Cohomology, and Other Results of Algebraic Topology

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Setup: Let R be a commutative ring w/ 1. X is a topological space.

Def/ The n^{th} cochain group $C^n(X;R)$ is defined as $\mathrm{Hom}_R(C_n(X;R), R)$ the R-module of R-linear maps $C_n(X;R) \to R$, where $C_n(X;R)$ is the free R-module generated by n-simplices $\sigma : \Delta^n \to X$ where Δ^n is the standard n-simplex $\Delta^n =$

$$= \{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_i \geq 0 \text{ and } x_0 + x_1 + \dots + x_n = 1 \}.$$

Def/ The coboundary map $\delta : C^n(X;R) \to C^{n+1}(X;R)$ defined on cochains $\varphi \in C^n(X;R)$ is given by

$$\delta(\varphi)(c) = \underbrace{(-1)^{n+1}}_{} \varphi(\partial c) \text{ for all } (n+1)\text{-chains } c \in C_{n+1}(X;R)$$

From Milnor, this sign seems outdated (Hatcher nor Bonahon use it)

Here, $\partial : C_{n+1}(X;R) \longrightarrow C_n(X;R)$ is the **boundary map** defined by

$$\partial(\sigma) = \sum_{i=0}^{n+1} (-1)^i (\sigma \circ f_i) \quad \text{for all } (n+1)\text{-simplices } \sigma,$$

where $f_i : \Delta^n \longrightarrow \Delta^{n+1}$ is the i^{th} face map.

Proposition: $\delta^2 = 0$.

Def/ Let $Z^n(X;R) := \ker \delta : C^n(X;R) \to C^{n+1}(X;R)$

and let $B^n(X;R) := \operatorname{im} \delta : C^{n-1}(X;R) \to C^n(X;R)$.

The **n^{th} cohomology group** $H^n(X;R)$ is defined to be $Z^n(X;R) / B^n(X;R)$.

Notation/ when the ring R is understood, we simply write $C^n(X)$, $H^n(X)$, etc.

Theorem: Given $[\varphi] \in H^n(X)$ and $[c] \in H_n(X)$, so $\varphi \in Z^n(X)$ and $c \in Z^n(X)$, then the scalar $\varphi(c) \in R$ is independent of the representatives. we therefore have a well-defined map of R-modules

$$H^n(X) \longrightarrow \mathrm{Hom}_R(H_n(X), R)$$

canonically defined by $[\varphi] \mapsto ([c] \mapsto \varphi(c))$.

The result is that this canonical map is an isomorphism <u>assuming</u> R is a PID and either $H_{n-1}(X) = 0$ or $H_{n-1}(X)$ is a free R-module.

Cor/ If $R = F$ is a field, then $H^n(X)$ is canonically isomorphic to $\mathrm{Hom}_F(H_n(X), F)$ for all n.

Universal Cover and the Fundamental Group

Def/ A covering space $(Y, p: Y \to X)$ of a space X is a <u>universal cover of X</u> if Y is simply connected.

Thm/ A universal cover is unique up to homeomorphism preserving the fibers.

Rmk/ we thus refer to <u>the</u> universal cover of X.

Thm/ Let $(Y, p: Y \to X)$ be the universal cover of a space X (X is assumed path connected, locally path connected, semi-locally simply connected). Let $\text{Aut}\, Y$ be the group $\{ \varphi : Y \to Y \text{ homeo s.t. } p \circ \varphi = p \}$.

Then $\pi_1(X) \cong \text{Aut}(Y)$.

Finite Cell Complexes (adapted from Bonahon's notes)

Def/ A <u>finite cell complex</u> is a space X which can be written as

$$X = X^{(0)} \cup X^{(1)} \cup \cdots \cup X^{(n)}$$

where

1) $X^{(0)}$ consists of finitely many points.

2) Each $X^{(i+1)}$ is obtained from $X^{(i)}$ by gluing finitely many copies $B_1^{i+1}, B_2^{i+1}, \ldots, B_{k_i}^{i+1}$ of the $(i+1)$-dim ball B^{i+1} along continuous maps $\varphi_j : S_j^i \to X^{(i)}$ where S_j^i = boundary of B_j^{i+1} ≅ copy of i-dim sphere S^i

Rmks/ • So each $X^{(i)}$ is contained in $X^{(i+1)}$.

• $X^{(i+1)}$ is determined by maps $\varphi_j^i : S_j^i \to X^{(i)}$

such that $X^{(i+1)} = X^{(i)} \amalg B_1^{i+1} \amalg \cdots \amalg B_{k_i}^{i+1} / \sim$

where for $x \in S_j^i \subset B_j^{i+1}$, $x \sim \varphi_j^i(x) \in X^{(i)}$.

• $\pi_1(X^{(0)}, x_0) \cong \{e\}$. • X is of dim n is $X = X^{(n)}$ for some $n \geq 0$.

Proposition / Definition

A finite graph is a finite cell complex of dimension 1 (i.e. $X = X^{(1)}$).

For a finite graph which is connected, has v vertices (i.e. $\#(X^{(0)}) = v$), and has e edges (i.e. $k_1 = e$), then

$\pi_1(X, x_0) \cong F^{e-v+1}$ the free group on $e-v+1$ generators, also written $\mathbb{Z}^{*(e-v+1)}$.

Cellular Homology (adapted from Hatcher)

Lemma/ Let $X = X^{(0)} \cup X^{(1)} \cup \dots \cup X^{(n)}$ be a finite cell complex. Over $\mathbb{Z}$,

(a) $H_p(X^{(k)}, X^{(k-1)}) \cong \begin{cases} 0, & p \neq k \\ \mathbb{Z}^{\#\{k\text{-cells}\}}, & p = k \end{cases}$ $\quad (\mathbb{Z}^0 \equiv 0)$

(b) $H_p(X^{(k)}) \cong 0$ for $p > k$.

(c) $H_p(X^{(k)}) \cong H_p(X)$ for $p < k$.

Def/ Let $X = X^{(0)} \cup X^{(1)} \cup \dots \cup X^{(n)}$ be a finite cell complex. The <u>cellular chain complex of X</u> is the complex

$$\cdots \longrightarrow H_{k+1}(X^{(k+1)}, X^{(k)}) \xrightarrow{d_{k+1}} H_k(X^{(k)}, X^{(k-1)}) \xrightarrow{d_k} H_{k-1}(X^{(k-1)}, X^{(k-2)}) \to \cdots$$

where the boundary maps d_k are defined by

$$\begin{tikzcd}
& H_k(X^{k-1}) \cong 0 \arrow[dr, "i_k^{(k)}"] & & & H_k(X^{k+1}, X^k) \cong 0 & & \\
& & H_k(X^k) \arrow[r, "i_k^{(k+1)}"] \arrow[dr, "p_k^{(k)}"] & H_k(X^{k+1}) \cong H_k(X) \arrow[ur, "p_k^{(k+1)}"] & & & \\
\cdots \arrow[r, dashed] & H_{k+1}(X^{k+1}, X^k) \arrow[ur, "\partial_{k+1}^{(k+1)}"] \arrow[rr, dashed, "d_{k+1}"] & & H_k(X^k, X^{k-1}) \arrow[rr, dashed, "d_k"] \arrow[dr, "\partial_k^{(k)}"] & & H_{k-1}(X^{k-1}, X^{k-2}) \arrow[r] & {} \\
& & & & H_{k-1}(X^{k-1}) \arrow[ur, "p_{k-1}^{(k-1)}"] & & \\
& & & H_{k-1}(X^{k-2}) \cong 0 \arrow[ur, "i_{k-1}^{(k-1)}"] & & &
\end{tikzcd}$$

Here we are using the long exact sequence in relative homology.

So $d_k := p_{k-1}^{(k-1)} \circ \partial_k^{(k)}$.

It is obvious $d_k \circ d_{k+1} = 0$.

So far we have not used any of the isomorphisms marked in red on the diagram.

We think of elts of $H_k(X^k, X^{k-1})$ as linear combinations of k-cells of X.

Define the <u>n^{th} cellular homology group of X (over $\mathbb{Z}$)</u>, denoted $H_n^{CW}(X)$, as the n^{th} homology of this complex.

Thm/ $H_n^{CW}(X) \cong H_n(X)$.

<u>Proof (Thm)</u>: Now we use the markings in red.

$$H_k(X) \cong H_k(X^{k+1}) \cong \ker p_k^{(k+1)} \cong \operatorname{im} i_k^{(k+1)} \cong H_k(X^k)/\ker i_k^{(k+1)}$$

$$\cong H_k(X^k)/\operatorname{im} \partial_{k+1}^{(k+1)}.$$

Since $p_k^{(k)}$ is injective, $\operatorname{im} \partial_{k+1}^{(k+1)} \cong \operatorname{im} p_k^{(k)} \circ \partial_{k+1}^{(k+1)}$

$$= \operatorname{im} d_{k+1},$$

and $H_k(X^k) \cong \operatorname{im} p_k^{(k)} = \ker \partial_k^{(k)}$.

Since $\rho_{k-1}{}^{(k-1)}$ is injective, $\ker d_k = \ker \partial_k{}^{(k)}$.

we conclude $H_k(X) \cong H_k(X^k) / \operatorname{im} \partial_{k+1}^{(k+1)} \cong \ker d^k / \operatorname{im} d^{k+1}$. $\square$

Rmks/ (Immediate consequences)

- we continue writing X^k instead of $X^{(k)}$.

- $H_k(X) = 0$ if X has no k-cells.

Proof: cellular chain complex:

$$\to H_{k+1}(X^{k+1}, X^k) \xrightarrow{d_{k+1}} H_k(X^k, X^{k-1}) \xrightarrow{d_k} H_{k-1}(X^{k-1}, X^{k-2}) \to$$

$$H_k(X^k, X^{k-1}) = H_k(X^{k-1}, X^{k-1}) \text{ since } X^k = X^{k-1}.$$

we are done since

$H_k(X^{k-1}, X^{k-1}) = 0$:

Recall $_{\text{generally}}$ $H_k(X, A) = Z_k(X, A) / B_k(X, A)$

where $Z_k(X, A) = \{ c_k \in C_k(X) : \partial c_k \in C_{k-1}(A) \}$

and $B_k(X, A) = \{ c_k \in C_k(X) : \exists c_{k+1} \in C_{k+1}(X) \text{ s.t. } \partial c_{k+1} - c_k \in C_k(A) \}$

Clearly when $A = X$, $Z_k(X, X) = B_k(X, X) = C_k(X)$. ✓ $\square$.

- If X has m_k k-cells, then $H_n(X)$ is generated by at most k elements.

<u>Proof</u>: $H_k(X^k, X^{k-1}) \cong \mathbb{Z}^{m_k}$ is free abelian of rank m_k.

Hence $\ker d_k \subset H_k(X^k, X^{k-1})$ is free abelian of rank $\leq m_k$ (since $\mathbb{Z}$ is a PID).

Hence $\ker d_k / \operatorname{im} d_{k+1}$ is generated by at most m_k elts (it is not necessarily free!). $\square$

- If no two cells of X are in adjacent dimensions, and there are m_k k-cells, then $H_k(X) \cong H_k(X^k, X^{k-1}) \cong \mathbb{Z}^{m_k}$.

<u>Proof</u>: $\to \underset{0}{H_{k+1}(X^{k+1}, X^k)} \xrightarrow{d_{k+1}} \underset{\mathbb{Z}^{m_k}}{H_k(X^k, X^{k-1})} \xrightarrow{d_k} \underset{0}{H_{k-1}(X^{k-1}, X^{k-2})} \to$

$$H_k(X) \cong \ker d_k / \operatorname{im} d_{k+1} \cong \mathbb{Z}^{m_k}/0. \qquad \square.$$

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example/ It can be shown $e^0 \cup e^2 \cup \cdots \cup e^{2n}$ is a cellular decomposition of $\mathbb{C}P^n$.

So $H_k(\mathbb{C}P^n) = \begin{cases} 0, & k \text{ odd} \\ \mathbb{Z}, & k \text{ even} \end{cases}$.

Goal/ Compute the boundary maps d_k in terms of degrees.

warm up/ (Computation of d_1 in a special case)

$$\begin{tikzcd} & H_0(X^0) \arrow[dr, "p_0^{(0)}"] & \\ H_1(X^1, X^0) \arrow[ur, "\partial_1^{(1)}"] \arrow[rr, "d_1"] & & H_0(X^0, \emptyset) \end{tikzcd}$$

$$\cong H_0(X^0)$$

Surely $p_0^{(0)} = \mathrm{id}_{H_0(X^0)}$ so $d_1 = \partial_1^{(1)}$

the connecting homomorphism $H_1(X^1, X^0) \to H_0(X^0)$

w/ formula $[c] \mapsto [\partial c]$.

$c \in C_1(X^1)$
$\partial c \in C_0(X^0)$

If X is connected, $H_0(X) = \mathbb{Z}$.

If in addition X has a single 0-cell, then $H_0(X^0) \cong \mathbb{Z}$.

so $\mathbb{Z} \cong H_0(X) \cong H_0(X^0)/\mathrm{im}\, d_1 \cong \mathbb{Z}/\mathrm{im}\, d_1$.

so d_1 must be the 0 map.

Thm/ (Cellular Boundary Formula)

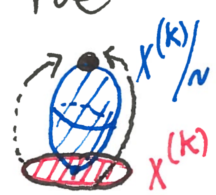
Let $X = X^{(0)} \cup X^{(1)} \cup \cdots \cup X^{(n)}$ be a finite cell complex, where for $n-1 \geq k \geq 0$,

$$X^{(k+1)} = X^{(k)} \amalg_{\varphi_1^k} B_1^{k+1} \amalg \cdots \amalg_{\varphi_{m_k}^k} B_{m_k}^{k+1}.$$

Here φ_j^k is an attaching map $S_j^k = \partial B_j^k \to X^{(k)}$.

We may view $H_k(X^{(k)}, X^{(k-1)})$ as freely generated over $\mathbb{Z}$ by $\{ B_j^k : j = 1, \ldots, m_k \}$.

For $i = 1, \ldots, m_k$ Let $\pi_i^k : X^{(k)} \to X^{(k)}/\sim \, \simeq S^k$ be the quotient map determined by collapsing



$$X^{(k)} - \mathrm{int} B_i^k = X^{(k-1)} \amalg_{\varphi_1^k} B_1^k \amalg \cdots \amalg_{\varphi_{i-1}^k} B_{i-1}^k \amalg_{\varphi_{i+1}^k} B_{i+1}^k \amalg \cdots \amalg_{\varphi_{m_k}^k} B_{m_k}^k$$

to a point (i.e. $\begin{matrix} x \neq y \\ x \sim y \end{matrix}$ iff $x \neq y$ & $x, y \in X^{(k)} - \mathrm{int} B_i^k$).

Then the boundary map d_k is calculated as

$$d_k(B_j^k) = \sum_{i=1}^{m_{k-1}} d_{ij} B_i^{k-1} \quad (j = 1, \ldots, m_k)$$

where d_{ij} is the degree of the composite map

$$S^{k-1} \cong S_j^{k-1} \xrightarrow{\varphi_j^{k-1}} X^{(k-1)} \xrightarrow{\pi_i^{k-1}} X^{(k-1)}/\sim \, \cong S^{k-1}.$$

Rmks/ • Recall that $H_n(S^n) \cong \mathbb{Z}$ which gives rise to the notion of degree: If $f: S^n \to S^n$ is a continuous map, then f induces a morphism on homology $H_n(f): H_n(S^n) \to H_n(S^n)$ which is determined by the number $d \in \mathbb{Z}$ to which $1 \in \mathbb{Z}$ is sent. d is defined to be the <u>degree</u> of f.

$$H_n(f) : \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}} \longrightarrow \underset{\mathbb{Z}}{\underset{\cong}{H_n(S^n)}}$$

Useful Facts About Degrees of maps $f: S^n \to S^n$

- If f is not surjective, $\deg f = 0$. (contractibility)
- $f \simeq g \Rightarrow \deg f = \deg g$ (homotopy equivalent maps induce the same map on homology)
- $\deg f = \deg g \Rightarrow f \simeq g$ (Hopf Theorem)
- $\deg fg = \deg f \cdot \deg g$ (covariance of homology)
- From last fact, f homotopy equivalence (e.g. homeo) $\Rightarrow \deg f = \pm 1$.
- There is a generator $[\sigma_n^N - \sigma_n^S]$ of $H_n(S^n)$ where σ_n^N / σ_n^S maps Δ^n homeomorphically (rel boundary $S^{n-1} \subset S^n$) to the upper/lower closed hemisphere, respectively.

Picture:

Consequently if r is the reflection $S^n \to S^n$ fixing the first n coordinates and taking the negative of the last, then $H_n(r)([\sigma^N - \sigma^S]) = [r \circ \sigma^N - r \circ \sigma^S] = -[\sigma^N - \sigma^S]$, hence $\deg r = -1$. It follows by composition that if $\mathcal{R}$ is the reflection through the origin, then $\deg \mathcal{R} = (-1)^{n+1}$.

Computing Degrees — The Local Degree Formula

Let $f : S^n \longrightarrow S^n$ be a map. Assume $y \in S^n$ satisfies $f^{-1}(y) = \{x_1, \ldots, x_m\}$ is finite. Assume there is a nbhd V of y and disjoint nbhds $U_1, \ldots, U_m$ of $x_1, \ldots, x_m$ s.t. $f(U_k) \subset V$ for all $k = 1, \ldots, m$. Hence $f(U_k - x_k) \subset V - y$.

We want to examine a big commutative diagram. We have an inclusion $U_k \xhookrightarrow{i_k} S^n$ s.t. $i_k(U_k - x_k) \subset S^n - f^{-1}(y)$, so we have an inclusion $(U_k, U_k - x_k) \xrightarrow{i_k} (S^n, S^n - f^{-1}(y))$. This induces a map $H_n(U_k, U_k - x_k) \xrightarrow{H_n(i_k)} H_n(S^n, S^n - f^{-1}(y))$. Similarly we have an inclusion $(S^n, S^n - f^{-1}(y)) \xrightarrow{j_k} (S^n, S^n - x_k)$ which induces a map $H_n(S^n, S^n - f^{-1}(y)) \xrightarrow{H_n(j_k)} H_n(S^n, S^n - x_k)$. The composition of these two maps gives a map $H_n(U_k, U_k - x_k) \xrightarrow{H_n(j_k \circ i_k)} H_n(S^n, S^n - x_k)$ which is also just the map induced by the inclusion $j_K \circ i_K$.

Claim: $H_n(j_K \circ i_K)$ is an isomorphism $H_n(U_K, U_K - X_K) \xrightarrow{\sim} H_n(S^n, S^n - X_K)$.

Proof (claim) we use Excision.

Let $X = S^n$ and $A = S^n - X_K$. Let $B = S^n - U_K$ so that

$U_K = X - B$ and $U_K - X_K = A - B$. we have

$\overline{B} = B \subset A = \operatorname{int} A$. The Excision thm

gives the desired isomorphism. $\square$.

Let i be the inclusion $(S^n, \emptyset) \xrightarrow{i} (S^n, S^n - f^{-1}(y))$

which induces a map $H_n(S^n) \cong H_n(S^n, \emptyset) \xrightarrow{H_n(i)} H_n(S^n, S^n - f^{-1}(y))$.

Again the composition $j_K \circ i$ induces a map $H_n(S^n) \xrightarrow{H_n(j_K \circ i)} H_n(S^n, S^n - X_K)$.

Claim: $H_n(j_K \circ i)$ is an isomorphism $H_n(S^n) \xrightarrow{\sim} H_n(S^n, S^n - X_K)$.

Proof (claim): we use LES in relative homology.

$$H_n(\underset{0}{S^n - X_K}) \xrightarrow{f} H_n(S^n) \xrightarrow{g} H_n(S^n, S^n - X_K) \xrightarrow{h} H_{n-1}(S^n - X_K) \xrightarrow{l}$$

$\xrightarrow{l} H_{n-1}(S^n)$. when $n > 1$, $H_n(S^n - X_K)$ and $H_{n-1}(S^n - X_K)$ are 0

since $S^n - X_K$ is contractible. when $n = 1$,

$$\underset{\mathbb{Z}}{H_0(S^1 - X_K)} \xrightarrow[\longrightarrow]{} \underset{\mathbb{Z}}{H_0(S^1)}$$ is just the identity, hence

its kernel is 0. so $h \equiv 0$ is the zero map, hence g is an isomorphism. $\square$.

So far we have succeeded in constructing a commutative diagram: (black)

$$\begin{tikzcd}
& H_n(U_k, U_k - x_k) \arrow[r, "H_n(f)"] \arrow[dl, "\sim"'] \arrow[d] & H_n(V, V-y) \arrow[d, "\wr"] \\
H_n(S^n, S^n - x_k) & H_n(S^n, S^n - f^{-1}(y)) \arrow[l] \arrow[r, "H_n(f)"] & H_n(S^n, S^n - y) \\
& H_n(S^n) \arrow[ul, "\sim"] \arrow[u] \arrow[r, "H_n(f)"] & H_n(S^n) \arrow[u, "\wr"']
\end{tikzcd}$$

The containments $f(U_k - x_k) \subset V - y$, $f(S^n - f^{-1}(y)) \subset S^n - y$, and the inclusions $(V, V-y) \to (S^n - y)$ and $(S^n, \phi) \to (S^n, S^n - y)$, together w/ the same methods using Excision and LES in relative homology allow us to complete the above commutative diagram (blue).

We thus obtain identifications $H_n(U_k, U_k - x_k) \cong H_n(S^n)$ and $H_n(V, V-y) \cong H_n(S^n)$. So the map $H_n(U_k, U_k - x_k) \xrightarrow{H_n(f)} H_n(V, V-y)$ induced by the restriction of f to U_k is a map $\mathbb{Z} \longrightarrow \mathbb{Z}$ characterized by the image d_k of 1, which we call the <u>local degree of f at x_k</u>.

Thm/ (Local Degree Formula)

Keeping w/ the current setup,

$$\deg f = \sum_k \deg f|_{x_k}$$

where $\deg f|_{x_k}$ is the local degree of f at x_k denoted previously by d_k.

Rmks/ • If each U_k is mapped homeomorphically onto V, then $\deg f|_{x_k} = \pm 1$ for each k.
One determines the sign depending on the situation.

example/ For a detailed calculation, see the solution to problem 3c of the Spring 2011 geometry exam.

Poincaré Duality & Orientations (adapted from Hatcher)

Prop/ Let $x \in \mathbb{R}^n$. Then $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \mathbb{Z}$.

Proof: This is clearly true when $n=0$ and $\mathbb{R}^0 = \{x\}$.

When $n \geq 1$, we have the LES in relative homology

$$\to \underset{\cong\, H_n(S^{n-1})}{H_n(\mathbb{R}^n - \{x\})} \to H_n(\mathbb{R}^n) \to \underset{\cong\, H_n(\mathbb{R}^n, S^{n-1})}{H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})} \to \underset{\cong\, H_{n-1}(S^{n-1})}{H_{n-1}(\mathbb{R}^n - \{x\})}$$

$$\to H_{n-1}(\mathbb{R}^n) \to H_{n-1}(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \to$$

For $n > 1$: This becomes

$$\to 0 \to 0 \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \xrightarrow{\sim} \mathbb{Z} \to 0 \to H_{n-1}(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \to$$

as desired.

For $n=1$: This becomes $(x,y) \overset{g}{\longmapsto} x+y$

$$\to 0 \to 0 \to H_1(\mathbb{R}, \mathbb{R} - \{x\}) \xrightarrow[\text{inclusion}]{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \to 0 \to$$

So $H_1(\mathbb{R}, \mathbb{R} - \{x\}) \cong \operatorname{Im} f = \ker g \cong \mathbb{Z}$. $\square$

Rmk/ Let x and y be two points of $\mathbb{R}^n$, $n \geq 1$. Then there is a canonical isomorphism $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong$
$\cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$.

Indeed, let U be a ball containing x and y. Then $\mathbb{R}^n - B$ is a deformation retract of $\mathbb{R}^n - \{x\}$ and of $\mathbb{R}^n - \{y\}$.
So $H_n(\mathbb{R}^n, \mathbb{R}^n - x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - B) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - y)$.
It is canonical since the map $H_n(\mathbb{R}^n, \mathbb{R}^n - B) \to H_n(\mathbb{R}^n, \mathbb{R}^n - x)$ is just induced by inclusion.

<u>Definition</u> A <u>local orientation of $\mathbb{R}^n$ at a point x</u> is a choice of generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - x) \cong \mathbb{Z}$. By the preceding remark, this forces a choice of generator of $H_n(\mathbb{R}^n, \mathbb{R}^n - y)$ for all $y \in \mathbb{R}^n$ via the canonical isomorphism $H_n(\mathbb{R}^n, \mathbb{R}^n - x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - y)$.

Def/ An <u>n-dimensional topological manifold</u> M is a topological space M which is Hausdorff and satisfies that every point $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$.

Prop/ Let $x \in M$, M an n-dim top mfld. Then $H_n(M, M-x) \cong \mathbb{Z}$.

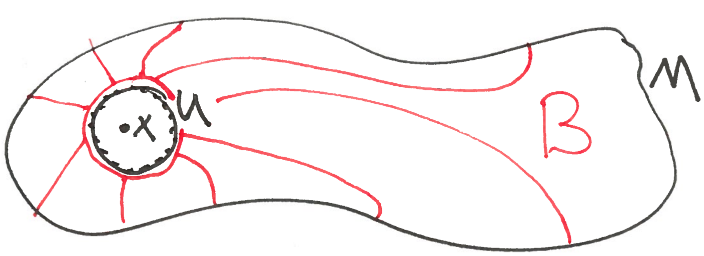
<u>Proof:</u> By excision:

$$H_n(M, M-x) \cong H_n(U, U-x) \cong \mathbb{Z}$$

where U is a small ball around x homeomorphic to $\mathbb{R}^n$. Here we are using the Excision thm

$$H_n(X-B, A-B) \cong H_n(X, A) \qquad (\bar{B} \subset A)$$

w/ $X = M$, $A = M-x$, $B = U^c$.



$\square$

Def/ A local orientation of an n-dim top mfld M at a point x in M is the choice of a generator of $H_n(M, M-x)$.

Rmks/ • For the same reason as for $\mathbb{R}^m$, a local orientation of M at x forces the choice of a generator of $H_n(M, M-y)$ at each y in a small ball around x.

• There are only two orientations at x.

• Following the next def, a connected mfld has only two orientations.

Def/ A global orientation of an n-dim top mfld M is the choice of a local orientation at each $x \in M$ s.t. these choices coincide for all y w/ the local orientation at y forced by the local orientation at any nearby x.

M is orientable if it admits a global orientation.

Prop/ Let M be a connected n-dim top mfld. If $\pi_1(M)$ does not have a subgroup of index 2, then M is orientable.

In particular, M is orientable if it is simply connected.

Proof/ Omitted. $\square$.

Thm/ Let M be a compact connected n-dim top mfld.

(a) If M is orientable, the projection-induced map $H_n(M) \longrightarrow H_n(M, M-x) \cong \mathbb{Z}$ is an isomorphism for all $x \in M$.

(b) If M is <u>not</u> orientable, the same map $H_n(M) \to H_n(M, M-x) \cong \mathbb{Z}$ is inj and is the zero map; that is, $H_n(M) = 0$.

(c) $H_k(M) = 0$ for $k > \dim M$.

Proof: Omitted. $\square$.

Thm/

Let M be a smooth manifold.

Then M is orientable as a manifold iff M is orientable as a top manifold.

Proof/ Omitted.

Thm/ (Poincaré Duality)

Recall the def of the k^{th} cohomology group $H^k(X;\mathbb{Z})$ of a topological space.

Let $X \equiv M$ be an orientable n-dim compact top mfld.

For each $k = 0, 1, \ldots, n$, $H^k(M;\mathbb{Z}) = H_{n-k}(M;\mathbb{Z})$.

Proof/ Omitted.

Reduced Homology

Def/ Let X be a topological space. The <u>augmented chain complex associated to X (with integer coefficients)</u> is the chain complex

$$\cdots \to C_k(X) \xrightarrow{\partial_k \equiv \tilde{\partial}_k} C_{k-1}(X) \xrightarrow{\partial_{k-1} \equiv \tilde{\partial}_{k-1}} \cdots \to C_1(X) \xrightarrow{\partial_1 \equiv \tilde{\partial}_1} C_0(X) \xrightarrow{\varepsilon \equiv \tilde{\partial}_0} \mathbb{Z} \to 0 \to \cdots$$

where ∂_k is the usual boundary map, and $\varepsilon : C_0(X) \to \mathbb{Z}$ is defined by

$$\varepsilon\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i .$$

Then it is easy to show $\varepsilon \circ \partial_1 = 0$, and so we define the <u>k^{th} relative homology group of X</u> to be $\tilde{H}_k(X;\mathbb{Z}) = \ker \tilde{\partial}_k / \operatorname{im} \tilde{\partial}_{k+1}$.

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Thm/ For $k>0$, $H_k(X,\mathbb{Z}) \cong \tilde{H}_k(X,\mathbb{Z})$.

For $k=0$, $H_0(X,\mathbb{Z}) \cong \mathbb{Z} \oplus \tilde{H}_k(X,\mathbb{Z})$.

Proof/ Omitted. $\square$.

example/ For a detailed example where you have to play around w/ long exact sequences and reduced homology, see problem 4 of the Spring 2011 Geometry exam.

Seifert - van Kampen Theorems (adapted from Bonahon)

Def/ Let A and B be groups. The <u>free product $A*B$ of A and B</u> is the group of all finite sequences $X_1 X_2 \cdots X_n$ s.t. (i) $X_i \in A$ or B, $X_i \neq 1_A$ or 1_B; and (ii) if $X_i \in A$ (resp. B) then $X_{i+1} \in B$ (resp. A). The group law is given by concatenation and simplification. The identity is given by the empty sequence.

Def/ The <u>free group F_n on n generators</u> is the group $\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ copies}}$ where $\mathbb{Z}$ is viewed multiplicatively, i.e. $\mathbb{Z} = \langle x \rangle$.

Def/ (Group Presentations)

Let F_n be the free group generated by $x_1, \ldots, x_n$.

Let $r_1, r_2, \ldots, r_m \in F_n$.

The <u>group presented by the generators x_i and relations r_i</u>, denoted $\langle x_1, \ldots, x_n \mid r_1 = \cdots = r_m = 1 \rangle$, is defined as the quotient F_n / K where $K = \langle g r_i g^{-1} : i = 1, \ldots, m,\ g \in F_n \rangle$ is the smallest normal subgroup of F_n containing the r_i. (K is also called the subgroup 'normally generated' by the r_i.)

Def/ A <u>finite presentation for a group G</u> is an isomorphism $G \cong F_n / K$.

(Culture)
Thm/ (A.A. Markov)

There exists no algorithm which given $r_1, \dots, r_m \in F_n$ determines whether the group $G = \langle x_1, \dots, x_n \mid r_1 = \dots = r_m = 1 \rangle$ is the trivial group or not.

Def/ (Amalgamated Products)

Let A, B, C be groups.
Let $i_A : C \to A$, $i_B : C \to B$ be group homomorphisms.
The <u>amalgamated product $A *_C B$ of A and B amalgamated along C via the group homomorphisms i_A and i_B</u> is the quotient $A * B / K$ where K is the subgroup normally generated by $\{ i_A(c) i_B(c)^{-1} \in A * B : c \in C \}$.

Rmks/ • The amalgamated product along i_A, i_B is the pushout in the category of groups. Specifically, first note that there are natural maps j_A, j_B defined by

$$\begin{tikzcd} A \arrow[r, hook] \arrow[rr, dashed, bend left, "j_A"] & A*B \arrow[r, two heads] & A*B/K \\ B \arrow[ru, hook] \arrow[rru, dashed, bend right, "j_B"'] & & \end{tikzcd}$$

Then for any group D and any group homomorphisms $A \to D$, $B \to D$ s.t. $\begin{tikzcd}[row sep=small] & A \arrow[dr] & \\ C \arrow[ur] \arrow[dr] & & D \\ & B \arrow[ur] & \end{tikzcd}$ is a commutative diagram there is a unique group homomorphism $A *_C B \xrightarrow{\varphi} D$ making the following diagram commute:

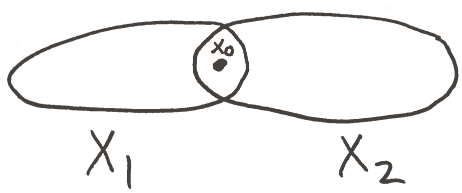
$$\begin{tikzcd} & A \arrow[dr, "j_A"] \arrow[drr, bend left] & & \\ C \arrow[ur, "i_A"] \arrow[dr, "i_B"'] & & A *_C B \arrow[r, dashed, "\exists ! \varphi"] & D \\ & B \arrow[ur, "j_B"'] \arrow[urr, bend right] & & \end{tikzcd}$$

- For all $c \in C$, $\overline{i_A(c)} = \overline{i_B(c)} \in A *_C B$.
- ★ In practice, if $A = \langle a_1, \dots, a_m \mid r_1 = \dots = r_k = 1 \rangle$ and $B = \langle b_1, \dots, b_n \mid s_1 = \dots = s_\ell = 1 \rangle$ and $C = \langle c_1, \dots, c_p \rangle$, then $A * B = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid r_1 = \dots = r_k = s_1 = \dots = s_\ell = 1 \rangle$ and $A *_C B = \langle a_1, \dots, a_m, b_1, \dots, b_n \mid r_1 = \dots = r_k = s_1 = \dots = s_\ell = i_A(c_1) i_B(c_1)^{-1} = \dots = i_A(c_p) i_B(c_p)^{-1} = 1 \rangle$.

Rmks/ • Always ask yourself what $i_A : C \to A$ and $i_B : C \to B$ are. Without this information, notations like $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$ are meaningless.

Thm/ (Seifert - van Kampen I)

Setup:



$i_1 : X_1 \cap X_2 \to X_1$ $\qquad$ $k_1 : X_1 \to X_1 \cup X_2$

$i_2 : X_1 \cap X_2 \to X_2$ $\qquad$ $k_2 : X_2 \to X_1 \cup X_2$

$x_0 \in X_1 \cap X_2$

induces $\rightsquigarrow$

$$\begin{tikzcd}
 & \pi_1(X_1; x_0) \arrow[dr] \arrow[drr, bend left, "k_{1*}"] & & \\
\pi_1(X_1 \cap X_2; x_0) \arrow[ur, "i_{1*}"] \arrow[dr, "i_{2*}"'] & & \pi_1(X_1;x_0) *_{\pi_1(X_1 \cap X_2, x_0)} \pi_1(X_2;x_0) \arrow[r, dashed, "\exists ! \varphi"] & \pi_1(X_1 \cup X_2; x_0) \\
 & \pi_1(X_2; x_0) \arrow[ur] \arrow[urr, bend right, "k_{2*}"'] & &
\end{tikzcd}$$

Here we are using that $k_1 \circ i_1 = k_2 \circ i_2 \Rightarrow k_{1*} \circ i_{1*} = k_{2*} \circ i_{2*}$, so that the outer diagram commutes and the UMP of the pushout can be used.

The result: If X_1, X_2 are open in $X_1 \cup X_2$, $x_0 \in X_1 \cap X_2$, and $X_1 \cap X_2$ is path connected, then the induced homomorphism φ gives a natural isomorphism

$$\pi_1(X_1; x_0) *_{\pi_1(X_1 \cap X_2; x_0)} \pi_1(X_2; x_0) \cong \pi_1(X_1 \cup X_2; x_0).$$

Thm/ (Seifert-van Kampen II)

setup: The same as for SvK I.

result: If X_1, X_2 are closed in $X_1 \cup X_2$, $x_0 \in X_1 \cap X_2$, X_1 and X_2 and $X_1 \cap X_2$ are path connected, and $X_1 \cap X_2$ is a neighborhood deformation retract in both X_1 and X_2, then the same conclusion as in SvK is true.

Covering Spaces (adapted from Bonahon)

Thm/ (path lifting property)

Let $p: \tilde{X} \to X$ be a covering space.

Let $\alpha: [0,1] \to X$ be a path.

Let $x_0 = \alpha(0)$ and let $\tilde{x}_0 \in p^{-1}(x_0)$.

Then there is a <u>unique</u> path $\tilde{\alpha}: [0,1] \to \tilde{X}$ s.t. $\tilde{\alpha}(0) = \tilde{x}_0$ and $\alpha = p \circ \tilde{\alpha}$.

Thm/ (homotopy lifting property)

Let $p: \tilde{X} \to X$ be a covering space.

Let α, β be two paths in X s.t. $\alpha(0) = \beta(0) = x_0$.

Let $\tilde{x}_0 \in p^{-1}(x_0)$. Let $\tilde{\alpha}, \tilde{\beta}$ the unique lifts of α, β w/ $\tilde{\alpha}(0) = \tilde{\beta}(0) = \tilde{x}_0$.

If $\alpha \sim \beta$ by a path homotopy $H: I \times I \to X$, then H lifts to a unique path homotopy $\tilde{H}: I \times I \to \tilde{X}$ between $\tilde{\alpha}$ and $\tilde{\beta}$ satisfying $p \circ \tilde{H} = H$.

Cor/ $\tilde\alpha$ and $\tilde\beta$ end at the same point in $\tilde X$.

Thm/ (lifting criterion)

Let $p: \tilde X \to X$ be a covering space.
Let $f: Y \to X$. Let Y be path connected and locally path connected.
Let $y_0 \in Y$, $x_0 = f(y_0) \in X$, $\tilde x_0 \in p^{-1}(x_0) \subset \tilde X$.
Then there exists a unique lift $\tilde f: Y \to \tilde X$ satisfying $p \circ \tilde f = f$ and $\tilde f(y_0) = \tilde x_0$ if and only if $f_*(\pi_1(Y;y_0)) \subset p_*(\pi_1(\tilde X;\tilde x_0)) \subset \pi_1(X;x_0)$.

Def/ A space X is <u>simply connected</u> if X is p.c., locally p.c., and $\pi_1(X;x_0) = 1$.

Cor/ If X is simply connected, then the lifting criterion is always met.

Def/ The covering spaces $p: \tilde X \to X$ and $p': \tilde X' \to X$ are isomorphic if there exists a homeomorphism $\varphi: \tilde X \to \tilde X'$ s.t.

$$\begin{tikzcd} \tilde X \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & \tilde X' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

commutes, i.e. $p' \circ \varphi = p$.

Thm/ Let $p:\tilde{X}\to X$ and $p':\tilde{X}'\to X$ be covering spaces w/ X p.c. and locally p.c. $\tilde{X},\tilde{X}'$ p.c.
Let $x_0\in X$, $\tilde{x}_0\in p^{-1}(x_0)\subset\tilde{X}$, and $\tilde{x}_0'\in(p')^{-1}(x_0)\subset\tilde{X}'$.
Then there exists an isomorphism $\varphi:\tilde{X}\to\tilde{X}'$ sending $\tilde{x}_0\mapsto\tilde{x}_0'$ if and only if $p_*(\pi_1(\tilde{X};\tilde{x}_0))=p'_*(\pi_1(\tilde{X}';\tilde{x}_0'))\subset\pi_1(X;x_0)$

Q: What if we remove the condition that $\varphi(\tilde{x}_0)=\tilde{x}_0'$?

A: Thm/ Let $p:\tilde{X}\to X$ and $p':\tilde{X}'\to X$ be covering spaces w/ X p.c. and locally p.c. $\tilde{X},\tilde{X}'$ p.c.
Let $x_0\in X$, $\tilde{x}_0\in p^{-1}(x_0)\subset\tilde{X}$, and $\tilde{x}_0'\in(p')^{-1}(x_0)\subset\tilde{X}'$.
Then there exists an isomorphism $\varphi:\tilde{X}\to\tilde{X}'$ if and only if $p'_*(\pi_1(\tilde{X}';\tilde{x}_0))$ is a conjugate of $p_*(\pi_1(\tilde{X};\tilde{x}_0))$ in $\pi_1(X;x_0)$.

Rmks/ • This theorem says there is an injection of the collection {isomorphism classes of path connected covering spaces rel. base point} $\xhookrightarrow{p_*}$ {subgroups of $\pi_1(X;x_0)$} when X is p.c. and locally p.c.

• Q: when is this a 1-1 correspondence?

A: Def/ X is <u>locally simply connected</u> if for every $x \in X$ and for each nbhd $U \ni x$ there exists a nbhd $V \ni x$ s.t. $V \subset U$, V p.c., locally p.c., $\pi_1(V;x) = 1$.

Def/ X is <u>semi-locally simply connected</u> if for every $x \in X$ and for each nbhd $U \ni X$ there exists a nbhd $V \ni x$ s.t. $V \subset U$, V p.c., locally p.c., and the homomorphism $i_* : \pi_1(V;x) \to \pi_1(X;x)$ induced by the inclusion $V \xrightarrow{i} X$ is the trivial homomorphism.

Rmks/ • So locally simply connected $\Rightarrow$ semi-locally simply connected

• So all topological manifolds are locally simply connected $\Rightarrow$ semi-locally simply connected.

Thm/ When X is path connected, locally path connected, and semi-locally simply connected, then the correspondence between path connected covering spaces rel. base point and subgroups of $\pi_1(X;x_0)$ is a 1-1 correspondence.

Cor/ (of the proof)

If X is p.c., locally p.c., and semi-locally simply connected, then there is a unique isomorphism class of path connected coverings $\tilde{X}$ (rel. base point or not) s.t. $\pi_1(\tilde{X}, \tilde{x}_0) = 1$.

Def/ Such an $\tilde{X}$ is called <u>the universal cover of X</u>. (Note: This whole time, because of the def of covering space, $\tilde{X}$ p.c. $\Longleftrightarrow$ $\tilde{X}$ p.c. & locally p.c. With this in mind, it is convenient to observe the universal cover $\tilde{X}$ is simply connected.)

Rmks/ • we put a partial ordering on p.c. coverings so that this 1-1 correspondence becomes <u>inclusion reversing</u>. Specifically, we say $(\tilde{X}, p) \geq (\tilde{X}', p')$ if p lifts w.r.t. p', i.e. there is $q: \tilde{X} \to \tilde{X}'$ s.t. $p' \circ q = p$ so that the following diagram commutes

$$\begin{tikzcd} \tilde{X} \arrow[rr, dashed, "q"] \arrow[dr, "p"'] & & \tilde{X}' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

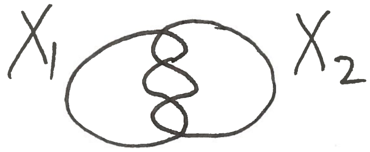
By the Lifting Criterion,

$\tilde{X} \geq \tilde{X}'$ iff $p_*(\pi_1(\tilde{X}; \tilde{x}_0)) \leq p'_*(\pi_1(\tilde{X}'; \tilde{x}_0'))$. In particular, the universal cover $\tilde{X}$ is the 'largest' cover.

Mayer – Vietoris Theorems

Thm (Mayer – Vietoris I)

Let X_1, X_2 open $\subset X_1 \cup X_2$.



$i_1 : X_1 \cap X_2 \to X_1$ $\quad j_1 : X_1 \to X_1 \cup X_2$

$i_2 : X_1 \cap X_2 \to X_2$ $\quad j_2 : X_2 \to X_1 \cup X_2$

Then there is a LES

$$\to H_{k+1}(X_1 \cup X_2) \xrightarrow{\delta_{k+1}} H_k(X_1 \cap X_2) \xrightarrow{(-H_k(i_1)) \oplus H_k(i_2)} H_k(X_1) \oplus H_k(X_2) \xrightarrow{H_k(j_1) + H_k(j_2)} H_k(X_1 \cup X_2) \xrightarrow{\delta_k} H_{k-1}(X_1 \cap X_2) \to$$

Thm (Mayer – Vietoris II)

Let X_1, X_2 closed $\subset X_1 \cup X_2$

X_1, X_2 a nbhd deformation retract in $X_1 \cup X_2$.

Then get same LES as in MV I.

Relative Homology
& Excision

RUSH-PRESBYTERIAN-ST. LUKE'S MEDICAL CENTER 1653 WEST CONGRESS PARKWAY, CHICAGO 60612 312/942-7083

DEPARTMENT OF FAMILY PRACTICE

RUSH UNIVERSITY

COLLEGII MED. RUSH. SIGILLUM CHICAGO MDCCCXXXVII

LINDA OLSON DOUGLAS, M.D.
DIRECTOR OF
PRECLERKSHIP EDUCATION

RUSH MEDICAL COLLEGE

$\boxed{\text{when } n=1}$

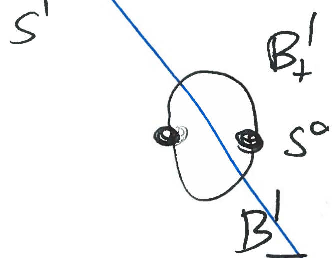
$H(i_+) \oplus (-H(i_-))$

1

$H_0(S^0) \to H_0(B^1_+) \oplus H_0(B^1_-)$

$R \oplus R \xrightarrow{f} R \oplus R$

$(x, y) \mapsto (x+y, -x-y)$



$i_+ : S^0 \hookrightarrow B^1_+$

$i_- : S^0 \hookrightarrow B^1_-$

arbitrary elt of $H_0(S^0)$ is class of elt in one component and class of elt in another component.

$[x + y] = [x]+[y]$

$\mapsto [x+y] = [x]+[y]$

$\ker f = \{(x,y) : x+y = 0\}$

$= \{(x,-x) : x \in R\} \cong R.$

	0	1	2	3	4
S^0	$R\oplus R$	0	0	0	0
S^1	R	R	0	0	0
S^2	R	0	R	0	0

$0 \to H_1(S^1) \to H_0(S^0) \xrightarrow{f} H_0(B^1_+) \oplus H_0(B^1_-)$

$H_1(S^1) \cong \ker f \cong R$

$\boxed{H_1(S^1) \cong R}$

conclusion

$H_1(S^n) \cong 0 \quad n>1$
$\cong R \quad n=1$

$H_k(S^n) \cong H_{k-1}(S^{n-1}) \cong \dots \cong H_1(S^{n-k+1})$

If $k<n$, $n-k+1>1 \Rightarrow H_k(S^n) = 0.$

If $k=n$, $H_k(S^n) = H_1(S^1) = R.$

If $k>n$, $H_k(S^n) \cong H_{k-n}(S^0) = 0.$

Relative Homology

Setup: X space, $A \subset X$ subspace. $i_A : A \to X$ inclusion.

$$S_n(X) = \{ \text{simplices } \sigma_n : \Delta_n \to X \}$$

we may view $S_n(A) \subset S_n(X)$ by $\sigma_n : \Delta_n \to A \to X$.

$$C_n(X) = \left\{ \text{chains } \sum_{i=1}^{m} a_i \sigma_i : a_i \in R,\ \sigma_i \in S_n(X) \right\}$$

so we may view $C_n(A) \subset C_n(X)$. Moreover, $C_n(A)$ is a submodule.

Consider the quotient $C_n(X,A) := C_n(X)/C_n(A)$.

The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ defined by

$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ f_i$ for $\sigma \in S_n(X)$ and extended linearly,

where $f_i : \Delta_{n-1} \to \Delta_n$ is the i^{th} face map, restricts to a

boundary map $C_n(A) \to C_{n-1}(A)$. Hence we get a map $\bar{\partial}_n : C_n(X,A) \to C_{n-1}(X,A)$.

$$\begin{tikzcd}
C_n(X) \arrow[r] \arrow[dr] & C_{n-1}(X) \arrow[r] & C_{n-1}(X)/C_{n-1}(A) \\
& C_n(X)/C_n(A) \arrow[ur, dashed] &
\end{tikzcd}$$

And $\bar{\partial}_{n-1} \circ \bar{\partial}_n = 0$.

$$\begin{tikzcd}
C_n(X) \arrow[r] \arrow[rr, bend left, "0"] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & C_{n-2}(X) \arrow[d] \\
C_n(X,A) \arrow[r] \arrow[rr, bend right, dashed, "0"'] & C_{n-1}(X,A) \arrow[r] & C_{n-2}(X,A)
\end{tikzcd}$$

Define the $\underline{n^{th} \text{ relative homology } R\text{-module of } X \text{ w.r.t. } A}$ 2

as $H_n(X,A) := \ker \bar{\partial}_n / \operatorname{im} \bar{\partial}_{n-1}$.

---

e.g. $A = \emptyset$. $C_n(\emptyset) = 0$. $C_n(X,\emptyset) = C_n(X)$. $H_n(X,\emptyset) = H_n(X)$.

Rmk/ we want a more geometric picture.

Let $Z_n(X,A) := \{ c_n \in C_n(X) : \partial_n c_n \in C_{n-1}(A) \}$

and let $B_n(X,A) := \left\{ c_n \in C_n(X) : \begin{array}{l} \exists c_{n+1} \in C_{n+1}(X) \text{ and} \\ \exists a_n \in C_n(A) \\ \text{s.t. } c_n = a_n + \partial_{n+1} c_{n+1} \end{array} \right\}$.

Clearly $B_n(X,A) \subset Z_n(X,A)$: $\partial_n(a_n + \partial_{n+1} c_{n+1}) = \partial_n(a_n) \in C_{n-1}(A)$. ✓.

$\underline{\text{Proposition}}$: $H_n(X,A) \cong Z_n(X,A) / B_n(X,A)$.

Proof/ Set $\bar{Z}_n(X,A) := \ker \bar{\partial}_n \subset C_n(X)/C_n(A)$.

Set $\bar{B}_n(X,A) := \operatorname{im} \bar{\partial}_{n+1} \subset C_{n+1}(X)/C_{n+1}(A)$.

Let π be the natural projection $C_n(X) \to C_n(X)/C_n(A)$.

Since $\bar{Z}_n(X,A) = \ker \bar{\partial}_n$, $\pi(Z_n(X,A)) \subset \bar{Z}_n(X,A)$:

Indeed, if $c_n \in C_n(X)$ is in $Z_n(X,A)$, then

$\bar{\partial}_n(\pi(c_n)) = \pi(\overbrace{\partial_n(c_n)}^{\in C_{n-1}(A)}) = 0 \in C_{n-1}(X)/C_{n-1}(A)$.

And $\pi(B_n(X,A)) \subset \bar{B}_n(X,A)$:

Indeed, if $c_n \in C_n(X)$ is in $B_n(X,A)$, w/ $c_n = \overbrace{\partial_{n+1} c_{n+1}}^{C_{n+1}(X) \ni} + \overbrace{a_n}^{C_n(A) \ni}$,

Then $\bar{\partial}_{n+1}(\pi(c_{n+1})) = \pi(\partial_{n+1}(c_{n+1})) = \pi(c_n - a_n) = \pi(c_n) \in C_n(X)/C_n(A)$.

There we have an induced map $\overline{\pi} : \dfrac{Z_n(X,A)}{B_n(X,A)} \to \dfrac{\overline{Z}_n(X,A)}{\overline{B}_n(X,A)}$.

$$\begin{tikzcd}
Z_n(X,A) \arrow[r, "\pi"] \arrow[dr] & \overline{Z}_n(X,A) \arrow[r] & \overline{Z}_n(X,A)/\overline{B}_n(X,A) \\
& Z_n(X,A)/B_n(X,A) \arrow[ur, dashed, "\overline{\pi}"] &
\end{tikzcd}$$

claim: $\pi^{-1}\,\overline{B}_n(X,A) = B_n(X,A)$, i.e. $\overline{\pi}$ is injective.

Let $c \in \pi^{-1}\,\overline{B}_n(X,A) \subset C_n(X)$.

so $\pi(c) \in \overline{B}_n(X,A)$. so $\exists\, c_{n+1} \in C_{n+1}(X)$ s.t.

$\pi(c) = \pi(\partial_{n+1}(c_{n+1})) \in C_n(X)/C_n(A)$.

so $c - \partial_{n+1}(c_{n+1}) = a_n \in C_n(A)$. Gives $\subseteq$.

Conversely, if $c = \partial_{n+1} c_{n+1} + a_n$,

then $\pi(c) = \pi(\partial_{n+1}(c_{n+1})) = \overline{\partial}_{n+1}\,\pi(c_{n+1})$.

so $\pi(c) \in \overline{B}_n(X,A)$. Gives $\supseteq$. ✓

claim: $\overline{\pi}$ is surjective. $\quad C_n(X)/C_n(A)$

$\cup$

we know $\pi(Z_n(X,A)) \subset \overline{Z}_n(X,A)$.

Is the converse true?

Let $\overline{\partial}_n(\pi(c)) = 0 \in C_{n-1}(X)/C_{n-1}(A)$. So $\pi(\partial_n(c)) = 0$.

So $\partial_n(c) \in C_{n-1}(A)$. So $c \in Z_n(X,A)$. ✓ So have $\supseteq$.

This finishes the proof of the Proposition. $\square$

Thm/ X p.c. Then $H_0(X,A) = \begin{cases} 0 & \text{if } A \neq \emptyset \\ R & \text{if } A = \emptyset \end{cases}$.

Proof/ we know the result is true when $A = \emptyset$.

Let $A \neq \emptyset$. we need to show $Z_0(X,A) \subset B_0(X,A)$.

In fact, we show $B_0(X,A) = C_0(X)$.

Fix $x_0 \in A$. Let $c \in C_0(X)$. $c = \sum_{i=1}^{k} a_i x_i$.

For each i, choose $\sigma_i : \Delta_1 \to X$ s.t. $\sigma_i(1) = x_i$ and $\sigma_i(0) = x_0$.
(since X p.c.)

Then $\partial \sigma_i = x_i - x_0$.

$$\text{so} \quad c = \underbrace{\sum_{i=1}^{k} a_i \partial \sigma_i}_{= \partial\left(\sum_{i=1}^{k} a_i \sigma_i\right) \in \partial_1 C_1(X)} - \underbrace{x_0 \sum_i a_i}_{\in C_0(A)} . \quad \text{so } c \in B_0(X,A).$$

$\square$

Thm/ If $\{X_i\}$ are the components of X,
then $H_n(X,A) = \bigoplus_i H_n(X_i, A_i)$ w/ $A_i \equiv X_i \cap A$.

Rmk/ we have a SES of chain complexes

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_n(A) \arrow[r, "i_n"] \arrow[d, "\partial_n"] & C_n(X) \arrow[r, "\pi_n"] \arrow[d, "\partial_n"] & C_n(X,A) \arrow[r] \arrow[d, "\bar{\partial}_n"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r] \arrow[d] & C_{n-1}(X) \arrow[r] \arrow[d] & C_{n-1}(X,A) \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

which we may write $0 \to C(A) \xrightarrow{i} C(X) \xrightarrow{\pi} C(X,A) \to 0.$

Therefore, we get a LES in homology

$$\to H_{n+1}(X,A) \to$$

$$\to H_n(A) \xrightarrow{H_n(i)} H_n(X) \xrightarrow{H_n(\pi)} H_n(X,A) \xrightarrow{\delta_n}$$

$$\to H_{n-1}(A) \to$$

we can understand δ_n concretely:

Let $[c_n] \in H_n(X,A) = Z_n(X,A) / B_n(X,A).$

so $c_n \in C_n(X)$, $\partial c_n \in C_{n-1}(A).$

$[[c_n]] \in C_n(X,A) = C_n(X)/C_n(A).$

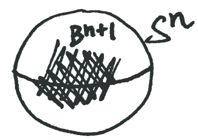
where in $C_{n-1}(A)$ is the natural place for c_n to go? $\partial_n c_n \in \ker \partial_{n-1}$, of course.

$$\boxed{\delta_n([c_n]) = [\partial c_n] \in H_{n-1}(A)}$$

Why is $[\partial c_n] \underset{\text{necessarily}}{\neq} 0$ in $H_{n-1}(A)$?

Why is $[\partial c_n] = 0$ in $H_{n-1}(X)$?
Because $c_n \in C_n(X)$.
But c_n need not be in $C_n(A)$.

e.g. Let $A = S^n \subset \mathbb{R}^{n+1}$



and let $X = B^{n+1} \subset \mathbb{R}^{n+1}$

we get the LES in relative homology

$$\longrightarrow H_{p+1}(B^{n+1}, S^n) \to H_p(S^n) \longrightarrow H_p(B^{n+1}) \longrightarrow H_p(B^{n+1}, S^n) \xrightarrow{\delta_p} H_{p-1}(S^n) \longrightarrow$$

<u>If $p > 1$</u> : Then $\underset{0}{H_p(B^{n+1})} \to H_p(B^{n+1}, S^n) \to H_{p-1}(S^n) \to \underset{0}{H_{p-1}(B^{n+1})}$

$$\boxed{H_p(B^{n+1}, S^n) \cong H_{p-1}(S^n) \quad (n \geq 0,\ p > 1)}$$

<u>If $p = 1$; if $n > 0$</u> :

$$\underset{0}{H_1(B^{n+1})} \longrightarrow H_1(B^{n+1}, S^n) \xrightarrow{\delta_1} \underset{R}{H_0(S^n)} \xrightarrow{H_0(i)} \underset{R}{H_0(B^{n+1})} \longrightarrow \underset{0}{H_0(B^{n+1}, S^n)}$$

connected $\downarrow$ $\neq \emptyset \downarrow$

$H_1(B^{n+1}, S^n) \cong \operatorname{im} \delta_1 \cong \ker H_0(i) : R \to R$ $\therefore$ $\boxed{H_1(B^{n+1}, S^n) \cong 0 \quad (n > 0, p = 1)}$

$H_0(i)\left(\left[\sum_k a_k x_k\right]\right) = \left[\sum_k a_k x_k\right].$

So the map $R \to R$ is the identity map, in particular, injective

If $p=1$; if $n=0$:

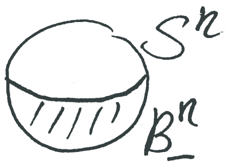
$$\underset{0}{H_1(B^{n+1})} \to H_1(B^{n+1}, S^n) \xrightarrow{\delta_1} \underset{R\oplus R}{H_0(S^n)} \xrightarrow{H_0(i)} \underset{R}{H_0(B^{n+1})} \to \underset{0}{H_0(\overset{\text{p.c.}}{B^{n+1}}, \overset{\neq\emptyset}{S^n})}$$

$$H_1(B^{n+1}, S^n) \cong \operatorname{im}\delta_1 \cong \ker H_0(i) : R\oplus R \to R$$
$$(x,y) \mapsto x+y$$

$$\ker H_0(i) = \{(x,-x)\} \subset R\oplus R \cong R.$$

$$\boxed{H_1(B^1, S^0) \cong R \quad (n=0,\ p=1)}$$

Next: $\quad B^n_- = \{x \in S^n : x_{n+1} \leq 0\}.$



$\cong B^n$ by projection along x_{n+1} axis.

Let $X = S^n$ and let $A = B^n_-$.

~~If $p>1$~~: Get LES in relative homology.

$$\to H_{p+1}(S^n, B^n_-) \to$$
$$\to H_p(B^n_-) \to H_p(S^n) \to H_p(S^n, B^n_-) \xrightarrow{\delta_p}$$
$$\to H_{p-1}(B^n_-) \to$$

If $p > 1$:

$$\underset{0}{H_p(B^n_-)} \longrightarrow H_p(S^n) \longrightarrow H_p(S^n, B^n_-) \longrightarrow \underset{0}{H_{p-1}(B^n_-)}$$

$$\boxed{H_p(S^n) \cong H_p(S^n, B^n_-) \quad (n \geq 0,\ p > 1)}$$

If $p = 1$, $n \geq 0$:

$$\underset{0}{H_1(B^n_-)} \longrightarrow H_1(S^n) \longrightarrow H_1(S^n, B^n_-) \xrightarrow{\delta_1} \underset{R}{H_0(B^n_-)}$$

$$H_0(B^n_-) \xrightarrow{H_0(i)} \underset{R \text{ or } R \oplus R}{H_0(S^n)}$$

either $\quad R \longrightarrow R$

$\qquad x \longmapsto x$

or $\quad R \longrightarrow R \oplus R$

$\qquad x \longmapsto (x, 0)$

$\therefore H_0(i)$ injective

$$H_1(S^n) \cong \ker \delta_1, \qquad \operatorname{im} \delta_1 \cong \ker H_0(i) = 0 \Rightarrow \delta_1 = 0.$$

$$\Rightarrow \boxed{H_1(S^n) \cong H_1(S^n, B^n_-) \quad (p = 1,\ n \geq 0)}$$

Def/ Let $A \subset X$ and $B \subset Y$.

By a map $f : (X, A) \to (Y, B)$ it is meant a continuous map $f : X \to Y$ s.t. $f(A) \subset B$.

Rmk/ Let $f : (X, A) \to (Y, B)$.

$f : X \to Y \rightsquigarrow C_n(f) : C_n(X) \to C_n(Y)$

defined by extending $\sigma : \Delta_n \to X \rightsquigarrow \tilde{\sigma} : \Delta_n \to X \xrightarrow{f} Y$.

Since $f(A) \subset B$, $C_n(f)(C_n(A)) \subset C_n(B)$.

So we get an induced map

$$\overline{C_n(f)} : C_n(X, A) \longrightarrow C_n(Y, B)$$

which commutes w/ boundaries, inducing a chain map and thus a map at the level of homology:

$$H_n(f) : H_n(X, A) \longrightarrow H_n(Y, B).$$

Thm/ (Excision Theorem)

Def/ Let $B \subset A \subset X$. Then the map

$$(X-B, A-B) \xrightarrow{i} (X, A)$$

induces a map on homology

$$H_n(i) : H_n(X-B, A-B) \longrightarrow H_n(X, A).$$

We say the inclusion i is an <u>excision</u>, or that <u>B can be excised from (X,A)</u>, if $H_n(i)$ gives an iso $H_n(X-B, A-B) \cong H_n(X,A)$ for all n.

The theorem is that if $\overline{B} \subset \operatorname{int} A$ (i.e. "B sits deeply inside A"), then B can be excised from (X, A).

Key Lemma:

Let $\mathcal{U}$ be an open covering of X.

Let $S_n^{\mathcal{U}}(X) = \left\{ \begin{array}{l} \text{simplices } \sigma : \Delta_n \to X \text{ such that} \\ \sigma \text{ maps } \Delta_n \text{ into one of the } U \in \mathcal{U} \end{array} \right\}$.

Let $C_n^{\mathcal{U}}(X) =$ free R-module generated by $S_n^{\mathcal{U}}(X)$.

The boundary ∂ maps $C_n^{\mathcal{U}}(X) \to C_{n-1}^{\mathcal{U}}(X)$

so we get homology groups $H_n^{\mathcal{U}}(X)$,

as well as an inclusion $H_n^{\mathcal{U}}(X) \hookrightarrow H_n(X)$.

[The theorem is that this inclusion gives an iso $H_n^{\mathcal{U}}(X) \cong H_n(X)$]

---

e.g. Return to the computation of $H_k(S^n)$.

Summary

(1) 

$$\boxed{\begin{array}{c} H_p(B^{n+1}, S^n) \cong H_{p-1}(S^n) \\ (p > 1,\ n \geq 0) \\ H_1(B^{n+1}, S^n) \cong \begin{cases} 0, & n > 0 \\ R, & n = 0 \end{cases} \end{array}}$$

(2) 

$$\boxed{\begin{array}{c} H_p(S^n) \cong H_p(S^n, B_-^n) \\ (p \geq 1,\ n \geq 0) \end{array}}$$

Lemma: $\operatorname{int}(B^n_-)$ can be excised from (S^n, B^n_-),

$H_k(B^n_+, S^{n-1}) \cong$

i.e. the map on homology $H_k(S^n - \operatorname{int}(B^n_-), B^n_- - \operatorname{int}(B^n_-)) \to H_k(S^n, B^n_-)$ induced by the inclusion $(S^n - \operatorname{int}(B^n_-), B^n_- - \operatorname{int}(B^n_-)) \to (S^n, B^n_-)$.

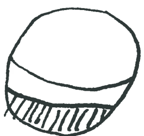
Say for $n \geq 1$.

Note: we can't immediately use the Excision Theorem because $B^n_- = \overline{\operatorname{int}(B^n_-)} \not\subset \operatorname{int}(B^n_-)$.

Proof: Let $B \subset \operatorname{int}(B^n_-)$ be defined by $B = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \dots + x_n^2 = 1$ and $x_n \leq -\frac{1}{2}\}$. Then $\overline{B} \subset \operatorname{int}(B^n_-)$.

By Excision, $H_k(S^n - B, B^n_- - B) \cong H_k(S^n, B^n_-)$.

We also have a theorem that homology is preserved under deformation retract.


B

Since $S^n - B$ deformation retracts onto B^n_+ and B^n_- deformation retracts to S^{n-1} under this larger deformation retraction, we gather

$$\begin{array}{c} H_k(B^n_+, S^{n-1}) \cong H_k(S^n, B^n_-) \\ \text{|||} \\ H_k(S^n - \operatorname{int}(B^n_-), B^n_- - \operatorname{int}(B^n_-)). \end{array}$$

$\square$

If $k \geq 1, n \geq 1$: $H_k(S^n) \cong H_k(S^n, B^n_-) \cong H_k(B^n_+, S^{n-1}) \cong \begin{cases} H_{k-1}(S^{n-1}) & (k > 1) \\ 0 & (k = 1) \end{cases}$

Let $k > n \geq 1$:

$H_k(S^n) \cong H_{k-1}(S^{n-1}) \cong \dots \cong H_{k-n}(S^0) \cong 0.$

Let $1 \leq k < n$:

$H_k(S^n) \cong H_{k-1}(S^{n-1}) \cong \dots \cong H_1(S^{n-k+1}) \cong H_1(S^{n-k+1}, B^{n-k+1}_-) \overset{n-k+1 \geq 1}{\cong} H_1(B^{n-k+1}_+, S^{n-k}) \overset{n-k>0}{\cong} 0.$

Let $1 \leq k = n$

$$H_n(S^n) \cong H_{n-1}(S^{n-1}) \cong \cdots \cong H_1(S^1) \cong H_1(S^1, B^1_-) \cong H_1(B^1_+, S^0) \cong R.$$

If $k \geqslant 1$, $n = 0$

$H_k(S^0) = 0.$

If $k = 0$, $n \geqslant 1$

$H_0(S^n) = R.$

If $k = 0$, $n = 0$

$H_0(S^0) = R \oplus R.$

Interior Products & The Lie Derivative, & Other Results of Differential Geometry.

Def/ Let X be a (smooth) vector field on a manifold M. The <u>interior product</u> associated w/ X is the map $i_X : \Omega^p(M) \longrightarrow \Omega^{p-1}(M)$ defined by the property that

$$i_X(\omega)(X_1, X_2, \dots, X_{p-1}) = \omega(X, X_1, X_2, \dots, X_{p-1}).$$

<u>Proposition</u> (i_X is an antiderivation of deg -1)

If α is a p-form and β is a q-form, then

$$i_X(\alpha \wedge \beta) = i_X(\alpha) \wedge \beta + (-1)^p \alpha \wedge i_X(\beta).$$

<u>Recall</u>: Let M be a manifold, and let X be a vector field on M. Fix a point $p \in M$. There exists $\varepsilon > 0$ and a neighborhood U of p together w/ a family $\theta_t : U \longrightarrow M$ of smooth maps which are diffeomorphisms onto their images, where $t \in (-\varepsilon, \varepsilon)$,

satisfying θ_0 = the inclusion $\mathcal{U} \hookrightarrow M$,

$$\frac{d}{dt}\theta_t(p) = X_{\theta_t(p)},$$ and there exists

$W \subset \mathcal{U}$ s.t. $\theta_t(W) \subset \mathcal{U}$ and $\theta_{-t}\big|_{\theta_t(W)} = \left(\theta_t\big|_W\right)^{-1}$.

This construction is called the <u>flow of X near p</u>.

Def/ Let M be a mfld. Let X, Y be v.f.'s on M. Let $p \in M$. The <u>Lie derivative of Y w.r.t. X at the point p</u> is defined to be

$$(\mathcal{L}_X Y)_p := \lim_{t \to 0} \frac{T_{\theta_t(p)}(\theta_{-t})(Y_{\theta_t(p)}) - Y_p}{t} = \frac{d}{dt}\Big|_{t=0} T_{\theta_t(p)}(\theta_{-t})(Y_{\theta_t(p)})$$

which is an elt of T_pM when it exists. Here θ is the flow of X near p.

Lemma/ If X and Y are smooth vector fields on a smooth mfld M, then for every $p \in M$ the Lie derivative $(\mathcal{L}_X Y)_p$ exists and the resulting v.f. is a smooth.

Theorem/ Let M be a mfld, and let X and Y be v.f.'s on M.
Then $\mathcal{L}_X Y = [X, Y]$.

Theorem/ Let M be a mfld, and let X and Y be v.f.'s on M, w/ corresponding flows θ and ψ.

By shrinking ε and $\mathcal{U}$ we may assume $\theta_s \circ \psi_t(q)$ and $\psi_s \circ \theta_t(q)$ make sense for all $s, t \in (-\varepsilon, \varepsilon)$ and $q \in \mathcal{U}$.

If these two values are equal for all parameters, including p, we say the flows θ and ψ <u>commute</u>.

The theorem is that

$[X, Y] = 0$ if and only if θ and ψ commute.

Def & Theorem/ There is a notion of the Lie derivative of a differential form ω w.r.t. a vector field X on a mfld M. Known as Élie Cartan's "magic" formula, this can be calculated as

$$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X \omega)$$

Gaussian Curvature

There is a def in problem 3 of Fall 2010.

Thm/ (Gauss-Bonnet)

If M is a compact oriented embedded surface w/out boundary in $\mathbb{R}^3$,

then $$\int_M K = 2\pi \chi(M)$$

where K is the Gaussian curvature and $\chi(M)$ is the Euler characteristic.

Thm/ A compact <u>embedded</u> surface w/out boundary in $\mathbb{R}^3$ must have a point of positive curvature. (Not true if 'embedded' replaced by 'immersed'.)

Orientations (adapted from Bonahon)

Def/ Let M be an m-dim mfld w/ atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$. The atlas $\mathcal{A}$ is <u>oriented</u> if for all $i, j \in I$ and for each $y \in \varphi_i(U_i \cap U_j)$ we have $\det T_y(\varphi_j \circ \varphi_i^{-1}) > 0$.

Def/ Two oriented atlases $\mathcal{A}, \mathcal{A}'$ are <u>compatible</u> <u>as oriented atlases</u> if $\mathcal{A} \cup \mathcal{A}'$ is oriented (in particular, $\mathcal{A}$ and $\mathcal{A}'$ are compatible as atlases).

Rmk/ • Compatibility as oriented atlases is an equivalence relation among oriented atlases.

Def/ An <u>orientation</u> for the mfld M is a choice of equivalence class of oriented atlases.

Lemma/ Every oriented atlas is contained in a unique maximal oriented atlas.

Rmk/
- Here, 'maximal' is w.r.t. the obvious partial ordering by inclusion on the collection of <u>oriented atlases</u>.
- An orientation for M is the same as a choice of maximal oriented atlas.
- In practice, an oriented atlas for M defines a unique orientation.

Def/ Let M be a mfld of dim m.

A <u>volume form</u> on M is a form $\omega \in \Omega^m(M)$ of top-degree m s.t. $\forall x \in M$,

$$\omega_x \neq 0 \in \underbrace{\mathrm{Alt}^m(T_x M)}_{\cong \mathbb{R}}.$$

Thm/ M admits an orientation if and only if M admits a volume form.

Def/ Two volume forms ω, ω' are equivalent, denoted $\omega \sim \omega'$, if there exists a real-valued function $f: M \to \mathbb{R}$ s.t.

$$\begin{cases} \omega' = f\omega \\ f > 0 \end{cases}.$$

Cor/ There is a bijection

$$\left\{ \begin{array}{c} \text{orientations} \\ \text{on } M \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{volume} \\ \text{forms} \end{array} \right\} / \sim .$$

Cor/ If M is connected, it admits either 0 or 2 orientations.

Def/ Let X and Y be two vector fields on an m-dim mfld M. The <u>Lie bracket</u> $[X, Y]$ of

X and Y is a vector field on M

defined by $[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f))$.

Here we are viewing $Y(f)$ and $X(f)$ as functions $M \to \mathbb{R}$ and f is also a function $M \to \mathbb{R}$.

Prop/ $[fX, gY] = f(Xg)Y - g(Yf)X + fg[X,Y]$.

Fact/ If X, Y are vector fields on a mfld (of dim 2). Then for fixed $p \in M$, there is a local coordinate system around p s.t. $X = \partial_x$ and $Y = \partial_y$ IF AND ONLY IF

$$[X, Y] = 0.$$

The obvious generalization to higher dimensions is also true.
