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$, show that the image $f(B^n)$ contains the ball B^n.

1. Stokes, $\int$ volume form $\neq 0$
2. Regular Value Theorem
3. $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta$, $\mathcal{L}_X = d \circ i_X + i_X \circ d$, $i_X(\omega) = \omega(X)$ for 1-forms
4. Use a cutoff function, a nbhd at $x \in M$, and a buffer nbhd.
5. n sheeted covering spaces $\overset{1}{\longleftrightarrow}$ $\rho : \pi_1(X) \to S_n/\sim$ where $\rho_1 \sim \rho_2$ iff $\rho = h\rho_2 h^{-1}$
6. $\mathbb{R}^5 - A \overset{h.e.}{\simeq} \bigvee_{i=1}^{4} S^4$
7. There is no retract $r : B^n \to S^{n-1}$.

Geo/Top – Sp'03 : 1.

(1.) M cpt, ornt, dim $2n$, $\partial M = \phi$, $\omega \in \Omega^2(M)$ with $d\omega = 0$ and $\omega \wedge \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$ is a volume form.
Prove that $H^2_{dR}(M) \neq 0$ by showing ω not exact.

Suppose ω is exact, i.e. $\exists\, \beta \in \Omega^1(M)$ s.t. $d\beta = \omega$.

Then $\int_M \omega \wedge \cdots \wedge \omega = \int_M d\beta \wedge \cdots \wedge d\beta = \int_M d(\beta \wedge d\beta \wedge \cdots \wedge d\beta)$

$= \int_{\partial M} \beta \wedge d\beta \wedge \cdots \wedge d\beta = \int_{\emptyset} \beta \wedge d\beta \wedge \cdots \wedge d\beta = 0$, a contradiction

since $\omega \wedge \cdots \wedge \omega$ is a volume form

[note that $d(\beta \wedge d\beta \wedge \cdots \wedge d\beta) = d\beta \wedge \cdots \wedge d\beta - \beta \wedge \cancel{d(d\beta \wedge \cdots \wedge d\beta)}^{0}$].

Therefore, since $d\omega = 0$, $[\omega] \in H^2(M)$ well-defined,
but ω not exact, so $[\omega] \neq 0$, hence $H^2(M) \neq 0$.

(2.) Show $SL_n = \{A \in M_n(\mathbb{R}) : \det A = 1\}$ is manifold. Dimension?

Consider the function $f : M_n(\mathbb{R}) \to \mathbb{R}$
$A \mapsto \det A$

Then $f^{-1}(1) = SL_n$, hence must only show 1 is a regular value. Choose $P \in f^{-1}(1)$ and consider

$$T_p f : T_p M_n(\mathbb{R}) \longrightarrow T_1 \mathbb{R}$$

$$T_p f = \left[\frac{\partial f}{\partial x_{11}} \ \frac{\partial f}{\partial x_{12}} \ \cdots \ \frac{\partial f}{\partial x_{nn}} \right] \Big|_p =$$

Now, $f(X) = f(x_{ij}) = \det(x_{ij}) = \sum_{j=1}^{n} (-1)^{j+1} x_{1j} \det(X_{1j})$

Then $\frac{\partial f}{\partial x_{1j}} = (-1)^{j+1} \det(X_{1j})$ so:

$$T_p f = \left[\det(P_{11}) \ -\det(P_{12}) \ \det(P_{13}) \ \cdots \ (-1)^{n+1} \det(X_{1n}) \ \cdots \right]$$

but since $\det(P) = 1 \neq 0$, not all of $\det(P_{1j})$ can be zero, hence $T_p f \neq 0$, hence $T_p f$ surjective and SL_n mfld of dimension $n^2 - 1$.

(3.) $\mathbb{R}^4 = \{(x_1, y_1, x_2, y_2)\}$, $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \in \Omega^2(\mathbb{R}^4)$

Given $f: \mathbb{R}^4 \to \mathbb{R}$, let X be given by

2.

$$X = \frac{\partial f}{\partial y_1}\frac{\partial}{\partial x_1} - \frac{\partial f}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial f}{\partial y_2}\frac{\partial}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial}{\partial y_2}$$

Compute $\mathcal{L}_X \omega$:

$$\left\{\begin{array}{ll} \text{Recall} & \mathcal{L}_X(\alpha\wedge\beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta \\ & i_X(\alpha\wedge\beta) = i_X\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge i_X\beta \\ & i_X(\omega) = \omega(X) \text{ for } \omega \in \Omega^1 \end{array}\right\}$$

Now: $\mathcal{L}_X\omega = \mathcal{L}_X(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$

$$= \mathcal{L}_X(dx_1\wedge dy_1) + \mathcal{L}_X(dx_2 \wedge dy_2)$$

$$= (\mathcal{L}_X dx_1 \wedge dy_1) + (dx_1 \wedge \mathcal{L}_X dy_1) + (\mathcal{L}_X(dx_2)\wedge dy_2) + (dx_2 \wedge \mathcal{L}_X dy_2)$$

And:

$$\mathcal{L}_X dx_1 = d\circ i_X(dx_1) + i_X \circ \overset{0}{\cancel{d(dx_1)}} = d(dx_1(X)) = d\left(\frac{\partial f}{\partial y_1}\right)$$

$$= \frac{\partial^2 f}{\partial y_1 \partial x_1}dx_1 + \frac{\partial^2 f}{\partial y_1^2}dy_1 + \frac{\partial^2 f}{\partial y_1 \partial x_2}dx_2 + \frac{\partial^2 f}{\partial y_1 \partial y_2}dy_2$$

$$\mathcal{L}_X dy_1 = d\circ i_X(dy_1) = d(dy_1(X)) = d\left(-\frac{\partial f}{\partial x_1}\right)$$

$$= -\frac{\partial^2 f}{\partial x_1^2}dx_1 - \frac{\partial^2 f}{\partial x_1 \partial y_1}dy_1 - \frac{\partial^2 f}{\partial x_1 \partial x_2}dx_1 - \frac{\partial^2 f}{\partial x_1 \partial y_2}dy_2$$

$$\mathcal{L}_X dx_2 = d(dx_2(X)) = d\left(\frac{\partial f}{\partial y_2}\right)$$

$$= \frac{\partial^2 f}{\partial y_2 \partial x_1}dx_1 + \frac{\partial^2 f}{\partial y_2 \partial y_1}dy_1 + \frac{\partial^2 f}{\partial y_2 \partial x_2}dx_2 + \frac{\partial^2 f}{\partial y_2 \partial y_2}dy_2$$

$$\mathcal{L}_X dy_2 = d(dy_2(X)) = d\left(-\frac{\partial f}{\partial x_2}\right)$$

$$= -\left(\frac{\partial^2 f}{\partial x_2 \partial x_1}dx_1 + \frac{\partial^2 f}{\partial x_2 \partial y_1}dy_1 + \frac{\partial^2 f}{\partial x_2^2}dx_2 + \frac{\partial f}{\partial x_2 \partial y_2}dy_2\right)$$

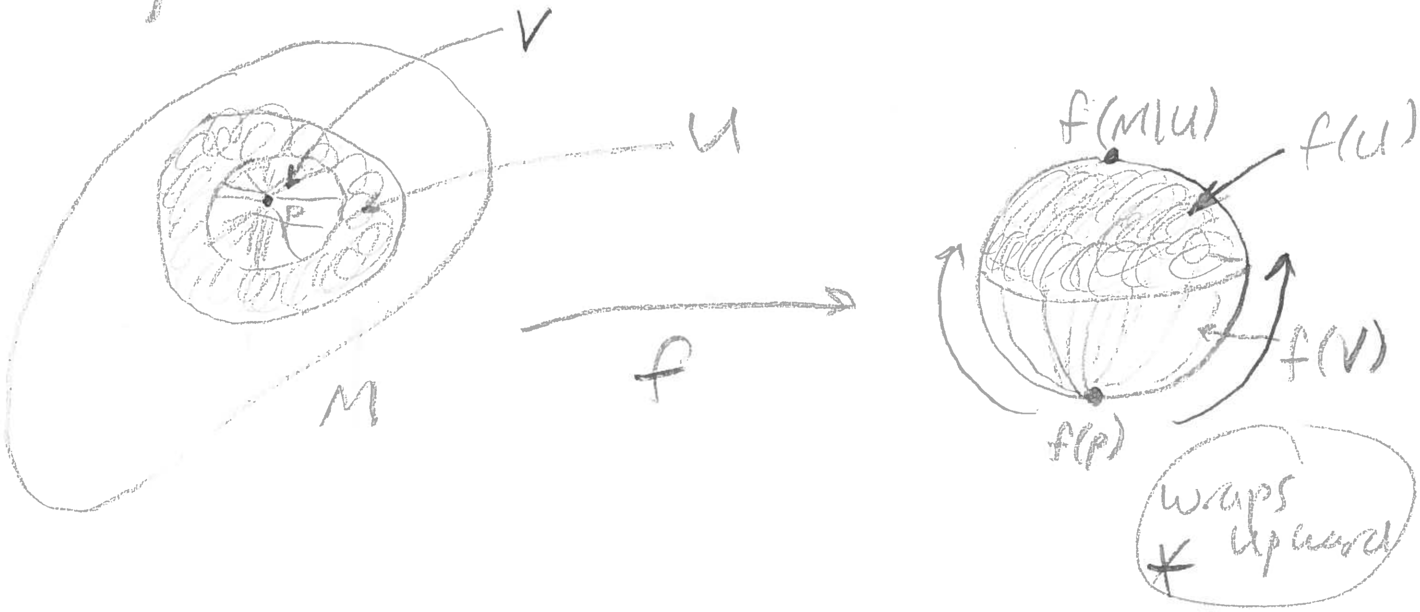
So now:

$$
\begin{aligned}
\mathcal{L}_X \omega &= d\left(\frac{\partial f}{\partial y_1}\right)\wedge dy_1 + dx_1 \wedge d\left(\frac{\partial f}{\partial x_1}\right) + d\left(\frac{\partial f}{\partial y_2}\right)\wedge dy_2 + dx_2 \wedge d\left(\frac{\partial f}{\partial x_2}\right) \\
&= d\left(\frac{\partial f}{\partial y_1}\right)\wedge dy_1 + d\left(\frac{\partial f}{\partial x_1}\right)\wedge dx_1 + d\left(\frac{\partial f}{\partial y_2}\right)\wedge dy_2 + d\left(\frac{\partial f}{\partial x_2}\right)\wedge dx_2 \\
&= \left(\underline{f_{y_1x_1}}dx_1 + \cancel{f_{y_1y_1}dy_1} + \underline{f_{y_1x_2}dx_2} + \underline{f_{y_1y_2}}dy_2\right)\wedge dy_1 \\
&\quad + \left(\cancel{f_{x_1x_1}dx_1} + \underline{f_{x_1y_1}}dy_1 + \underline{f_{x_1x_2}dx_2} + \underline{f_{x_1y_2}}dy_2\right)\wedge dx_1 \\
&\quad + \left(\underline{f_{y_2x_1}}dx_1 + \underline{f_{y_2y_1}}dy_1 + \underline{f_{y_2x_2}dx_2} + \cancel{f_{y_2y_2}dy_2}\right)\wedge dy_2 \\
&\quad + \left(\underline{f_{x_2x_1}}dx_1 + \underline{f_{x_2y_1}dy_1} + \cancel{f_{x_2x_2}dx_2} + \underline{f_{x_2y_2}}dy_2\right)\wedge dx_2 \\
&= (f_{y_1x_1} - f_{x_1y_1})\,dx_1\wedge dy_1 + (f_{y_2x_1} - f_{x_1y_2})\,dx_1\wedge dy_2 \\
&\quad + (f_{x_2x_1} - f_{x_1x_2})\,dx_1\wedge dx_2 + (f_{y_2y_1} - f_{y_1y_2})\,dy_1\wedge dy_2 \\
&\quad + (f_{x_2y_1} - f_{y_1x_2})\,dy_1\wedge dx_2 + (f_{y_2x_2} - f_{x_2y_2})\,dx_2\wedge dy_2
\end{aligned}
$$

(4.) M cpt, ornt, $n$dim mfld, $\partial M = \emptyset$, $n \geq 1$.
Show $\exists$ $f: M \to S^n$ of degree $\underline{1}$

3.

Choose $p \in M$, and cpt neighborhoods $p \in V \subseteq U$.
Then define $f(p) = s \in S^n$, the south pole,
$f(V) = S \subseteq S^n$ the southern hemisphere, $f(U) = N \subseteq S^n$
the northern hemisphere, and $f(M \setminus U) = n \in S^n$, the
north pole:



(5.) $X = T^2 = S^1 \times S^1$; determine the equivalence classes of all 3-sheeted covering spaces.

Recall that 3-sheeted covering spaces are in correspondence with homomorphisms $f: \pi_1(T^2) \to S_3$,

$\pi_1(T^2) = \pi_1(S^1) \oplus \pi_1(S^1) = \mathbb{Z}\langle\alpha\rangle \oplus \mathbb{Z}\langle\beta\rangle$; homomorphisms are determined by their action on generators, so consider $f(\alpha), f(\beta)$.

Since $\pi_1(T^2)$ is abelian, $f(\alpha)f(\beta) = f(\alpha\beta) = f(\beta\alpha) = f(\beta)f(\alpha)$, hence they must map to commuting permutations.

See that $S_3 = \{id, (12), (13), (23), (123), (123)^2 = (132)\}$

No 2-cycle commutes with a 3-cycle, hence we have the following possibilities:

	$f(\alpha)$	$f(\beta)$
7	id	id
	(12)	id
	(13)	id
	(23)	id
	id	(12)
	id	(13)
	id	(23)
4	(123)	id
	$(123)^2$	id
	id	(123)
	id	$(123)^2$

	$f(\alpha)$	$f(\beta)$
7	(12)	(12)
	(23)	(23)
	(13)	(13)
	(123)	(123)
	(132)	(132)
	(123)	(132)
	(132)	(123)

Total = 18

(6.) $H_i(\mathbb{R}^5 \setminus \{p_1,p_2,p_3,p_4\}) = ?$

Recall that $\mathbb{R}^5 \setminus \{p_1,p_2,p_3,p_4\} \simeq \bigvee_{i=1}^{4} S^4$,

and the wedge of the spheres has S^4 forming a good pair with the basepoint for all 4 copies,

hence:

$$\tilde{H}_i\left(\bigvee_{i=1}^{4} S^4\right) = \tilde{H}_i(S^4) \oplus \tilde{H}_i(S^4) \oplus \tilde{H}_i(S^4) \oplus \tilde{H}_i(S^4)$$

$$= \begin{cases} 0 & 0 \le i \le 3,\ i > 4 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & i = 4 \end{cases}$$

Recall $\tilde{H}_i(\bigvee^4 S^4) \cong H_i(\bigvee^4 S^4) \quad \forall i > 0$

and since $\bigvee^4 S^4$ is path-connected, $H_0(\bigvee^4 S^4) \cong \mathbb{Z}$.

So:

$$\boxed{H_i(\bigvee^4 S^4) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & i = 1,2,3,\ i > 4 \\ \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} & i = 4 \end{cases}}$$

(7.) $B^n \subseteq \mathbb{R}^n$ closed unit ball, $\partial B^n = S^{n-1}$

If $f: B^n \to \mathbb{R}^n$ s.t. $f(x) = x$ for all $x \in S^{n-1}$, show that $f(B^n)$ contains B^n.

Suppose $B^n \not\subseteq f(B^n)$, i.e. suppose $p \in B^n$ and $p \notin f(B^n)$, WLOG, let $p = 0$ (can translate for any other point). Then $f(B^n) \subseteq \mathbb{R}^n \setminus \{0\}$. Now consider the following maps:

$$S^{n-1} \xhookrightarrow{i} B^n \xrightarrow{f} \mathbb{R}^n \setminus \{0\} \xrightarrow{r} S^{n-1}$$

$$x \longmapsto \underset{\in S^{n-1}}{x} \longmapsto \underset{\in S^{n-1}}{f(x) = x} \longmapsto \frac{x}{\|x\|} = x$$

$\|x\| = 1$ since $x \in S^{n-1}$

Therefore $r \circ f \circ i \equiv id_{S^{n-1}}$, hence $(r \circ f \circ i)^*: H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ is the identity map.

But note that $i^*: H_{n-1}(S^{n-1}) \to H_{n-1}(B^n)$ is the zero map since a $(n-1)$-cycle in S^{n-1} is clearly an $(n-1)$-boundary in B^n, so $i^* = 0$, which is a contradiction since we have

$$id^* = (r \circ f \circ i)^* = r^* \circ f^* \circ i^* = r^* \circ f^*(0) \equiv 0$$

and $id^* \neq 0$ clearly.

Hence each point of B^n must be in $f(B^n)$, i.e. $B^n \subseteq f(B^n)$.

Graduate Exam in Topology/Geometry

February 2002

1. Let $P^n(\mathbb{R})$ be the projective n-space, namely the quotient space of the sphere S^n by the equivalence relation $\sim$ defined by $x \sim y \Leftrightarrow x = \pm y$.
   (a) Show that $P^n(\mathbb{R})$ is a manifold.
   (b) Show that $P^n(\mathbb{R})$ is orientable if and only if n is odd.

2. In the set $M(n)$ of all $n \times n$ matrices, identified to $\mathbb{R}^{n^2}$, consider the subset $O(n)$ consisting of the orthogonal matrices, namely those matrices A for which AA^t is the identity (where A^t denotes the transpose). Show that $O(n)$ is a submanifold of $M(n) = \mathbb{R}^{n^2}$, and that the tangent space $T_{\mathrm{Id}}O(n)$ at the identity Id is equal to the space of all antisymmetric matrices (namely those matrices for which $A^t = -A$).

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x, y, z) = (\alpha x + \beta y, \gamma x + \delta y, \varepsilon z)$, where α, β, γ, δ, ε are constants with $\alpha\delta - \beta\gamma = 1$. Find the matrix of $f^* : \Lambda^2\mathbb{R}^3 \to \Lambda^2\mathbb{R}^3$ associated to the basis $dy \wedge dz$, $dz \wedge dx$, $dx \wedge dy$.

4. Let $P^2(\mathbb{R})$ be the real projective plane.
   (a) If $x \in P^2(\mathbb{R})$, compute the fundamental group $\pi_1(P^2(\mathbb{R}) - \{x\})$.
   (b) Show that any map $f : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ which is not surjective is homotopic to a constant map. (Hint: use a covering space).

5. Let B^2 be the closed 2-dimensional ball, with boundary the circle S^1. Let $X = S^1 \times B^2$ and let $\partial X = S^1 \times S^1$. Compute the relative homology groups $H_n(X, \partial X)$ with coefficients in Z. (You are allowed to use whatever you may know about the homology of the torus ∂X).

6. Let X be the figure eight  , union of two circles C_1 and C_2 meeting in one point. Let $p : \tilde{X} \to X$ be a covering space such that $\tilde{X}$ is connected and such that the preimage $p^{-1}(x)$ of each $x \in X$ consists of 2 points. Compute the fundamental group of $\tilde{X}$.

7. What are the compact connected surfaces S for which there exists an immersion $S \to S$ which is not a diffeomorphism? (Hint: Euler characteristic).

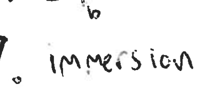
1. a) $\mathbb{Z}_2$ acts on S^n freely, discontinuously. b) HW problem, top dimensional form α can be written as $\alpha = f\omega$ ω volume form.

2. Regular Value Theorem w/ $T_xM = \ker T_xf$.

3. $f^*\omega = \sum_{i_1 < \dots < i_k} g_{i_1 \dots i_k} \circ f \; df_{i_1} \wedge \dots \wedge df_{i_k}$ where $f = (f_1, \dots, f_n)$ $\omega = \sum_{i_1 < \dots < i_k} g_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$

4. a) $P^2(\mathbb{R}) - \{x\}$ def. ret. to S^1. b) f^* is trivial, lifting criterion

5. Long Exact Sequence for Relative Homology

6.  has fund. group $\mathbb{Z} * \mathbb{Z} * \mathbb{Z}$.

7. immersion $\Rightarrow$ covering map (surjective by connectivity) if $p: \tilde{X} \to X$ is n-sheeted, $\chi(\tilde{X}) = n\chi(X)$, so $\chi(S) = 0$ $\Rightarrow$ $S =$ Torus or Klein Bottle
$\chi(M_g) = 2 - 2g$ $\chi(N_g) = 2 - g$.

Geo/Top · Spring 02

(1.) $\mathbb{R}P^n = S^n / x \sim \alpha(x)$

(a) Show $\mathbb{R}P^n$ is smooth mfld

(b) Show $\mathbb{R}P^n$ ornt. $\iff$ n odd

(a) See that since $\langle\alpha\rangle \cong \mathbb{Z}_2$, we have that $\mathbb{R}P^n = S^n/\mathbb{Z}_2$,
ie. we must only show the action $\mathbb{Z}_2 \curvearrowright S^n$ is
free and discontinuous

- Free: α is the only non-identity element in $\mathbb{Z}_2$, and clearly $\alpha(p) = -p \neq p$ for all $p \in S^n$, hence $\mathbb{Z}_2 \curvearrowright S^n$ is free.
- Discontinuous: clearly for any cpt $K \subseteq S^n$ the set $\{\gamma \in \mathbb{Z}_2 : \gamma K \cap K \neq \phi\}$ is finite since $|\mathbb{Z}_2| = 2 < \infty$, so action is trivially discontinuous

$\mathbb{Z}_2 \curvearrowright S^n$ free & discont $\Rightarrow$ $S^n/\mathbb{Z}_2 = \mathbb{R}P^n$ is a manifold

(b) $\mathbb{R}P^n$ ornt $\iff$ n odd.

($\Rightarrow$) $\mathbb{R}P^n$ orientable. This induces an orientation on S^n via quotient $\pi: S^n \to \mathbb{R}P^n$ by letting $(v_1, \dots, v_n) \in T_pS^n$ be positively-ornt. iff $(T_p\pi(v_1), \dots, T_p\pi(v_n))$ positively oriented.
Now recall that $\alpha(v_i) = v_i$ on $\mathbb{R}P^n$ since α identity map on $\mathbb{R}P^n$.
hence $(T_p\pi(v_1), \dots, T_p\pi(v_n)) = (T_p\pi(\alpha(v_1)), \dots, T_p\pi(\alpha(v_n)))$,
ie. $(v_1, \dots, v_n)$ & $(\alpha(v_1), \dots, \alpha(v_n))$ positively oriented on S^n
$\Rightarrow$ α orientation preserving $\Rightarrow$ n odd

($\Leftarrow$) n odd. Then $\alpha: S^n \to S^n$ is orientation preserving.

Now define $(w_1, \dots, w_n) \subseteq T_{\pi(p)} \mathbb{R}P^n$ to be $+$-ornt. if $\exists$ $+$-ornt basis $(v_1, \dots, v_n) \subseteq T_p S^n$ s.t. $(T_p\pi(v_1), \dots, T_p\pi(v_n)) = (w_1, \dots, w_n)$ — Define

Now suppose $(v_1, \dots, v_n)$ & $(v_1', \dots, v_n')$ are ornt. basis for $T_p S^n$ & $T_{-p} S^n = T_{\alpha(p)} S^n$ (resp.) in the same ornt class ($+$).

well-def.

Now since $\pi \circ \alpha = \pi$,

$$T_p \pi(v_i) = T_p(\pi \circ \alpha)(v_i) = T_{\alpha(p)}\pi \circ T_p\alpha(v_i) = T_{-p}\pi\left(T_p\alpha(v_i)\right)$$

But α orientation preserving, hence $T_p\alpha(v_i)$ & v_i' are in the same orientation class in $T_{-p} S^n$, hence $T_p\pi(v_i)$ & $T_{-p}\pi(v_i')$ are in the same orientation class in $T_{\pi(p)}\mathbb{R}P^n$.

Hence only $+$-class ornt bases may be mapped to what we defined as the $+$-class ornt basis for $T_{\pi(p)}\mathbb{R}P^n$, i.e. our ornt. is well-defined

(2.) Show $O_n = \{A \in M_n : AA^T = I\} \subseteq M_n$ submfld and $T_I(O_n) = \text{AntiSym}_n$

Let $f: M_n \to \text{Sym}_n$ where $\text{Sym}_n = \{A^T = A\}$
$X \longmapsto XX^T$

(since $\text{im} f = \{XX^T\}$ and $(XX^T)^T = XX^T$, we have $\text{im} f \subseteq \text{Sym}_n$)

So $f^{-1}(I) = O_n$; now consider $P \in f^{-1}(I)$ and:

$$T_p f : T_p M_n \to T_I \text{Sym}_n$$

For $A \in T_p M_n$, let $\alpha(t) = At + P$ be representative curve; then $T_p f(A) = (f \circ \alpha)'(0)$:

$$(f \circ \alpha)(t) = f(At+P) = (At+P)(At+P)^T = (At+P)(A^T t + P^T)$$

$$= AA^T t^2 + AP^T t + PA^T t + PP^T$$

$$\Rightarrow (f \circ \alpha)'(t) = 2AA^T t + AP^T + PA^T$$

$$\Rightarrow T_p f(A) = (f \circ \alpha)'(0) = AP^T + PA^T.$$

Now, recall that $P \in f^{-1}(I)$, hence $PP^T = I$; now consider any symmetric matrix $C \in T_I \text{Sym}_n \cong \text{Sym}_n$. Then: $\frac{1}{2} CP \in T_p M_n \cong M_n$ and:

$$T_p f(\tfrac{1}{2} CP) = (\tfrac{1}{2} CP) P^T + P(\tfrac{1}{2} CP)^T$$

$$= \tfrac{1}{2} C + \tfrac{1}{2} PP^T C^T = \tfrac{1}{2} C + \tfrac{1}{2} C^T = \tfrac{1}{2} C + \tfrac{1}{2} C = C,$$

hence $T_p f$ is surjective for all $P \in f^{-1}(I)$, so O_n submfld

Now recall that $T_I O_n = \ker T_I f = \{A : T_I f(A) = 0\}$

$= \{A : A + A^T = 0\} = \{A = -A^T\} = \underline{\text{AntiSym}_n}$

(3) $f: \mathbb{R}^3 \to \mathbb{R}^3$ given by $f(x,y,z) = (\alpha x+\beta y, \gamma x+\delta y, \varepsilon z)$ where $\alpha\delta-\beta\gamma = 1$; find the matrix of $f^*: \Lambda^2\mathbb{R}^3 \to \Lambda^2\mathbb{R}^3$ associated to basis $dy\wedge dz$, $dz\wedge dx$, $dx\wedge dy$.

Consider the value of f^* on the basis elements:

$$f^*(dy\wedge dz) = f^*dy \wedge f^*dz = d(y\circ f)\wedge d(z\circ f)$$
$$= d(\gamma x+\delta y)\wedge d(\varepsilon z)$$
$$= (\gamma dx + \delta dy)\wedge \varepsilon dz$$
$$= \gamma\varepsilon dx\wedge dz + \delta\varepsilon dy\wedge dz$$
$$= \underline{\delta\varepsilon dy\wedge dz - \gamma\varepsilon dz\wedge dx}.$$

$$\Rightarrow f^*\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\delta\varepsilon\\-\gamma\varepsilon\\0\end{pmatrix}$$

$$f^*(dz\wedge dx) = f^*dz\wedge f^*dx = d(z\circ f)\wedge d(x\circ f)$$
$$= d(\varepsilon z)\wedge d(\alpha x+\beta y)$$
$$= \varepsilon\alpha dz\wedge dx + \varepsilon\beta dz\wedge dy$$
$$= \varepsilon\alpha dz\wedge dx - \varepsilon\beta dy\wedge dz$$

$$\Rightarrow f^*\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-\varepsilon\beta\\\varepsilon\alpha\\0\end{pmatrix}$$

$$f^*(dx\wedge dy) = f^*dx\wedge f^*dy = d(x\circ f)\wedge d(y\circ f)$$
$$= d(\alpha x+\beta y)\wedge d(\gamma x+\delta y)$$
$$= (\alpha dx+\beta dy)\wedge(\gamma dx+\delta dy)$$
$$= \alpha\gamma \cancel{dx\wedge dx} + \alpha\delta dx\wedge dy + \beta\gamma dy\wedge dx + \beta\delta \cancel{dy\wedge dy}$$
$$= (\alpha\delta-\beta\gamma)dx\wedge dy = dx\wedge dy$$

$$\Rightarrow f^*\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

So $$f^* = \begin{bmatrix}\delta\varepsilon & -\varepsilon\beta & 0\\ -\gamma\varepsilon & \varepsilon\alpha & 0\\ 0 & 0 & 1\end{bmatrix}$$

④ (a) $\pi_1(\mathbb{RP}^2 \setminus \{pt\})$

(b) Show that any map $f: \mathbb{RP}^2 \to \mathbb{RP}^2$ which is not surjective is homotopic to constant map

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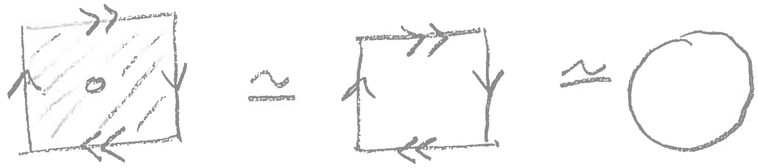
(a)  , i.e. $\mathbb{RP}^2 \setminus \{pt\}$

deformation retracts to circle, hence:

$$\pi_1(\mathbb{RP}^2 \setminus \{pt\}) \cong \pi_1(S^1) \cong \mathbb{Z}$$

(b) Suppose that f is not surjective, i.e. suppose $\operatorname{im} f = \mathbb{RP}^2 \setminus \{pt\} \simeq S^1$; hence we have the induced map $f_* : \pi_1(\mathbb{RP}^2) \to \pi_1(\mathbb{RP}^2 \setminus pt) \cong \pi_1(S^1)$

$$\Rightarrow f_* : \mathbb{Z}_2 \to \mathbb{Z} \Rightarrow f_* \equiv 0 \text{ since } \mathbb{Z}$$

has no finite-order elements

Hence for covering space $p: \mathbb{R} \to S^1$, we have $f_*(\pi_1(\mathbb{RP}^2)) = 1 \leq p_*(\pi_1(\mathbb{R})) = 1$, hence we have a lift:

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ \mathbb{RP}^2 \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & \mathbb{RP}^2 \setminus \{pt\} \end{tikzcd}$$

Now $\mathbb{R}$ is contractible, hence $\exists$ homotopy $h_t : \mathbb{R} \to \mathbb{R}$ s.t. $h_0 = id_{\mathbb{R}}$, $h_1 = \text{const.}$; then see that $h_0 \circ \tilde{f} = \tilde{f}$ and $h_1 \circ \tilde{f} = \text{const.}$, hence $h_t \circ \tilde{f} := g_t$ is homotopy of $\tilde{f}$ to const, hence, since $f = p \circ \tilde{f}$, we have $p \circ g_0 = p \circ \tilde{f} = f$ and $p \circ g_1 = \text{const}$, hence $p \circ g_t$ is homotopy of f to const, i.e. $f \simeq \text{const}$

(5.) B^2, $\partial B^2 = S^1$, $X = S^1 \times B^2$, $\partial X = S^1 \times S^1$

Compute $H_n(X, \partial X)$:

Consider the Long Exact Sequence of Relative Homology:

$$\cdots 0 \to H_3(X,\partial X) \to H_2(\partial X) \to H_2(X) \to H_2(X,\partial X)$$
$$\to H_1(\partial X) \xrightarrow{i^*} H_1(X) \to H_1(X,\partial X) \to H_0(\partial X) \to H_0(X)$$
$$\to H_0(X,\partial X) \to 0$$

Recall that $X = S^1 \times B^2 \simeq$ solid $T^2 \simeq S^1$
$\&$ $\partial X = S^1 \times S^1 \cong T^2$.

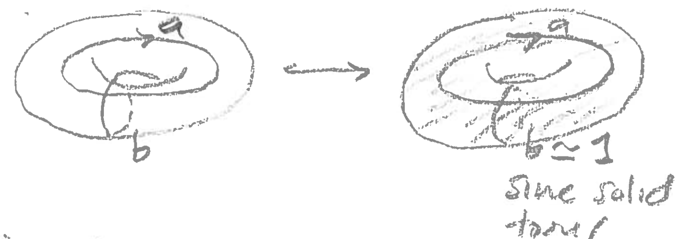
So: $H_i(X) = \begin{cases} \mathbb{Z} & i = 0,1 \\ 0 & i > 1 \end{cases}$ and $H_i(\partial X) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1 \\ \mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$

and then:

$$0 \to H_3(X,\partial X) \to \mathbb{Z} \to 0 \to H_2(X,\partial X) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i^*} \mathbb{Z}$$
$$\to H_1(X,\partial X) \to \mathbb{Z} \xrightarrow{i_0^*} \mathbb{Z} \to H_0(X,\partial X) \to 0$$

Clearly $H_3(X,\partial X) \cong \mathbb{Z}$ by exactness.

Now consider the inclusion map $i : \partial X \hookrightarrow X$ and the induced homomorphism $i^* : H_1(\partial X) \cong \mathbb{Z}\langle a \rangle \oplus \mathbb{Z}\langle b \rangle \to H_1(X) \cong \mathbb{Z}\langle a \rangle$



hence $i^*(a) = a$
$i^*(b) = 0$, hence $\ker i^* = \mathbb{Z}$
and $\underline{\operatorname{im} i^* = \mathbb{Z}}$.

Furthermore, the induced hom
$i^* : H_0(\partial X) \to H_0(X)$ is injective
since i is inclusion $\&$ both path-connected.

$\Rightarrow \ker i^* = 0$ here.

Now consider the subsequence:

$$0 \longrightarrow H_2(X,\partial X) \xrightarrow{\alpha} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{i^*} \mathbb{Z} \xrightarrow{\beta} H_1(X,\partial X) \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{i_0^*} \mathbb{Z} \xrightarrow{\delta} H_0(X,\partial X) \to 0$$

Now, $\ker i^* = \mathbb{Z} \Rightarrow \underline{\mathbb{Z} = \operatorname{im}\alpha \cong H_2(X,\partial X)}$ since α inj. by exactness

We also have $\ker i_0^* = 0 \Rightarrow \operatorname{im} i_0^* = \mathbb{Z} \Rightarrow \ker\delta = \mathbb{Z}$
$\Rightarrow \operatorname{im}\delta = 0$, but δ surjective by exactness, hence $\underline{H_0(X,\partial X)=0}$

On the other hand, $\ker i_0^* = 0 \Rightarrow \operatorname{im}\gamma = 0 \Rightarrow$
$\ker\gamma = H_1(X,\partial X)$, but $\operatorname{im} i^* = \mathbb{Z} \Rightarrow \ker\beta \doteq \mathbb{Z}$
$\Rightarrow \operatorname{im}\beta = 0 \Rightarrow \ker\gamma = 0 \Rightarrow \underline{H_1(X,\partial X) = 0}$

So we have $\boxed{H_i(X,\partial X) \cong \begin{cases} 0 & i=0,1 \\ \mathbb{Z} & i=2,3 \\ 0 & i>3. \end{cases}}$

(6.) $X = S^1 \vee S^1$, $p : \tilde{X} \to X$ covering space s.t. $|p^{-1}(x)| = 2$

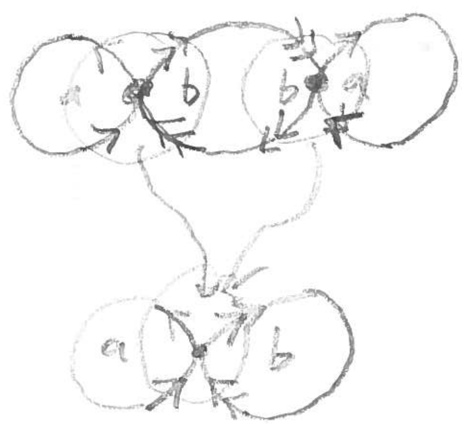
Compute $\pi_1(\tilde{X})$.

#1: By Seifert-van Kampen, $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} = F_2$, the free group on 2 generators.
Then a 2-sheeted connected cover corresponds to an index 2 sbg. $H \le F_2$, i.e. $H = F_3$ ($H \le F_{n+1}$, $[F_{n+1} : H] = k \Rightarrow H = F_{kn+1}$)

So $\pi_1(\tilde{X}) = F_3$

#2: 

$= S^1 \vee S^1$; a 2-sheeted cover will be compact (X cpt, $\tilde{X}$ finite sheeted $\Rightarrow$ $\tilde{X}$ cpt) and have 2 wedge points:



So $\tilde{X} =$ 

& $\pi_1(\tilde{X}) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = F_3$

⑦ What are the cpt, connected surfaces S for which $\exists$ $f: S \to S$ immersion that is not a diffeomorphism?

Since $f: S \to S$ is immersion, $T_pf: T_pS \cong \mathbb{R}^2 \to T_{f(p)}S \cong \mathbb{R}^2$ is injective for all p, hence invertible by dimension, hence local diffeo. at p by Inverse Fn. Thm, hence local diffeo at all $p \in M$, hence f a local diffeo.

- A local diffeo. is an open/closed map, hence since S is open/closed, $f(S)$ is open/closed, hence $f(S) = S$ by connectivity of S
- For $p \in S$, $f^{-1}(p)$ is closed subset of cpt S, hence cpt, hence finite # of points, i.e. $f^{-1}(p) = \{q_1, \dots, q_k\}$.
- Now, choose neighborhoods $U_{q_1}, \dots, U_{q_k}$ s.t. f is diffeo at $q_1, \dots, q_k$, hence $p \in f(U_{q_i}) \approx U_{q_i}$

Now consider the nbhd:

$$p \in V = f(U_{q_1}) \cap \dots \cap f(U_{q_k})$$

This is clearly evenly-covered by disjoint nbhds $V_{q_i} \subseteq U_{q_i}$ (since restriction of homeomorphism is homeo. onto image)

$\Rightarrow$ Therefore, f is a covering map, hence for f not diffeomorphism, f will be n-sheeted cover for $n > 1$, and then; since $\chi(\tilde{X}) = n\chi(X)$, we have:

$\chi(S) = n\chi(S) \Rightarrow \chi(S) = 0$ since $n > 1$

Then $0 = \chi(M_g) = 2 - 2g \Rightarrow g = 1 \Rightarrow S = \text{Torus}$
or $0 = \chi(N_g) = 2 - g \Rightarrow g = 2 \Rightarrow S = \text{Klein bottle}$

Qualifying Exam in Geometry/Topology Fall 2000

1. Let ω be a 1-form defined on the sphere $S^2 = \{x \in R^3 \| x| = 1\}$. Assume ω is invariant under rotations, i.e. $\phi^*\omega = \omega$ for any $\phi \in SO(3)$, show $\omega = 0$.

2. Show the set $M = \{x \in R^4 | x_1 x_2 = x_3 x_4, |x| = 1\}$ is a smooth orientable surface.

3. Let M, N be smooth manifolds of dimension n, and $\pi : M \to N$ be a smooth map which is onto and has rank n at each point. Prove or disprove the statements:

(a) π is locally a diffeomorphism;

(b) π is a covering map. → See other test regarding this too

4. Let S^1 be the unit circle in $R^2 = R^2 \times \{0\} \subset R^3$. Compute the fundamental group of $R^3 - S^1$.

5. Compute the homology of $R^3 - S^1$ with coefficients in Z.

6. Let $f : RP^2 \to T^2$ be a continuous map from the projective plane RP^2 to the torus $T^2 = S^1 \times S^1$.

(a) Show that the induced homomorphism $f_* : \pi_1(RP^2) \to \pi_1(T^2)$ is trivial.

(b) Show that f is homotopic to a constant map.

1. $\forall x \in S^2$, $v \in T_x S^2$, $\exists \varphi \in SO(3)$ s.t. $\varphi(x) = x$ and $\varphi(v) = -v$. Then $\omega_x v = \omega_x(-v) \Rightarrow 2\omega_x(v) = 0 \Rightarrow \omega_x(v) = 0 \;\forall x, \forall v$.

2. Regular Value Theorem

3. a) Tangent map is an isomorphism $\Rightarrow \pi$ is a local diffeomorphism.

b) Counter example: $\pi : \mathbb{R} \to S^1$ $\pi(t) = t^2$.

4. $\mathbb{R}^3 - S^1 \overset{h.e.}{\cong} S^1 \vee S^2$

5. $\tilde{H}_n(\vee_\alpha X_\alpha) \cong \oplus_\alpha \tilde{H}_n(X_\alpha)$ provided (X_α, x_α) are good pairs, ie, x_α is a def. retract of some neighborhood in X_α.

6. a) Any homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is trivial

b) Lifting Criterion

Geo/Top - Fall 2000:

(1.) $\omega \in \Omega^1(S^2)$, $\phi^*\omega = \omega$ for all $\phi \in SO_3 \Rightarrow \omega = 0$

- Recall that SO_3 acts transitively on TS^2, i.e. the action has one orbit, hence for any two $(p,v), (q,w) \in TS^2$, $\exists \phi \in SO_3$ such that $\phi^*(p,v) = (\phi(p), T_p\phi(v)) = (q,w)$.
- In particular, let $\phi_0 \in SO_3$ be the isometry s.t. $\phi_0^*(p,v) = (p,-v)$; Then:

$$\omega_p(v) = \phi_0^*(\omega_p(v)) = \omega_{\phi_0(p)}(T_p\phi_0(v))$$
$$= \omega_p(-v)$$

$$\Rightarrow \omega_p(v) = -\omega_p(v) \Rightarrow \omega_p(v) = 0$$

But p, v arbitrary, hence $\omega \equiv 0$.

(2.) Show $M = \{x \in \mathbb{R}^4 : x_1 x_2 = x_3 x_4, |x| = 1\}$ smooth, orient surface

Consider the map $f: S^3 \longrightarrow \mathbb{R}$
$(x_1, x_2, x_3, x_4) \mapsto x_1 x_2 - x_3 x_4$.

Then clearly $f^{-1}(0) = M \subseteq S^3$ (by the $|x| = 1$ condition)

Now for $p \in f^{-1}(0)$, consider $T_p f : T_p S^3 \to T_0 \mathbb{R}$

$$T_p f = [x_2 \;\; x_1 \;\; -x_4 \;\; -x_3]\big|_p$$

$p \in S^3$, hence not all $x_i = 0$ since $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$, therefore $T_p f \neq 0$, hence surjective by linearity.

So $M \subseteq S^3 \subseteq \mathbb{R}^4$ submanifold

(3.) M, N manifolds, dim n, $\pi: M \to N$ onto, rank n at every point. T/F:

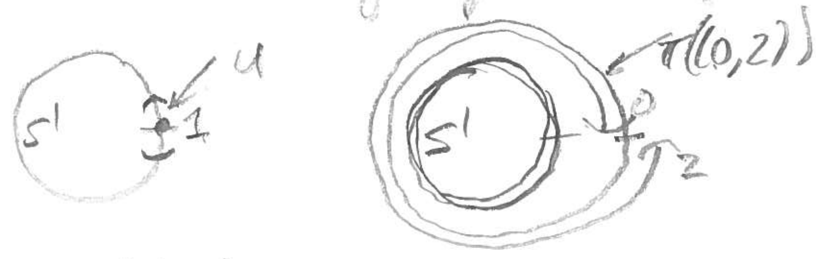
(a) π is locally a diffeomorphism:

Since $T_p\pi : T_pM \to T_{\pi(p)}N$ is rank n for all $p \in M$ and $T_pM \cong \mathbb{R}^n$, $T_{\pi(p)} \cong \mathbb{R}^n$, $T_p\pi$ is invertible by linearity. Hence π a local diffeo. at p by inverse fn. theorem, hence π a local diffeo. at all points, hence π local diffeo. (TRUE)

(b) π covering map?

FALSE; consider the map $\pi: (0,2) \longrightarrow S^1$
$t \longmapsto e^{2\pi i t}$

Clearly π is surjective and a local diffeomorphism but consider the inverse image of a neighborhood of 1 in S^1:



The inverse image $f^{-1}(U) = (0,\varepsilon) \cup (1-\varepsilon, 1+\varepsilon) \cup (2-\varepsilon, 2)$ but clearly $(0,\varepsilon)$ & $(2-\varepsilon, 2)$ do not map homeomorphically onto U since $f(0,\varepsilon)$, $f(2-\varepsilon,2)$ do not contain 1,

hence no nbhd of 1 is evenly covered, hence π not covering map.

④ Compute $\pi_1(\mathbb{R}^3 \setminus S^1)$:

$\mathbb{R}^3\setminus S^1 \simeq B^3 \setminus S^1 \simeq$  $= S^2 \vee S^1$

So $\pi_1(\mathbb{R}^3\setminus S^1) \cong \pi_1(S^2 \vee S^1)$
$\cong \pi_1(S^2) * \pi_1(S^1) \cong \mathbb{Z}$.

⑤ Compute $H_i(\mathbb{R}^3\setminus S^1)$:

Since $\mathbb{R}^3\setminus S^1 \simeq S^2 \vee S^1$, we have:
$\tilde{H}_i(\mathbb{R}^3\setminus S^1) \cong \tilde{H}_i(S^2\vee S^1) \cong \tilde{H}_i(S^2) \oplus \tilde{H}_i(S^1)$ since the spheres are both a good pair with base point.

Hence $\tilde{H}_i(\mathbb{R}^3\setminus S^1) \cong \begin{cases} 0 & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases} \Rightarrow \boxed{H_i(\mathbb{R}^3\setminus S^1) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & i>2 \end{cases}}$

⑥ $f: \mathbb{R}P^2 \to T^2$
(a) $f_*: \pi_1(\mathbb{R}P^2) \to \pi_1(T^2)$ is trivial
(b) $f \simeq$ const.

(a) We know $\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2$ & $\pi_1(T^2) \cong \mathbb{Z}\oplus\mathbb{Z}$
Then the induced hom. is $f_*: \mathbb{Z}_2 \to \mathbb{Z}\oplus\mathbb{Z}$; $\mathbb{Z}\oplus\mathbb{Z}$ has no torsion elts, hence $f_*(1)=0$, hence $f_* \equiv 0$.

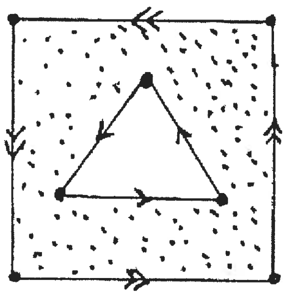
(b) Consider the covering $p: \mathbb{R}^2 \to T^2$. Then since $p_*(\pi_1(\mathbb{R}^2))=0$ & $f_* \equiv 0$, we have $f_*(\pi_1(\mathbb{R}P^2)) \subseteq p_*(\pi_1(\mathbb{R}^2))$, hence we have lift:

$$\begin{tikzcd} & \mathbb{R}^2 \arrow[d, "p"] \\ \mathbb{R}P^2 \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & T^2 \end{tikzcd}$$

; $\tilde{f}$ map into contractible space, hence $\tilde{f} \simeq$ const. by homotopy g_t; hence $f \simeq$ const. by $p \circ g_t$.

Geometry/Topology Graduate Exam
Fall 1999

1. Let Y be the space obtained by removing an open triangle from the interior of a compact square in $\mathbb{R}^2$. Let X be the quotient space of Y by the equivalence relation which identifies all four edges of the square and which identifies all three edges of the triangle according to the diagram below. Compute the fundamental group of X.



check

2. Let X be the space described in 1. Compute the homology groups $H_n(X; \mathbb{Z})$ of X with coefficients in $\mathbb{Z}$.

3. Give an example of a path connected space X which admits no covering $p : \widetilde{X} \to X$ with $\widetilde{X}$ simply connected.

4. Let X be a path connected manifold with $\pi_1(X; x_0) = \mathbb{Z}/5$, and consider a covering space $\pi : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 6 points. Show that $\widetilde{X}$ has either 2 or 6 connected components.

5. You may know that there exist continuous surjective maps $f : [0,1] \to [0,1]^2$ from the interval onto the square. Show that there exists no *continuously differentiable* surjective map $f : [0,1] \to [0,1]^2$.

6. Consider the map $\varphi : S^1 \times S^1 \to S^1 \times S^1$ defined by $\varphi(u, v) = (u^5, v^{-3})$, where we identify S^1 to the unit circle in the complex plane $\mathbb{C}$. Compute the degree of φ.

7. Let $\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$ be a closed (namely $d\omega = 0$) differential form of degree n on $\mathbb{R}^{n+1} - \{0\}$. Consider the homomorphism $i^* : \Omega^n(\mathbb{R}^{n+1} - \{0\}) \to \Omega^n(S^n)$ induced by the inclusion map $i : S^n \to \mathbb{R}^{n+1} - \{0\}$. Show that the form ω is exact (namely there exists $\alpha \in \Omega^{n-1}(\mathbb{R}^{n+1} - \{0\})$ such that $\omega = d\alpha$) if and only if $\int_{S^n} i^*(\omega) = 0$.

1. Van Kampen
2. Cellular Homology
3. Shrinking Wedge of circles (not semilocally simply connected, $\forall$ U around basept, $\pi_1(U) \to \pi_1(X)$ not trivial)
4. Classification of covering spaces
5. Corollary to Sard's Theorem.
6. Each point has 15 preimages, each of which is orientation reversing.
7. Stokes, $H^n_{dR}(S^n) \cong \mathbb{R}$ via I.

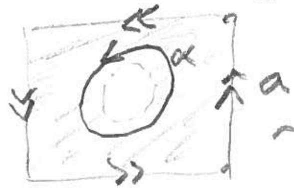
Geo/Top Fall '99:

(1.) $X =$  ; $\pi_1(X)$.

Let $A =$ , $B =$ , $A \cap B \simeq$  $= S^1$; so $A \cup B = X$

Now consider the inclusions; see that $A \simeq S^1$ & $B \simeq S^1$, so:

$$i_1^*: \underset{\mathbb{Z}\langle\alpha\rangle}{\overset{\parallel}{\pi_1(A\cap B)}} \longrightarrow \underset{\mathbb{Z}\langle a\rangle}{\overset{\parallel}{\pi_1(A)}} \qquad i_2^*: \underset{\mathbb{Z}\langle\alpha\rangle}{\overset{\parallel}{\pi_1(A\cap B)}} \longrightarrow \underset{\mathbb{Z}\langle b\rangle}{\overset{\parallel}{\pi_1(B)}}$$

 $\rightsquigarrow i_1^*(\alpha) = a^4$  $\rightsquigarrow i_2^*(\alpha) = b^3$

So apply Seifert-van Kampen:

$$\pi_1(X) = \pi_1(A) * \pi_1(B) \Big/ \langle i_1^*(\alpha)\, i_2^*(\alpha)^{-1} \rangle$$

$$= \mathbb{Z}\langle a\rangle * \mathbb{Z}\langle b\rangle \Big/ \langle a^4 b^{-3} \rangle = \underline{\langle a, b : a^4 b^{-3} = 1 \rangle}$$

(2.) $\underline{H_i(X) \text{ for same } X}$:

Apply Mayer-Vietoris with the same decomposition:

$$0 \longrightarrow H_2(X) \longrightarrow H_1(A\cap B) \longrightarrow H_1(A) \oplus H_1(B) \longrightarrow H_1(X)$$
$$\longrightarrow H_0(A\cap B) \to H_0(A) \oplus H_0(B) \to H_0(X) \longrightarrow 0$$

$$H_i(A) \cong H_i(B) \cong H_i(A\cap B) \cong H_i(S^1) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i > 1 \end{cases}$$

$\underline{H_0(X) \cong \mathbb{Z}}$ since X path-connected

So now we have:

$$0 \to H_2(X) \xrightarrow{\psi} \mathbb{Z}\langle\alpha\rangle \xrightarrow{(i^*,j^*)} \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle \xrightarrow{\varphi} H_1(X) \xrightarrow{\alpha} \mathbb{Z}$$
$$\xrightarrow{\beta} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \longrightarrow 0$$

- Consider the inclusions again:

$i^*: H_1(A\cap B) \to H_1(A)$ $\qquad$ $j^*: H_1(A\cap B) \longrightarrow H_1(B)$

$\mathbb{Z}\langle\alpha\rangle \qquad \mathbb{Z}\langle a\rangle$ $\qquad$ $\mathbb{Z}\langle\alpha\rangle \longrightarrow \mathbb{Z}\langle b\rangle$

$i^*(\alpha) = a^4$ $\qquad$ $j^*(\alpha) = b^3$ $\Big\}$ same reasoning as for van kampen

So $\operatorname{im}(i^*,j^*) = \langle(a^4, b^3)\rangle \cong \mathbb{Z} \Rightarrow (i^*,j^*)$ injective

$\Rightarrow \ker\varphi = \langle(a^4, b^3)\rangle \cong$

- Now, γ is surjective, hence $\ker\gamma = \mathbb{Z} \Rightarrow \operatorname{im}\beta = \mathbb{Z} \Rightarrow \ker\beta = 0$

$\Rightarrow \operatorname{im}\alpha = 0 \Rightarrow \varphi$ surjective.

Hence by 1st iso theorem, $H_1(X) \cong \mathbb{Z}\langle a\rangle \oplus \mathbb{Z}\langle b\rangle / \langle a^4, b^3\rangle$

$\cong \underline{\mathbb{Z}_4 \oplus \mathbb{Z}_3}.$

- Now, ψ must be injective, hence $\operatorname{im}\psi = H_2(X)$,

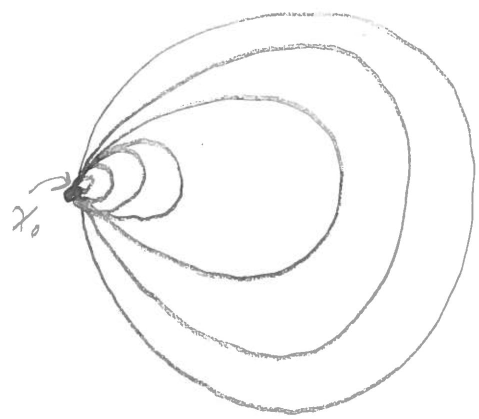
and $\operatorname{im}\psi = \ker(i^*,j^*) = 0$, so $H_2(X) \cong 0$

So we have: $H_i(X) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}_4\oplus\mathbb{Z}_3 & i=1 \\ 0 & i>1 \end{cases}$

③ Example of path-connected X with no universal cover:

A space has a universal cover if it is semi-locally simply-connected, i.e. each point $p \in X$ has a neighborhood U for which the inclusion $\pi_1(U, p) \hookrightarrow \pi_1(X, p)$ is trivial.

Now consider the shrinking wedge of circles:



$= X$

Any neighborhood of x_0 will contain a circle, hence the inclusion $\pi_1(U, x_0) \hookrightarrow \pi_1(X, x_0)$ may never be trivial, hence X has no universal cover.

(4.) X path-connected, $\pi_1(X, x_0) = \mathbb{Z}_5$, $p: \tilde{X} \to X$ covering space with $|p^{-1}(x_0)| = 6$. Show that $\tilde{X}$ has either 2 or 6 connected components

First, $\tilde{X}$ cannot be connected since $\mathbb{Z}_5$ has no index 6 subgroup (by classification of connected covering spaces).

Now, recall that a covering space can be given by the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$, i.e. 6-sheeted covering spaces are classified by the homomorphisms

$$f: \pi_1(X, x_0) \longrightarrow S_6 \rightsquigarrow f: \mathbb{Z}_5\langle a \rangle \longrightarrow S_6.$$

So $f(a)$ must have order 5, hence it is id or a 5-cycle. Recall that these permutations are given by lifts of loops in X to paths between the members of fiber $f^{-1}(x_0) := \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_6\}$.

- Case 5-cycle: A 5-cycle permutes 5 $\tilde{x}_i$ and fixes one, hence 5 of them must be in a path component. This leaves the last one to be in its own component (not connected!), hence 2 components
- Case id: every pt $\tilde{x}_i$ is fixed, hence there must only be a loop based at each in $\tilde{X}$, hence each resides in its own path component, i.e. 6 components

(5.) $\exists$ cont, surjective $f:[0,1]\to[0,1]^2$ (space-filling curve)
Show $\nexists$ cont, differentiable, surjective $f:[0,1]\to[0,1]\times[0,1]$

Suppose $f:[0,1]\to[0,1]\times[0,1]$ is surjective + differentiable.
Now consider the tangent map: $T_pf: T_pI\to T_pI^2$,
i.e. $T_pf:\mathbb{R}\to\mathbb{R}^2$, hence T_pf may never be surjective,
hence f has no regular pts, hence f has no regular values, hence $\operatorname{im} f$ meas. 0 by Sard.
But $I\times I$ is not measure 0, hence f not surjective.

(6.) $\varphi: S^1\times S^1\to S^1\times S^1$, $\varphi(z,w)=(z^5,w^{-3})$ $(S^1\subseteq\mathbb{C})$.
Deg φ = ?

We have the volume form $d\theta_1\wedge d\theta_2$ on the torus,
so consider the induced map on top-dim'l de Rham.

$\varphi^*: H^2(S^1\times S^1)\longrightarrow H^2(S^1\times S^1)$

$$\begin{aligned}\varphi^*(d\theta_1\wedge d\theta_2) &= \varphi^*d\theta_1\wedge\varphi^*d\theta_2\\ &= d(\theta_1\circ\varphi)\wedge d(\theta_2\circ\varphi)\\ &= d(5\theta_1)\wedge d(-3\theta_2)\\ &= -15\, d\theta_1\wedge d\theta_2\end{aligned} \Rightarrow \underline{\deg\varphi = -15}$$

$\varphi(z,w)=\varphi(e^{i\theta_1},e^{i\theta_2}) = (e^{i5\theta_1},e^{-i3\theta_2})$

(7.) $\omega \in \Omega^n(\mathbb{R}^{n+1}\setminus\{0\})$ closed

Consider $i^*: \Omega^n(\mathbb{R}^{n+1}\setminus\{0\}) \longrightarrow \Omega^n(S^n)$

Show ω exact $\Longleftrightarrow \int_{S^n} i^*(\omega) = 0$

($\Rightarrow$) Suppose ω exact. Then $[\omega] = 0 \in H^n(\mathbb{R}^{n+1}\setminus\{0\})$,

and $H^n(\mathbb{R}^{n+1}\setminus\{0\}) \cong H^n(S^n)$, hence $[i^*(\omega)] = 0 \in H^n(S^n)$

now apply top-dimensional de Rham isomorphism:

$$
\begin{aligned}
I : H^n(S^n) &\longrightarrow \mathbb{R} \\
\underset{\substack{\parallel \\ 0}}{[i^*(\omega)]} &\longmapsto \underline{\underline{\int_{S^n} i^*(\omega) = 0}}
\end{aligned}
$$

($\Leftarrow$) Suppose $\int_{S^n} i^*(\omega) = 0$

By the top-dim. de Rham iso, this means $[i^*(\omega)] = 0 \in H^n(S^n)$,

but again $H^n(S^n) \cong H^n(\mathbb{R}^{n+1}\setminus\{0\})$,

hence $[\omega] = 0$ since $[i^*(\omega)] = i^*([\omega]) = 0$

and i^* is isomorphism.

Geometry/Topology qualifying exam
Spring 1999

1. Let M be an embedded compact surface in $\mathbb{R}^3$, namely a non-empty 2-dimensional submanifold of $\mathbb{R}^3$. Show that there exists an infinite number of vertical lines $L = \{x\} \times \{y\} \times \mathbb{R}$ which meet M and are not tangent to it in the following sense: $M \cap L$ is non-empty and, for every $x \in M \cap L$, the plane tangent to M at x is not vertical.

2. Let N be a n-dimensional submanifold of the m-dimensional manifold M, and let $i : N \to M$ be the inclusion map. Suppose that N is closed in M. Show that, if $\alpha \in \Omega^p(N)$ is a degree p differential form on N, there exists a form $\beta \in \Omega^p(M)$ on M such that $i^*(\beta) = \alpha$. If $d\alpha = 0$, can you always choose β so that $d\beta = 0$?

3. In $B^2 \times B^2$, let X be the union of the torus $S^1 \times S^1$ and of the disk $B^2 \times \{x\}$ (where B^2 is the closed unit disk in $\mathbb{R}^2$ and S^1 is its boundary circle). Compute the fundamental group of X.

4. For X as in Problem 3, compute the homology modules $H_p(X; R)$, with coefficients in an arbitrary unitary ring R.

5. Prove or disprove: A surjective map $p : \widetilde{X} \to X$ is a covering map if and only if, for every $\widetilde{x} \in \widetilde{X}$, there is a neighborhood $\widetilde{U}$ of $\widetilde{x}$ such that the restriction $p_{|\widetilde{U}} : \widetilde{U} \to p\left(\widetilde{U}\right)$ is a homeomorphism.

6. Let X be a path connected space such that $\pi_1(X; x_0) = 1$ and $\pi_2(X; x_0) = 1$. Recall that the second property means that, for every continuous map $\alpha : [0,1] \times [0,1] \to X$ such that $\alpha(s,t) = x_0$ when $s \in \{0,1\}$ or $t \in \{0,1\}$, there is a homotopy $H : [0,1] \times [0,1] \times [0,1] \to X$ such that $H(s,t,u) = x_0$ when $s \in \{0,1\}$ or $t \in \{0,1\}$ or $u = 1$. Consider the 2-dimensional torus $T^2 = S^1 \times S^1$. Show that every continuous map $f : T^2 \to X$ is homotopic to a constant map. (Possible hint: Write the torus as a square with identifications of its sides.)

1. Use Sard's Theorem on $\pi : M \to \{xy\text{-plane}\}$

2. Use a bump function to extend α smoothly

3. Van Kampen

4. Cellular Homology

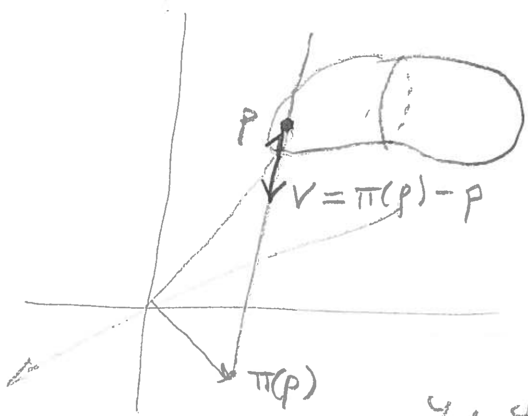
5. False: Consider countably many copies of $\mathbb{R}$ covering a shrinking wedge of circles.

6. f is homotopic to a map sending edges of to x_0 by $\pi_1(X, x_0) = 1$.

Then use $\pi_2(X, x_0) = 1$.

(1.) $M \subseteq \mathbb{R}^3$ emb, cpt, 2-mfld

Show $\{L = \{x\} \times \{y\} \times \mathbb{R} : L \text{ not tangent to } M\} \stackrel{?}{=} \infty$



$\pi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \subseteq \mathbb{R}^3$

$(x,y,z) \longmapsto (x,y,0)$

$T_p\pi : T_pM^{\cong \mathbb{R}^2} \longrightarrow T_p\mathbb{R}^2{}^{\cong \mathbb{R}^2}$

π linear $\Rightarrow T_p\pi = \pi \Rightarrow T_p\pi(\pi(p)-p) = \pi(p) - \pi(p) = 0$

Let $\mathcal{L}$ be our family & consider the complement:

$\mathcal{L}^c = \{L : L \text{ tangent to } M\} = \{V : V = \pi(p) - p \in T_pM \text{ for some } p\}$

$\subseteq \{V : V \in \ker T_p\pi \text{ for some } p\}$

For any map, $\ker T_p\pi \neq 0 \iff p$ critical point of π, so $? = \coprod_{\substack{p \in M \\ p \text{ crit of } \pi}} \ker T_p\pi$

$0 \stackrel{=}{\underset{\text{Sard}}{}} \mu(\{p \in M : p \text{ critical}\}) = \mu(\{p \in M : \ker T_p\pi \neq 0\})$

indexed by meas. 0 set, hence meas. 0 ?

(2.) $N \subseteq M$, $\dim N = n$, $\dim M = m$, $i: N \to M$ inclusion

Suppose N closed in M.
Show that, if $\alpha \in \Omega^p(N)$, $\exists \beta \in \Omega^p(M)$ s.t.
$i^*(\beta) = \alpha$. If $d\alpha = 0$, can you always choose β
s.t. $d\beta = 0$?

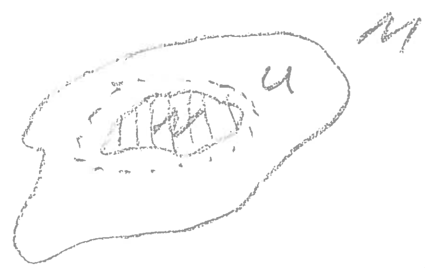
WTS: $i^*: \Omega^p(M) \longrightarrow \Omega^p(N)$ surjective.
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \alpha$

(#1)

$\alpha_p \in \mathrm{Alt}^p(T_pN) \Rightarrow \alpha_p : T_pN \otimes \cdots \otimes T_pN \longrightarrow \mathbb{R}$

$\beta_p : T_pM \otimes \cdots \otimes T_pM \to \mathbb{R}$

$i^*(\beta) = \beta_{i(p)}(T_p i\, v)$



$N \subseteq M$ closed.

$\beta_p(v) = \alpha_p(v)$ for $v \in T_pN \subseteq T_pM$ subspace, where $T_pN \cong \mathbb{R}^n$, $T_pM \cong \mathbb{R}^m$

$= \underbrace{\quad}_{U} 0$

$= 0$ for $v \in (T_pN)^\perp$

Then $i^*(\beta) = \alpha$

(#2) : i^* surj. on forms $\Rightarrow$ i^* surj on Homology

$\Rightarrow$ $d\alpha = 0 \Rightarrow \alpha \in H^p(N) \Rightarrow \exists \beta \in H^p(M)$ s.t. $i^*(\beta) = \alpha$

But $0 = d\alpha = d\, i^*(\beta) = i^*(d\beta) =$

$\Rightarrow d\beta = 0$ since hom.

③ $X = (S^1 \times S^1 \amalg B^2) \big/ {}_{S^1 \times \{x\} \sim \partial B^2}$

$\pi_1(X) = ?$

Let $A = T^2$, $B = B^2$, $A \cap B = S^1$, $A \cup B = X$

$i_1 : A \cap B \hookrightarrow A$

$i_1^* : \pi_1(A \cap B) \longrightarrow \pi_1(A) \cong \mathbb{Z}\langle a \rangle \oplus \mathbb{Z}\langle b \rangle$

$i_1^*(\alpha) = a$

$i_2^* : \pi_1(A \cap B) \longrightarrow \pi_1(B) = 1 \Rightarrow i_2^*(\alpha) = 1$

$\Rightarrow \pi_1(X) = \pi_1(A) * \pi_1(B) \big/ \langle a \rangle$

$\cong \mathbb{Z}\langle a \rangle \oplus \mathbb{Z}\langle b \rangle \big/ \langle a \rangle \cong \mathbb{Z}\langle b \rangle \cong \mathbb{Z}$

4.) $H_p(X; \mathbb{R})$ for $X = S^1 \times S^1 \sqcup B^2 / \partial B^2 \sim S^1 \times \{x\}$.

$A = B^2$, $B = T^2$, $A \cap B = S^1$, $A \cup B = X$.

Mayer-Vietoris :

$$0 \to H_2(S^1) \to H_2(A) \oplus H_2(B) \to H_2(X) \to H_1(S^1)$$
$$\to H_1(A) \oplus H_1(B) \to H_1(X) \to H_0(S^1) \to H_0(A) \oplus H_0(B)$$
$$\to H_0(X) \to 0$$

X p.c. $\Rightarrow$ $H_0(X) \cong \mathbb{Z}$; #3 $\Rightarrow$ $H_1(X) \cong \mathbb{Z}$.

$$\Rightarrow \quad 0 \to \mathbb{Z} \xrightarrow{\psi} H_2(X) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z}$$
$$\xrightarrow{\alpha} \mathbb{Z} \to 0.$$

α surj. $\Rightarrow$ $\ker \alpha = \mathbb{Z}$ $\Rightarrow$ $\mathrm{im}\, \beta = \mathbb{Z}$ $\Rightarrow$ $\ker \beta = 0$ $\Rightarrow$ $\mathrm{im}\, \gamma = 0$

$\Rightarrow$ $\ker \gamma = \mathbb{Z}$ $\Rightarrow$ $\mathrm{im}\, \delta = \mathbb{Z}$ $\Rightarrow$ $\ker \delta = \mathbb{Z}$ $\Rightarrow$ $\mathrm{im}\, \varepsilon = \mathbb{Z}$ $\Rightarrow$ $\ker \varepsilon = 0$

$\Rightarrow$ $\mathrm{Im}\, \phi = 0$

and ψ is injective, hence $\mathrm{Im}\, \psi = \mathbb{Z}$ $\Rightarrow$ $\ker \phi = \mathbb{Z}$

S

$H_2(X) = \mathbb{Z}$

(5) T/F: surj. $p:\tilde{X}\to X$

p covering $\iff$ p local diffeo

✗ $\rightsquigarrow$ consider $p:(0,2)\longrightarrow S^1$

$t \longmapsto e^{2\pi i t}$

$1\in S^1$ is not evenly covered.

(6) X p.c., $\pi_1(X)=1$, $\pi_2(X)=1$.

Consider T^2; show every $f:T^2\to X$ is homotopic to const.

$T^2 =$  $\longrightarrow X$ $\rightsquigarrow$ restricted to edge, ~~map~~ maps homotopic to const,

$\|$

$S^1\times S^1$

~~$f(s,t) = (\alpha(s), \beta(t))$~~

hence $f\simeq f'$ sending edges to const.

$\pi_2(X)=1 \Rightarrow$ every $\alpha: I\times I\to X$ $\exists$ α_t w/ $\alpha_0=\alpha$, $\alpha_1=x_0$

$f\simeq[f]\in\pi_2(X,x_0)$

$\Rightarrow f\simeq$ const.