Math 540, Topology

Instructor: Francis Bonahon

Fall 2016

Course outline

This is an outline of what we have covered in the semester so far, with main definitions and statements. No proofs!

Version as of: *December 1, 2016*

CHAPTER 1

A crash course in pointset topology

This material will not be covered in class, and is only intended as review.

1. Topological spaces

DEFINITION. A *topological space* consists of a set X and of a family $\mathcal{T}$ of subsets of X such that:

1. the empty set $\varnothing$ and the whole space X are both elements of $\mathcal{T}$;
2. if $\{U_i\}_{i\in I}$ is a (possibly infinite) family of elements of $\mathcal{T}$, their union $\bigcup_{i\in I} U_i$ is also in $\mathcal{T}$.
3. if $\{U_i\}_{i=1,2,\dots,n}$ is a *finite* family of elements of $\mathcal{T}$, their intersection $\bigcap_{i=1,2,\dots,n} U_i$ is also in $\mathcal{T}$.

The family $\mathcal{T}$ is then called a *topology* on the set X. The elements of $\mathcal{T}$ are called *open subsets* of X for this topology.

Recall that, given a family $\{X_i\}_{i\in I}$ of sets X_i indexed by a set I, their union $\bigcup_{i\in I}$ is the set of all x for which there exists an $i \in I$ with $x \in X_i$. Their intersection $\bigcap_{i\in I}$ is the set of all x such that $x \in X_i$ for every $i \in I$. Note that I does not need to be finite.

FUNDAMENTAL EXAMPLE. Let (X, d) be a *metric space*, namely a set X endowed with a function $d\colon X \times X \to \mathbb{R}$ such that

1. $d(x, y) \geqslant 0$ for every x, y;
2. $d(x, y) = 0$ if and only if $x = y$;
3. $d(y, x) = d(x, y)$ for every x, y;
4. $d(x, z) \leqslant d(x, y) + d(y, z)$ for every x, y, z.

Given x in such a metric space X and given $\varepsilon > 0$, let the open ball of radius ε centered at x be $B(x, \varepsilon) = \{y \in X; d(x, y) < \varepsilon\}$.

Define $\mathcal{T}$ by the property that $U \subset X$ is an element of $\mathcal{T}$ if and only if, for every $x \in U$, there exists a ball $B(x, \varepsilon)$ which is contained in U. Then, $\mathcal{T}$ is a topology for X.

DEFINITION. If x is an element of a topological space X, a *neighborhood* of x is a subset $W \subset X$ for which there exists an open subset U of X with $x \in U \subset W$.

Note that the terminology is not standard. This one is used in most research articles. Many textbook require a neighborhood to be open, namely just an open subset containing x.

DEFINITION. A topological space X is *Hausdorff* if, for every x, $y \in X$ with $x \neq y$, there exists a neighborhood U of x and a neighborhood V of y such that $U \cap V = \varnothing$.

For instance, the topology of a metric space is always Hausdorff.

CONVENTION. All topological spaces encountered in the course will be implicitly assumed to be Hausdorff, except for those that are constructed explicitly (and for which we will need to prove the Hausdorff property).

2. Continuous functions

DEFINITION. Let $f\colon X \to Y$ be a map between two topological spaces. The map f is *continuous* if, for every open subset U of Y, its pre-image $f^{-1}(U)$ is an open subset of X.

Recall that the pre-image of U is $f^{-1}(U) = \{x \in X; f(x) \in U\}$.

For metric spaces, this definition of continuity turns out to be equivalent to a more familiar one:

PROPOSITION. ($\varepsilon - \delta$ definition of continuity) Let $f : X \to Y$ be a map between metric spaces (X, d_X) and (Y, d_Y). Then f is continuous in the above sense if and only if, for every $x \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for every $x' \in X$ with $d_X(x, x') < \delta$.

DEFINITION. A *homeomorphism* between two topological spaces X and Y is a bijection $f\colon X \to Y$ such that f and f^{-1} are both continuous.

If $f\colon X \to Y$ is a homeomorphism and U is a subset of X, it follows from definitions that U is open in X if and only if $f(U)$ is open in Y. Therefore, f establishes a 'dictionary', which enables us to translate any topological property of X to a similar property of Y. A homeomorphism is therefore an 'isomorphism of topological spaces'.

3. The subspace topology

Let X be a topological space, with topology $\mathcal{T}$. If X' is a subset of X, it is immediate that

$$\mathcal{T}' = \{U' \subset X'; \exists U \in \mathcal{T}, U' = X' \cap U\}$$

is a topology for X'.

DEFINITION. The topology $\mathcal{T}'$ is the *subspace topology* induced on X' by the topology of X.

4. The quotient topology

Let $p\colon X \to Y$ be a surjective map from a topological space X to a set Y. This situation typically arises when (and exactly when) Y is the quotient of X under an equivalence relation $\sim$, namely when Y is the set of equivalence classes of $\sim$, and when p is the natural projection.

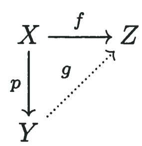
The topology of X induces a topology $\mathcal{T}$ on Y, defined by

$$\mathcal{T} = \{U \subset Y; p^{-1}(U) \text{ open in } X\}.$$

DEFINITION. The topology $\mathcal{T}$ is the *quotient topology* induced on Y by the topology of X.

The quotient topology is specially designed for the following property.

PROPOSITION. Let $f\colon X \to Z$ be a map such that $p(x) = p(x') \Rightarrow f(x) = f(x')$. Then there is a unique continuous map $g\colon Y \to Z$ such that $f = g \circ p$.



CHAPTER 2

The fundamental group

Let X be a topological space.

Throughout the course, we will assume that all topological spaces considered have the *Hausdorff Property.* This means that, for any two distinct points $x, y \in X$, there exists U and V open such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

Similarly, every map will be assumed to be continuous, unless you really have to prove that it is continuous.

Advice for the course: Draw plenty of pictures. They help in understanding what we are talking about.

1. Paths and homotopies

DEFINITION. A *path* in X is a continuous map $\alpha\colon [0,1] \to X$, where $[0,1]$ is the interval in $\mathbb{R}$.

STUPID EXAMPLE. The *constant path* at $x \in X$ is the map $c_x\colon [0,1] \to X$ defined by $c_x(s) = x$ for every $s \in [0,1]$.

This example will become not so stupid after all, and play an important role in the fundamental group of X.

A LESS TRIVIAL EXAMPLE. The path in $X = \mathbb{R}$ defined by $\alpha(s) = s - s^2$.

DEFINITION. Let α and β be two paths in X such that the final point $\alpha(1)$ is equal to the initial point $\beta(0)$. The *composition* of α and β is the path $\alpha * \beta\colon [0,1] \to X$ defined by

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) \text{ if } 0 \leqslant s \leqslant \frac{1}{2} \\ \beta(2s-1) \text{ if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}$$

Namely, $\alpha * \beta$ is defined by chaining α and β together, but speeding them up so that each of them is covered in half the time.

To prove that $\alpha * \beta$ is continuous, it is convenient to use the

PASTING LEMMA. Let X and Y be two topological spaces. Suppose that Y is the union of two *closed* subsets C_1 and C_2. Let $f_1\colon C_1 \to X$ and $f_2\colon C_2 \to X$ be two continuous maps which coincide on $C_1 \cap C_2$, namely such that $f_1(x) = f_2(x)$ for every $x \in C_1 \cap C_2$. Then, there exists a unique continuous map $f\colon Y \to X$ such that $f_{|C_1} = f_1$ and $f_{|C_2} = f_2$.

Question: Do we really need C_1 and C_2 to be closed?

DEFINITION. A *path homotopy* between the two paths α_0 and α_1 is a continuous map H: $[0,1] \times [0,1] \to X$ such that

$$\begin{cases} H(s,0) = \alpha_0(s), \ \forall s \in [0,1] \\ H(s,1) = \alpha_1(s), \ \forall s \in [0,1] \\ H(t,0) = \alpha_0(0) = \alpha_1(0), \ \forall t \in [0,1] \\ H(t,1) = \alpha_0(1) = \alpha_1(1), \ \forall t \in [0,1] \end{cases}$$

We then say that α_0 and α_1 are *path homotopic*, and we write $\alpha_0 \simeq \alpha_1$.

This definition is better understood if you write $H(s,t) = \alpha_t(s)$. We then have a family of paths α_t: $[0,1] \to X$, depending continuously (whatever this means) on a "time" parameter $t \in [0,1]$, all with the same starting point $\alpha_0(0) = \alpha_1(0)$ and the same end point $\alpha_0(1) = \alpha_1(1)$, and moving from α_0 when $t = 0$ to α_1 when $t = 1$.

REMARK. If two paths are path homotopic, they necessarily have the same end points.

EXAMPLE. Any two paths α and β in $\mathbb{R}^n$ with the same initial point $\alpha(0) = \beta(0)$ and the same final point $\alpha(1) = \beta(1)$ are path homotopic by the path homotopy $H(s,t) = (1-t)\alpha(s) + t\beta(s)$.

SORT OF EXAMPLE. In $\mathbb{R}^2 - \{(0,0)\}$, it appears that a path going from $(-1,0)$ to $(1,0)$ and passing "above" the hole $(0,0)$ is not path homotopic to a path joining the same points but passing "below" the hole. We will prove later on that this is indeed the case.

Question: Why can't we use the path homotopy $H(s,t) = (1-t)\alpha(s) + t\beta(s)$ in $\mathbb{R}^2 - \{(0,0)\}$? Where do things go wrong?

LEMMA. On the set of all paths in X, the relation $\simeq$ (= "is path homotopic to") is an equivalence relation.

We will usually denote by $[\alpha]$ the equivalence class of the path α.

2. The fundamental group

Pick a point $x_0 \in X$, called a base point. We will restrict attention to paths going from x_0 to x_0.

DEFINITION. The *fundamental group of X based at x_0* is the set of equivalence classes

$$\pi_1(X; x_0) = \{\alpha\colon [0,1] \to X \text{ path}; \alpha(0) = \alpha(1) = x_0\}/\simeq .$$

PROPOSITION. There is a unique group law $*$ on $\pi_1(X; x_0)$ defined by the property that $[\alpha] * [\beta] = [\alpha * \beta]$.

Note that we need the property that $\alpha(1) = \beta(0) = x_0$ for the composition $\alpha * \beta$ to be defined.

Here are the key steps of the proof of the Proposition. First we have to show that the law is well defined.

LEMMA. If $\alpha \simeq \alpha'$, $\beta \simeq \beta'$ and $\alpha(1) = \beta(0)$, then $\alpha * \beta \simeq \alpha' * \beta'$.

Then we have to show that the law is transitive.

LEMMA. If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$.

Note that in general the paths $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$ are not equal, they are just path homotopic.

We then have to find an identity element. For this, given $x \in X$, let $c_x\colon [0,1] \to X$ denote the *constant path* defined by $c_x(s) = x$ for every $s \in [0,1]$.

LEMMA. $c_{\alpha(0)} * \alpha \simeq \alpha \simeq \alpha * c_{\alpha(1)}$ for every path α.

This proves that $[c_{x_0}]$ is an identity element for $*$ in $\pi_1(X; x_0)$. We will also write $[c_{x_0}] = 1$ to express this fact.

Finally, we need an inverse. The *opposite* of a path α is the path obtained by traveling backwards, namely defined by $\bar{\alpha}(s) = \alpha(1-s)$.

LEMMA. $\alpha * \bar{\alpha} \simeq c_{\alpha(0)}$ and $\bar{\alpha} * \alpha \simeq c_{\alpha(1)}$ for every path α.

Therefore, $[\alpha]^{-1} = [\bar{\alpha}]$ in $\pi_1(X; x_0)$.

This proves that $*$ is a group law for $\pi_1(X; x_0)$.

STUPID EXAMPLE. If $X = \{x_0\}$, then $\pi_1(X; x_0)$ is the trivial group, usually denoted by 1.

3. Dependence on base points

PROPOSITION. If there is a path γ going from x_0 to y_0, the map $[\alpha] \mapsto [\gamma * \alpha * \bar{\gamma}]$ defines a group isomorphism $T_\gamma\colon \pi_1(X; y_0) \to \pi_1(X; x_0)$.

Note that $[\gamma * (\alpha * \bar{\gamma})] = [(\gamma * \alpha) * \bar{\gamma}]$, so that we do not have to worry about where we put the parentheses.

COROLLARY. If x_0 and y_0 are in the same path component of X, the fundamental groups $\pi_1(X; x_0)$ and $\pi_1(X; y_0)$ are isomorphic.

REMARK. If X_0 is the path connected component of X that contains x_0, then $\pi_1(X; x_0) = \pi_1(X_0; x_0)$.

4. Group homomorphisms induced by maps

DEFINITION. The *group homomorphism induced by* the map $f\colon X \to Y$ is the map $f_*\colon \pi_1(X; x_0) \to \pi_1(Y; y_0)$ defined by $f_*([\alpha]) = [f \circ \alpha]$, where $y_0 = f(x_0)$.

LEMMA. f_* is well-defined, and is really a group homomorphism.

DEFINITION. A *homotopy* between the maps $f_0, f_1\colon X \to Y$ is a map $H\colon X \times [0,1] \to Y$ such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for every $x \in X$. The maps f_0 and f_1 are then said to be *homotopic*. The homotopy is *base point preserving* if $H(x_0, t) = f_0(x_0) = f_1(x_0)$ for every $t \in [0,1]$.

Again, this definition is best understood by considering the family of maps $f_t\colon X \to Y$ defined by $f_t(x) = H(x, t)$.

PROPOSITION. If f and g are homotopic by a base point preserving homotopy then, for $y_0 = f(x_0) = g(x_0)$, the two homomorphisms f_* and $g_*\colon \pi(X; x_0) \to \pi_1(Y; y_0)$ are equal.

How about homotopies which do not fix base points?

PROPOSITION. If f and $g\colon X \to Y$ are homotopic by a homotopy H, let γ be the path from $y_0 = f(x_0)$ to $z_0 = g(x_0)$ defined by $\gamma(t) = H(x_0, t)$. Then $f_*\colon \pi_1(X; x_0) \to \pi_1(Y; y_0)$ and $g_*\colon \pi_1(X; x_0) \to \pi_1(Y; z_0)$ are related by the property that $g_* = T_\gamma \circ f_*$, where $T_\gamma\colon \pi_1(Y; y_0) \to \pi_1(Y; z_0)$ is the change of base point isomorphism defined by γ.

THEOREM. Let the product $X \times Y$ of the two topological spaces X and Y be endowed with the product topology, and let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the two projection maps defined by $p(x, y) = x$ and $q(x, y) = y$. Then the product map

$$p_* \times q_*\colon \pi_1\big(X \times Y; (x_0, y_0)\big) \to \pi_1(X; x_0) \times \pi_1(Y; y_0)$$

is a group isomorphism.

5. Homotopy equivalences and deformation retracts

DEFINITION. The map $f\colon X \to Y$ is a *homotopy equivalence* if there exists a map $g\colon Y \to X$ such that $g \circ f$ is homotopic to the identity Id_X and $f \circ g$ is homotopic to Id_Y. In this case, g is a *homotopy inverse* for f.

PROPOSITION. If $f\colon X \to Y$ is a homotopy equivalence and $y_0 = f(x_0)$, then $f_*\colon \pi(X; x_0) \to \pi(Y; y_0)$ is an isomorphism.

A good source of homotopy equivalences comes from the following definition.

DEFINITION. The space X *deformation retracts to* $A \subset X$, or A is a *deformation retract of* X, if there exists a homotopy $H\colon X \times [0, 1] \to X$ such that

$$\begin{cases} H(x, 0) = x \text{ for every } x \in X \\ H(x, 1) \in A \text{ for every } x \in X \\ H(a, t) = a \text{ for every } a \in A \text{ and every } t \in [0, 1] \end{cases}$$

The map $r\colon X \to A$ defined by $r(x) = H(x, 1)$ is the associated *retraction by deformation.*

LEMMA. If A is a deformation retract of X, the inclusion map $i\colon A \to X$ is a homotopy equivalence, with homotopy inverse the retraction $r\colon X \to A$. As a consequence, $i_*\colon \pi_1(A; x_0) \to \pi_1(X; x_0)$ is an isomorphism.

EXAMPLE. The space X is *contractible* if it deformation retracts to a point. Then, $\pi_1(X; x_0) \cong \pi_1(\{x_0\}; x_0) = 1$.

EXAMPLE OF THE EXAMPLE. A subset $X \subset \mathbb{R}^n$ is *star-shaped* if there exists $x_0 \in X$ such that, for every $x \in X$, the line segment $[x, x_0]$ is completely contained in X. Then X is contractible.

EXAMPLE. The *n-dimensional sphere* $S^n = \{x \in \mathbb{R}^{n+1}; \|x\| = 1\}$ is a deformation retract of $\mathbb{R}^n - \{0\}$, so that $i_*: \pi_1(S^n; x_0)$ is isomorphic to $\pi_1(\mathbb{R}^{n+1} - \{0\}; x_0)$ for every $x_0 \in S^n$. Of course, this is pretty useless at this point since we don't know either one of these fundamental groups (except when $n = 0$).

A weaker form of deformation retraction is the following.

DEFINITION. A *retraction* of X to a subset $A \subset X$ is a map $r: X \to A$ such that $r(a) = a$ for every $a \in A$. In this case, the space X *retracts to* A, or A is a *retract of* X.

PROPOSITION. If A is a retract of X and $x_0 \in A$, the homomorphism $i_*: \pi_1(A; x_0) \to \pi_1(X; x_0)$ induced by the inclusion map $i: A \to X$ is injective.

CHAPTER 3

The fundamental group of the circle

The main tool to compute the fundamental group of the circle S^1 is the map $p\colon \mathbb{R} \to S^1$ defined by $p(t) = (\cos t, \sin t)$.

A *lift* of a path α in S^1 is a path $\widetilde{\alpha}$ in $\mathbb{R}$ such that $p \circ \widetilde{\alpha} = \alpha$.

The two key tools are the following.

PROPOSITION. (Path Lifting Property) Let α be a path in S^1, beginning at $x_0 = \alpha(0)$. Then, for every $\widetilde{x}_0 \in p^{-1}(x_0)$, there exists a unique lift $\widetilde{\alpha}$ of α beginning at $\widetilde{\alpha}(0) = \widetilde{x}_0$.

Namely, given a path α in the circle, we know that each of its points can be written as $\alpha(t) = \big(\cos t(s), \sin t(s)\big)$ for some $t(s) \in \mathbb{R}$. However, we also know that there are many possible choices for $t(s)$, which is unique only modulo 2π. The Path Lifting Property asserts that we can choose $t(s)$ (called $\widetilde{\alpha}(s)$ in the statement) to depend continuously on s, and that $t(s)$ is then uniquely determined once we have made a choice for $t(0)$.

PROPOSITION. (Homotopy Lifting Property) Let α and β be two path homotopic path in S^1, beginning at $\alpha(0) = \beta(0) = x_0$. Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be their two lifts beginning at $\widetilde{\alpha}(0) = \widetilde{\beta}(0) = \widetilde{x}_0 \in p^{-1}(x_0)$. Then $\widetilde{\alpha}$ and $\widetilde{\beta}$ are path homotopic.

Similar properties will also play a critical role in our analysis of covering spaces (and will be proved there). The Homotopy Lifting Property has the following important consequence.

COROLLARY. Under the hypotheses of the Homotopy Lifting Property, $\widetilde{\alpha}$ and $\widetilde{\beta}$ have the same final point $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$.

THEOREM. Pick a base point x_0 in the circle S^1. Then there is a group isomorphism $\rho\colon \pi_1(S^1; x_0) \to \mathbb{Z}$ defined by the property that $\rho([\alpha]) = \frac{1}{2\pi}\big(\widetilde{\alpha}(1) - \widetilde{\alpha}(0)\big)$ for every $[\alpha] \in \pi_1(S^1; x_0)$ and every lift $\widetilde{\alpha}$ of α.

COROLLARY. $\pi_1(\mathbb{R}^2 - \{0\}; x_0)$ is isomorphic to $\mathbb{Z}$.

COROLLARY. There is no homeomorphism between the line $\mathbb{R}$ and the plane $\mathbb{R}^2$.

Consider the closed disk $B^2 = \{x \in \mathbb{R}^2, \|x\| \leqslant 1\}$.

COROLLARY. There is no retraction $r\colon B^2 \to S^1$, namely no such map such that $r(x) = x$ for every $x \in S^1$.

COROLLARY. (2–dimensional Brouwer Fixed Point Theorem) For every map $f\colon B^2 \to B^2$, there exists at least one point $x \in B^2$ such that $f(x) = x$.

CHAPTER 4

Computing fundamental groups

The key tool here is the van Kampen Theorem which, under suitable hypotheses, computes the fundamental group of a union $X_1 \cup X_2$ in terms of the fundamental groups of X_1, of X_2 and of the intersection $X_1 \cap X_2$. The statement of this theorem requires some algebraic preliminaries.

1. Algebraic preliminaries: Free products, presentations, and amalgamated products

1.1. Free products and free groups.

DEFINITION. The *free product* $A * B$ of the groups A and B consists of all finite sequences $x_1 x_2 \dots x_n$ where:

1. $x_i \in A - \{1\}$ or $x_i \in B - \{1\}$ for every i;
2. if $x_i \in A$ then $x_{i+1} \in B$ and, conversely, if $x_i \in B$ then $x_{i+1} \in A$.

We allow the empty sequence where $n = 0$.

EXAMPLE. Let A and B by both isomorphic to $\mathbb{Z}$. If is convenient to pick a generator $a \in A$ and a generator $b \in B$ and to denote these groups multiplicatively, so that $A = \{a^n; n \in \mathbb{Z}\}$ and $B = \{b^n; n \in \mathbb{Z}\}$. Then b^4 and $ab^{-3}a^{-5}b^2a^{-1}$ are typical elements of $A * B$.

We endow $A * B$ with a multiplication law $*$ defined by concatenation and simplification, namely

$$(x_1 x_2 \dots x_m)(y_1 y_2 \dots y_n) = \begin{cases} x_1 x_2 \dots x_m y_1 y_2 \dots y_n \text{ if } x_n \text{ and } y_1 \text{ are not} \\ \qquad\qquad\qquad\qquad \text{in the same set } A \text{ or } B \\ x_1 x_2 \dots x_{m-k-1}(x_{m-k} y_{k+1}) y_{k+2} \dots y_n \text{ otherwise,} \\ \qquad \text{where } k \text{ is the first } i \text{ such that } x_{m-i} \neq y_{i+1}^{-1}. \end{cases}$$

PROPOSITION. $*$ is a group law on $A * B$.

The only difficult (or at least annoying) part of the proof is that of the associativity, because of the cancellations. The identity element consists of the empty sequence, which will consequently be denoted by 1. The inverse of $x_1 x_2 \dots x_n$ is $x_n^{-1} x_{n-1}^{-1} \dots x_1^{-1}$.

DEFINITION. The *free group on n generators* is the free product $F_n = \mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z}$.

In practice, if we choose a generator x_i for the i–th cyclic group $\mathbb{Z}$, each element of F_n is of the form $x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_k}^{n_k}$ where $n_i \in \mathbb{Z} - \{0\}$ and $x_{i_{j+1}} \neq x_{i_j}$.

REMARK. The group F_0 is trivial. The group F_1 is isomorphic to $\mathbb{Z}$. The group F_n is non-commutative if $n \geqslant 2$.

More generally:

DEFINITION. The *free group $F(S)$ generated by* a set S consists of all finite sequences $x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$ of elements $x_i \in S$ and of superscripts $n_i \in \mathbb{Z} - \{0\}$, where any two consecutive x_i and $x_{i+1} \in S$ are distinct. The same concatenation/simplification law turns $F(S)$ into a group, with identity element the empty sequence.

Note that we have a natural inclusion map $i\colon S \to F(S)$, which to $x \in S$ associates $i(x) = x^1 \in F(S)$.

THEOREM. (Universal Property of Free Groups)

1. Every map $f\colon S \to G$ from the set S to a group G has a unique extension to a group homomorphism $\varphi\colon F(S) \to G$, in the sense that $\varphi \circ i = f$.
2. Up to isomorphism, the free group $F(S)$ is the only group satisfying (1). Namely, suppose that we are given another group F' and a map $i'\colon S \to F$ such that every map $f'\colon S \to G'$ from S to a group G' uniquely extends to a group homomorphism $\varphi'\colon F' \to G'$ with $\varphi' \circ i' = f'$. Then there exists a group homomorphism $\psi\colon F(S) \to F'$ such that $i' = \psi' \circ i$.

Similarly, note that the free product $A * B$ comes with natural inclusion homomorphisms $i_A\colon A \to A * B$ and $i_B\colon A * B$.

THEOREM. (Universal Property of Free Products)

1. Given two group homomorphisms $j_A\colon A \to G$ and $j_B\colon B \to G$, there exists a unique group homorphism $k\colon A * B \to G$ such that $j_A = k \circ i_A$ and $j_B = k \circ i_B$.
2. Up to isomorphism, the free $A * B$ is the only group satisfying (1). Namely, suppose that we are given another group H' and homomorphisms $i'_A\colon A \to H$ and $i'_B\colon B \to H$ such that, for every wo group homomorphisms $j_A\colon A \to G$ and $j_A\colon A \to G$, there exists a unique group homorphism $h'\colon H \to G$ such that $j_A = h \circ i'_A$ and $j_B = h \circ i'_B$. Then there exists an isomorphism $\varphi\colon H \to A * B$ such that $i_A = \varphi \circ i'_A$ and $i_B = \varphi \circ i'_B$.

1.2. Group presentations.

DEFINITION. If $x_1, x_2, \dots, x_n$ are the standard generators of F_n and if we are given $r_1, r_2, \dots, r_m \in F_n$, the *group defined by the generators* $x_1, x_2, \dots, x_n$ and by the relators $r_1, r_2, \dots, r_m$ is the quotient of F_n/K of F_n by the subgroup K normally generated by the r_i. Recall that the fact that K is normally generated by the r_i means that K is the smallest normal subgroup containing all the r_i; equivalently, every element of K can be written as a product of conjugates $\gamma r_i^{\pm 1} \gamma^{-1}$ with $\gamma \in F_n$.

In this case, we write $F_n/K = \langle x_1, x_2, \dots, x_n; r_1 = r_2 = \cdots = r_m = 1 \rangle$.

DEFINITION. A *(finite) presentation* for a group G is an isomorphism

$$\varphi\colon G \to F_n/K = \langle x_1, x_2, \dots, x_n; r_1 = r_2 = \cdots = r_m = 1 \rangle.$$

In practice, we have a homomorphism $\psi\colon F_n \to G$, defined as the composittion of the quotient map $F_n \to F_n/K$ and of $\varphi^{-1}\colon F_n/K \to G$. The fact that φ^{-1} is surjective implies that every $g \in G$ can be written as $g = g_{i_1}^{n_1} g_{i_2}^{n_2} \dots g_{i_k}^{n_k}$ with $n_i \in \mathbb{Z}$ and $g_i = \psi(x_i)$, namely that the g_i generate G. The fact that φ^{-1} is injective implies that, every time $g_{i_1}^{n_1} g_{i_2}^{n_2} \dots g_{i_k}^{n_k} = 1$ in G, the corresponding element $x_{i_1}^{n_1} x_{i_2}^{n_2} \dots x_{i_k}^{n_k} \in F_n$ is actually in K. In other words, any relation $g_{i_1}^{n_1} g_{i_2}^{n_2} \dots g_{i_k}^{n_k} = 1$ between the generators g_i can be deduced by the usual rules of algebra from the property that $\psi(r_1) = \psi(r_2) = \dots = \psi(r_m) = 1$

EXAMPLES.

1. $\langle a; a^n = 1\rangle \cong \mathbb{Z}/n$.
2. $\langle a, b; \rangle = \mathbb{Z} * \mathbb{Z}$.
3. The group $\langle a, b; aba^{-1}b^{-1} = 1\rangle$ is isomorphic to the direct product $\mathbb{Z} \times \mathbb{Z}$.
4. Let Δ_{2n} be the dihedral group, namely the subgroup of the orthogonal group $O(2)$ generated by the rotation A of angle $\frac{2\pi}{n}$ and by the reflection B across a line L. Then Δ_{2n} admits the presentation $\langle A, B; A^n = B^2 = (AB)^2 = 1\rangle$.

THEOREM. (Markov) There exists no algorithm which, given m relators r_1, r_2, …, r_m in the free group F_n and given an element $x \in F_n$, decides whether the element of $G = \langle x_1, x_2, \dots, x_n; r_1 = r_2 = \dots = r_m = 1\rangle$ defined by x is trivial or not.

One can similarly define group presentations with infinitely many generators and/or infinitely many relations by replacing the free group F_n by the group $F(S)$ freely generated by a possibly infinite set S, and taking as relators the elements of a possibly infinite subset R of $F(S)$.

1.3. Amalgamated products.

DEFINITION. If we are given three groups A, B and C and two group homomorphisms $i_A\colon C \to A$ and $i_B\colon C \to B$, the *free product of A and B amalgamated along C using the homomorphisms i_A and i_B* is the quotient $A *_C B = (A * B)/K$ of the free product $A * B$ by the subgroup K normally generated by all the elements of the form $i_A(c)i_B(c)^{-1}$ with $c \in C$. Note that the notation $A *_C B$ is somewhat incomplete, inasmuch as $A *_C B$ depends also on the homomorphisms i_A and i_B, not just on the groups A, B and C.

In practice, if

$$A = \langle x_1, x_2, \dots, x_n; r_1 = r_2 = \dots = r_m = 1\rangle,$$

if

$$B = \langle y_1, y_2, \dots, y_p; s_1 = s_2 = \dots = s_q = 1\rangle,$$

and if C is generated by c_1, c_2, …, c_t, then $A *_C B$ has the presentation

$$\begin{aligned} A *_C B = \langle x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_p;& \\ & r_1 = r_2 = \dots = r_m = s_1 = s_2 = \dots = s_q = 1, \\ & i_A(c_1)i_B(c_1)^{-1} = i_A(c_2)i_B(c_2)^{-1} = \dots = i_A(c_t)i_B(c_t)^{-1} = 1\rangle. \end{aligned}$$

Note that the amalgamated product comes with two natural group homomorphism $j_A\colon A \to A *_C B$ and $j_B \to A *_C B$. For instance, j_A is the composition of the inclusion map $A \to A * B$ with the quotient map $A * B \to A *_C B = A * B/K$.

THEOREM. (Universal Property of Amalgamated Products)

1. For any two group homomorphisms $k^A\colon A \to G$ and $k_B\colon B \to G$ such that $k_A(i_A(c)) = k_B(i_B(c))$ for each $c \in C$, there exists a unique group homomorphism $k\colon A *_C B \to G$ such that $k_A = k \circ j_A$ and $k_B = k \circ j_B$.
2. Up to isomorphism, $A *_C B$ is uniquely determined by Property (1).

EXERCISE. Write a detailed statement of the uniqueness property in (2) above.

2. The van Kampen Theorem(s)

If X_1 and X_2 are two subspaces of the topological space X and if we pick a base point x_0 in $X_1 \cap X_2$, the appropriate inclusion maps induce homomorphisms

$$i_{1*}\colon \pi_1(X_1 \cap X_2; x_0) \to \pi_1(X_1; x_0) \qquad i_{2*}\colon \pi_1(X_1 \cap X_2; x_0) \to \pi_1(X_2; x_0)$$
$$k_{1*}\colon \pi_1(X_1; x_0) \to \pi_1(X_1 \cup X_2; x_0) \qquad k_{2*}\colon \pi_1(X_2; x_0) \to \pi_1(X_1 \cup X_2; x_0)$$

such that $k_{1*} \circ i_{1*} = k_{2*} \circ i_{2*}$. The Universal Property of Amalgamated Products then provides a group homomorphism

$$k\colon \pi_1(X_1; x_0) *_{\pi_1(X_1 \cap X_2; x_0)} \pi_1(X_2; x_0) \to \pi_1(X_1 \cup X_2; x_0).$$

THEOREM. (van Kampen Theorem I) Suppose that X_1 and X_2 are open in X, and that X_1, X_2 and $X_1 \cap X_2$ are all path connected. Then the above homomorphism

$$k\colon \pi_1(X_1; x_0) *_{\pi_1(X_1 \cap X_2; x_0)} \pi_1(X_2; x_0) \to \pi_1(X_1 \cup X_2; x_0).$$

is an isomorphism.

The proof uses the following result from real analysis and/or pointset topology.

PROPOSITION. (Lebesgue Number Lemma) Let $\mathcal{U} = \{U_i; i \in I\}$ be an open covering of the compact metric space (X, d). (Namely, each U_i is open in X, and $X = \bigcup_{i \in I} U_i$.) Then, there exists a number $\varepsilon > 0$ such that, for every $x \in X$, the ball $B_d(X, \varepsilon)$ is completely contained in some U_i.

The number ε is a *Lebesgue number* for the covering $\mathcal{U}$.

Because we tend to prefer compact spaces, and because compact spaces tend to provide closed subsets, it is desirable to have a version of the van Kampen Theorem involving compact subsets. However, an additional condition is then necessary.

DEFINITION. A subset A is a *neighborhood deformation retract* in X if there exists an open subset U of X such that U deformation retracts to A. (In particular, U contains A.)

In general, most "reasonable" subsets are neighborhood deformation retracts. However, the Cantor set is not a neighborhood deformation retract in $\mathbb{R}$.

THEOREM. (van Kampen Theorem II) Suppose that X_1 and X_2 are closed in X, and that X_1, X_2 and $X_1 \cap X_2$ are all path connected. Suppose in addition that

$X_1 \cap X_2$ is a neighborhood deformation retract in both X_1 and X_2. Then the homomorphism

$$k\colon \pi_1(X_1;x_0) *_{\pi_1(X_1\cap X_2;x_0)} \pi_1(X_2;x_0) \to \pi_1(X_1 \cup X_2; x_0).$$

is an isomorphism.

EXAMPLE. The figure eight has fundamental group $\langle a, b; \rangle = \mathbb{Z} * \mathbb{Z}$.

EXAMPLE. The sphere $S^n = \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}; \sum_{i=0}^n x_i^2 = 1\}$ has fundamental group

$$\pi_1(S^n; x_0) = \begin{cases} 0 & \text{if } n \neq 1 \\ \mathbb{Z} & \text{if } n = 1 \end{cases}$$

EXAMPLE. The torus $S^1 \times S^1$ has fundamental group $\langle a, b; aba^{-1}b^{-1} = 1\rangle \cong \mathbb{Z} \times \mathbb{Z}$.

EXAMPLE. The Klein bottle has fundamental group $\langle a, b; abab^{-1} = 1\rangle$.

EXAMPLE. Let K_0 be the circle $S^1 \subset \mathbb{R}^2 \subset \mathbb{R}^3$. Then the complement $\mathbb{R}^3 - K_0$ has fundamental group isomorphic to $\mathbb{Z}$.

EXAMPLE. Let K_1 be the closed curve in $\mathbb{R}^3$ parametrized by

$$t \mapsto \big((R + r\cos 3t)\cos 2t, (R + r\cos 3t)\sin 2t, r\sin 3t\big)$$

with $R > r > 0$. The fundamental group of $\mathbb{R}^3 - K_1$ has presentation $\langle a, b; a^2 = b^3\rangle$.

COROLLARY. There is no homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ sending the overhand knot K_1 (also called trefoil knot) to the unknot K_0.

3. Finite cell complexes

Recall the definition of the n-dimensional ball $B^n = \{x \in \mathbb{R}^n; \|x\| \leqslant 1\}$, delimited by the $(n-1)$-dimensional sphere $S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$ in $\mathbb{R}^n$.

DEFINITION. An *n-dimensional cell complex* is a space X which can be written as $X = X^{(0)} \cup X^{(1)} \cup \cdots \cup X^{(n)}$ in such a way that:

1. $X^{(0)}$ consists of finitely many points;
2. each $X^{(i+1)}$ is obtained by gluing to $X^{(i)}$ finitely many copies B_1^{i+1}, B_2^{i+1}, $\dots$, $B_{k_i}^{i+1}$ of the $i+1$-dimensional ball B^{i+1} by continuous maps $\varphi_j\colon S_j^i \to X^{(i)}$ from the corresponding spheres $S_j^i \subset B_j^{i+1}$ to $X^{(i)}$; namely, given such maps φ_j, $X^{(i+1)}$ is the quotient space

$$X^{(i+1)} = X^{(i)} \sqcup B_1^{i+1} \sqcup B_2^{i+1} \sqcup \cdots \sqcup B_{k_i}^{i+1}/\sim$$

   quotient of the disjoint union of $X^{(i)}$ and the B_j^{i+1} by the equivalence relation $\sim$ that identifies each $x \in S_j^i$ to $\varphi_j(x) \in X^{(i)}$.

The images of the balls B_j^{i+1} are the $(i+1)$*-dimensional cells* of X, and $X^{(i)}$ is its *i-dimensional skeleton.*

DEFINITION. A *finite graph* is a 1–dimensional cell complex. Traditionally, the 0–dimensional cells of a graph are also called its *vertices*, and its 1–dimensional cells are its *edges*.

THEOREM. If $X = X^{(1)}$ is a connected graph with v vertices and e edges, its fundamental group $\pi_1(X; x_0)$ is isomorphic to the free group F_n where $n = e - v + 1$.

THEOREM. Let X be an n–dimensional cell complex, whose 2–skeleton $X^{(2)}$ is obtained by attaching 2–dimensional cells B_1^2, B_2^2, ..., B_k^2 to the 1–skeleton $X^{(1)}$ by gluing maps $\varphi_i \colon S_i^1 \to X^{(1)}$. Then, the fundamental group of X is isomorphic to the quotient of the fundamental group of the graph $X^{(1)}$ by the subgroup normally generated by the $\varphi_{i*}(a_i)$, where a_i is a generator of the fundamental group $\pi_1(S_i; x_i) \cong \mathbb{Z}$. In particular, the 1-skeleton $X^{(1)}$ provides generators, the 2–dimensional cells give relators, and the higher dimensional cells do not contribute to the fundamental group of X.

CHAPTER 5

Covering spaces

1. Definitions

DEFINITION. A *covering map* is a map $p\colon \widetilde{X} \to X$ such that, for every $x \in X$, there exists an open neighborhood U of x for which:

1. the preimage $p^{-1}(U)$ can be written as union of disjoint open subsets $\{\widetilde{U}_i\}_{i \in I}$;
2. for every $i \in I$, the restriction $p_{|\widetilde{U}_i}\colon \widetilde{U}_i \to U_i$ is a homeomorphism.

A *covering* is a triple $(X, \widetilde{X}, p)$ consisting of two topological spaces X and $\widetilde{X}$ and of a covering map $p\colon \widetilde{X} \to X$. The space $\widetilde{X}$ is the *total space* or *covering space* of the covering, while X is its *base*.

A subset $U \subset X$ that satisfies the properties of the definition above is said to be *evenly covered*.

EXAMPLES.

1. The map $p\colon \mathbb{R} \to S^1$ defined by $p(t) = (\cos t, \sin t)$, which we used to compute the fundamental group of the circle, is a covering map.
2. The restriction $p\colon [0, 6\pi) \to S^1$ of the above map is not a covering map. Note that $p^{-1}(x)$ consists of 3 points for every $x \in S^1$.
3. The map $p\colon \mathbb{C} - \{0\} \to \mathbb{C} - \{0\}$ defined by $p(z) = z^7$ is a covering map.
4. The map $p\colon \mathbb{C} \to \mathbb{C}$ defined by $p(z) = z^7$ is not a covering map.
5. Let X be a topological space, and let F be a set endowed with the discrete topology. The projection $p\colon X \times F \to X$ to the second factor is a covering map, called the *trivial covering* with base X and fiber F. Note that every covering is locally of this type, but not necessarily globally.

DEFINITION. Two coverings $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ over the same base X are *isomorphic* if there exists a homeomorphism $\varphi\colon \widetilde{X} \to \widetilde{X}'$ such that $p' \circ \varphi = p$.

Our goal is to classify all coverings of a given topological space X, up to isomorphism.

2. Lifting properties

The theory of covering spaces with base X turns out to be strongly connected to the fundamental group $\pi_1(X; x_0)$ because of the following two fundamental properties.

THEOREM.(Path Lifting Property) Let $p\colon \widetilde{X} \to X$ be a covering map, and let $\alpha\colon [0,1] \to X$ be a path beginning at $x_0 = \alpha(0)$. Then, for every $\widetilde{x}_0 \in p^{-1}(x_0)$, there exists a unique path $\widetilde{\alpha}\colon [0,1] \to \widetilde{X}$ such that

1. $\widetilde{\alpha}$ *lifts* α in the sense that $p \circ \widetilde{\alpha} = \alpha$;
2. $\widetilde{\alpha}(0) = \widetilde{x}_0$.

THEOREM. (Homotopy Lifting Property) Let $p\colon \widetilde{X} \to X$ be a covering map, and let α, $\beta\colon [0,1] \to X$ be two paths beginning at the same point $x_0 = \alpha(0) = \beta(0)$. For $\widetilde{x}_0 \in p^{-1}(x_0)$, let $\widetilde{\alpha}$, $\widetilde{\beta}\colon [0,1] \to \widetilde{X}$ be the two lifts beginning at $\widetilde{x}_0 = \widetilde{\alpha}(0) = \widetilde{\beta}(0)$ provided by the Path Lifting Property. If, in addition, α and β are path homotopic, then $\widetilde{\alpha}$ and $\widetilde{\beta}$ are path homotopic; in particular, $\widetilde{\alpha}$ and $\widetilde{\beta}$ end at the same point $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$.

COROLLARY. The homomorphism $p_*\colon \pi_1(\widetilde{X};\widetilde{x}_0) \to \pi_1(X;x_0)$ induced by a covering map $p\colon \widetilde{X} \to X$ is injective.

The (proofs of the) Path Lifting Property and Homotopy Lifting Property show us how to lift maps $\alpha\colon [0,1] \to X$ and $H\colon [0,1] \times [0,1] \to X$ to a covering space $\widetilde{X}$. The following generalizes these lifting properties, and explains what is special about $[0,1]$ and $[0,1] \times [0,1]$.

THEOREM. (Lifting Criterion) Let $p\colon \widetilde{X} \to X$ be a covering map, and consider a map $f\colon Y \to X$ where Y is path connected and locally path connected. Choose base points $x_0 \in X$, $\widetilde{x}_0 \in \widetilde{X}$ and $y_0 \in Y$ such that $x_0 = p(\widetilde{x}_0) = f(y_0)$. The following are equivalent.

1. There exists a lift $\widetilde{f}\colon Y \to \widetilde{X}$ of f (namely a map $\widetilde{f}$ such that $p \circ \widetilde{f} = f$) such that $\widetilde{f}(y_0) = \widetilde{y}_0$.
2. In $\pi_1(X;x_0)$, the subgroup $f_*\big(\pi_1(Y;y_0)\big)$ is contained in $p_*\big(\pi_1(\widetilde{X};\widetilde{x}_0)\big)$.

In particular, a lift always exists if Y is simply connected, in the following sense.

DEFINITION. A space X is *simply connected* if it is path connected and locally path connected, and if $\pi_1(X;x_0) = 1$ for some (or all) base point $x_0 \in X$.

Although apparently much more general than the Path and Homotopy Lifting Properties, this Lifting Criterion actually is a relatively simple consequence of these properties.

3. Connected coverings

We now restrict attention to coverings $p\colon \widetilde{X} \to X$ where the total space $\widetilde{X}$ is connected. It may useful to remember that a connected space that is locally path connected is also path connected.

THEOREM. Assume that X is connected and locally path connected. Let $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ be two coverings where $\widetilde{X}$ and $\widetilde{X}'$ are path connected, and pick two base points $\widetilde{x}_0 \in \widetilde{X}$ and $\widetilde{x}'_0 \in \widetilde{X}'$ such that $p(\widetilde{x}_0) = p'(\widetilde{x}'_0) = x_0$. There exists a covering isomorphism $\varphi\colon \widetilde{X}_0 \to \widetilde{X}'$ sending $\widetilde{x}_0$ to $\widetilde{x}'_0$ if and only if $p_*\bigl(\pi_1(\widetilde{X};\widetilde{x}_0)\bigr) = p'_*\bigl(\pi_1(\widetilde{X}';\widetilde{x}'_0)\bigr)$ in $\pi_1(X_0, x_0)$.

In other words, when X is connected and locally path connected, the map which to a path connected covering $p\colon \widetilde{X} \to X$ and base point $\widetilde{x}_0 \in \widetilde{X}$ associates the subgroup $p_*\bigl(\pi_1(\widetilde{X};\widetilde{x}_0)\bigr)$ is injective, up to covering isomorphism sending base point to base point. We will show later in this section that this map is surjective, under a (mild) additional hypothesis on X.

First, let us consider what happens if we do not require covering isomorphisms to send base point to base point.

THEOREM. Assume that X is connected and locally path connected. Let $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ be two coverings where $\widetilde{X}$ and $\widetilde{X}'$ are path connected, with base points $\widetilde{x}_0 \in \widetilde{X}$ and $\widetilde{x}'_0 \in \widetilde{X}'$ such that $p(\widetilde{x}_0) = p'(\widetilde{x}'_0) = x_0$. These two coverings are isomorphic if and only if the subgroup $p'_*\bigl(\pi_1(\widetilde{X};\widetilde{x}_0)\bigr)$ is conjugate to $p_*\bigl(\pi_1(\widetilde{X}';\widetilde{x}'_0)\bigr)$ in $\pi_1(X_0, x_0)$.

DEFINITION. A space X is *semi-locally simply connected* if it, for every $x \in X$ and every neighborhood U of x, there exists a smaller neighborhood $V \subset U$ of x which is path connected and such that the homomorphism $\pi_1(V;x) \to \pi_1(X;x)$ induced by the inclusion map has trivial image (namely every path from x to x in V is path homotopic to the constant path c_x in X, although not necessarily in V).

EXAMPLES.

1. Every topological manifold is semi-locally simply connected. (Recall that a topological manifold of dimension n is a space where every point admits a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.)
2. Consider the Hawaiian rings $X = \bigcup_{n\in\mathbb{N}} C_n$, where C_n is the circle of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0)$ in the plane. This space X is not semi-locally simply connected.

THEOREM. Let X be path connected and semi-locally simply connected. For every subgroup $G \subset \pi_1(X;x_0)$, there exists a path connected covering $p\colon \widetilde{X} \to X$ such that $p(\widetilde{x}_0)$.

COROLLARY. Up to isomorphism, a space X that is path-connected and semi-locally simply connected space admits a unique covering $p\colon \widetilde{X} \to X$ where $\widetilde{X}$ is simply connected.

This simply connected covering is the *universal covering space* of X.

By realizing a free group as the fundamental group of a 1-dimensional cell complex, one obtains a purely algebraic result.

COROLLARY. (Nielsen-Schreier Theorem) Every subgroup of a free group is isomorphic to a free group.

4. Coverings with a given fiber

The *fiber* of a covering $p\colon \widetilde{X} \to X$ with base point $x_0 \in X$ is the pre-image $F = p^{-1}(x_0)$.

Consider a point $\widetilde{x} \in F$ and an element $[\alpha] \in \pi_1(X; x_0)$. Use the Path Lifting Property to lift the path $\alpha\colon [0,1] \to X$ to a path $\widetilde{\alpha}\colon [0,1] \to \widetilde{X}$ with $p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}(0) = \widetilde{x}$. Then $\widetilde{\alpha}(1)$ is another element of the fiber F, usually distinct from the starting point $\widetilde{\alpha}(0) = \widetilde{x}$.

Let $\rho([\alpha])\colon F \to F$ be the map which to $\widetilde{x} \in F$ associates the element $\widetilde{\alpha}(1) \in F$ defined as above. The Homotopy Lifting Property shows that $\rho([\alpha])$ is independent of the choice of the path α used to represent $[\alpha] \in \pi_1(X; x_0)$.

PROPOSITION. This defines a group antihomomorphism $\rho\colon \pi_1(X; x_0) \to \mathrm{Bij}(F)$, valued in the group $\mathrm{Bij}(F)$ of all bijections $F \to F$. This is called the *monodromy antihomomorphism* of the covering.

The fact that ρ is an antihomomorphism just means that $\rho([\alpha] * [\beta]) = \rho([\beta]) * \rho([\alpha])$ for every $[\alpha]$, $[\beta] \in \pi_1(X; x_0)$.

THEOREM. Let X be a path connected and semi-locally simply connected space, let F be a set, and consider an antihomomorphism $\rho\colon \pi_1(X; x_0) \to \mathrm{Bij}(F)$. Then, there exists a covering $p\colon \widetilde{X} \to X$ with fiber $p^{-1}(x_0) = F$ and whose monodromy antihomomorphism is equal to ρ. In addition, this covering is unique up to isomorphism in the following sense: For any other covering $p'\colon \widetilde{X}' \to X$ with monodromy ρ, there is a covering isomorphism $\varphi\colon \widetilde{X} \to \widetilde{X}'$ (with $p' \circ \varphi = p$) which coincides with the identity on $F = p^{-1}(x_0) = p'^{-1}(x_0)$.

COROLLARY. If X is simply connected, every covering $p\colon \widetilde{X} \to X$ is isomorphic to the trivial bundle $X \times F \to X$.

CHAPTER 6

Singular homology

Homology is the science of the statement "B is the boundary of A", motivated by Stokes's theorem.

1. Simplices

DEFINITION. The *standard simplex* of dimension n is

$$\Delta_n = \{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1}; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geqslant 0, \forall i\}.$$

Namely Δ_n is the convex hull of the $n+1$ points that are the tips of the coordinate vectors of $\mathbb{R}^{n+1}$.

We can embed the $(n-1)$–dimensional simplex Δ_{n-1} in the n–dimensional simplex Δ_n, but there are $n-1$ natural ways to do so.

DEFINITION. For $i = 0, \ldots, n$, the *i–face map* is the map $F_i\colon \Delta_{n-1} \to \Delta_n$ defined by

$$F_i(t_0, t_1, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}),$$

where the 0 is inserted in the $(i+1)$–th position.

Note that the boundary $\partial\Delta_n$ is the union of the images $F_i(\Delta_{n-1})$ of the $n+1$ face maps.

DEFINITION. A *(singular) n–simplex* in a topological space X is a continuous map $\sigma\colon \Delta_n \to X$.

The set of all singular n–simplices is denoted by $\mathcal{S}_n(X)$.

2. Cochains

This is here just as a motivation for the definition of singular chains in the next section.

Let R be a commutative ring with a unit denoted by 1.

DEFINITION. A *singular n–cochain* valued in R is a map $c\colon \mathcal{S}_n(X) \to R$.

The set $C^n(X)$ of singular cochains valued in R is clearly an R–module.

HISTORIC EXAMPLE. If X is a manifold and if we restrict attention to differentiable singular simplices, a differential form $\omega \in \Omega^n(X)$ of degree n defines a cochain

$c_\omega: \mathcal{S}_n(X) \to \mathbb{R}$ by

$$c_\omega(\sigma) = \int_{\Delta_n} \sigma^*(\omega)$$

for every differentiable simplex $\sigma: \Delta_n \to X$.

DEFINITION. The *coboundary* of a cochain $c \in C^n(X;R)$ is the cochain $dc \in C^{n+1}(X;R)$ defined by the formula

$$dc(\sigma) = \sum_{i=0}^{n+1} (-1)^i c(\sigma \circ F_i) \in R$$

for every $(n+1)$–simplex σ. Note that each face map $F_i: \Delta_n \to \Delta_{n+1}$ defines an n–simplex $\sigma \circ F_i: \Delta_n \to X$.

The formula is specially designed so that $dc_\omega = c_{d\omega}$ in the historical example above, where $d\omega$ is the exterior derivative of the differential form ω. The sign $(-1)^i$ is then justified by the possible discrepancy, on the face $F_i(\Delta_n)$ of the boundary $\partial\Delta_{n+1}$, between the boundary orientation of $\partial\Delta_{n+1}$ and the orientation coming from the orientation of Δ_n.

These observations were designed as an introduction to what follows. Cochains will reappear later when we discuss cohomology.

3. Chains

From now on, we fix a commutative ring R with a unit element 1. This ring R will be $\mathbb{Z}$ most of the time, $\mathbb{R}$ or $\mathbb{Q}$ sometimes, $\mathbb{Z}_2$ sometimes too, and a general $\mathbb{Z}_n$ much less frequently.

DEFINITION. A *singular n–chain* in X and with coefficients in R is an element of the free R–module $C_n(X)$ generated by the set $\mathcal{S}_n(X)$. Recall that the free R–module $C_n(X)$ consists of all the maps $c: \mathcal{S}_n(X) \to R$ such that $c(\sigma) = 0$ for all but finitely many elements of $\mathcal{S}_n(X)$.

In practice, there is a canonical inclusion map $\chi: \mathcal{S}_n(X) \to C_n(X)$, which to $\sigma \in \mathcal{S}_n(X)$ associates the characteristic function χ_σ defined by $\chi_\sigma(\sigma') = 0$ if $\sigma' \neq \sigma$ and $\chi_\sigma(\sigma) = 1$. If we identify $\sigma \in \mathcal{S}_n(X)$ to $\chi_\sigma \in C_n(X)$, then any $a \in C_n(X)$ can be written as

$$c = \sum_{\sigma \in \mathcal{S}_n(X)} a_\sigma \sigma = \sum_{i=1}^{k} a_i \sigma_i$$

where $a_\sigma = c(\sigma) \in R$, where the set of $\sigma \in \mathcal{S}_n(X)$ with $c(\sigma) \neq 0$ is contained in the finite set $\{\sigma_1, \sigma_2, \dots, \sigma_k\}$, and where $a_i = c(\sigma_i) = a_{\sigma_i} \in R$. In particular, the first sum makes sense since only finitely many of its terms are non-zero.

DEFINITION. The *n–dimensional boundary map* is the unique linear map $\partial_n : C_n(X) \to C_n(X)$ such that, for every simplex $\sigma \in \mathcal{S}_n(X) \subset C_n(X)$,

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma \circ F_i.$$

Note the analogy with the coboundary map of the previous section.

In practice, we usually drop the subscript n, and write $\partial_n = \partial$.

LEMMA. $\partial \circ \partial = 0$, or more precisely $\partial_{n-1} \circ \partial_n(c) = 0$ for every $c \in C_n(X)$.

4. Homology modules

DEFINITION. The set of *closed n-chains*, or the set of *n-cycles*, is the kernel

$$Z_n(X) = \{c \in C_n(X); \partial_n c = 0\}$$

of the boundary map ∂_n. The set of *exact n-chains*, or the set of *n-boundaries* is the image

$$B_n(X) = \{c \in C_n(X); \exists c' \in C_{n+1}(X), c = \partial_{n+1} c'\}$$

of ∂_{n+1}. Note that $Z_n(X)$ and $B_n(X)$ are both submodules of $C_n(X)$, and that $B_n(X) \subset Z_n(X)$ since $\partial \circ \partial = 0$.

DEFINITION. The *n-th homology module* of X with coefficients in R is

$$H_n(X; R) = Z_n(X)/B_n(X).$$

We just write $H_n(X)$ when the coefficient ring R is clear from the context. When $R = \mathbb{Z}$, the $H_n(X)$ are the *homology groups* of X. (Recall that a $\mathbb{Z}$-module is just an abelian group),

SILLY EXAMPLE. If $X = \varnothing$, then $H_n(X) = 0$ for every n.

EXAMPLE. Let X consist of a single point x_0. Then

$$H_n(\{x_0\}) = \begin{cases} R \text{ if } n = 0 \\ 0 \text{ if } n \neq 0 \end{cases}$$

PROPOSITION. Let $\{X_i\}_{i \in I}$ be the set of path connected components of X. Then

$$H_n(X) \cong \bigoplus_{i \in I} H_n(X_i).$$

This enables us to restrict attention to path connected spaces.

THEOREM. If X is path connected and non-empty, then $H_0(X)$ is isomorphic to R.

5. Homomorphisms induced by continuous maps

Let $f\colon X \to Y$ be a (continuous) map. It determines a (set-theoretic) map $\mathcal{S}_n(X)\colon \mathcal{S}_n(X) \to \mathcal{S}_n(Y)$ defined by $\sigma \mapsto f \circ \sigma$. This map then defines an R-module homomorphism (= a linear map) $C_n(f)\colon C_n(X) \to C_n(Y)$, which to a chain $\sum_i a_i \sigma_i \in C_n(X)$, with $a_i \in R$ and $\sigma_i \in \mathcal{S}_n(X)$, associates $\sum_i a_i f \circ \sigma_i \in C_n(Y)$.

The following is automatic.

LEMMA. $\partial_{n-1} \circ C_n(f) = C_{n-1}(f) \circ \partial_m$.

COROLLARY. The map $C_n(f)\colon C_n(X) \to C_n(Y)$ sends $Z_n(X)$ to $Z_n(Y)$, and $B_n(X)$ to $B_n(Y)$. As a consequence, it induces a linear map $H_n(f)\colon H_n(X) \to H_n(Y)$.

DEFINITION. The above map $H_n(f)\colon H_n(X) \to H_n(Y)$ is the *homomorphism induced in n–th homology* by f.

6. Homotopy invariance

Recall that two maps f_0, f_1: $X \to Y$ are *homotopic* if there exists a continuous map H: $X \times [0,1] \to Y$, called a *homotopy* such that $H(x,0) = f_0(x)$ and $H(x,1) = f_1(x)$ for every $x \in X$.

THEOREM. (Homotopy Invariance) If f_0, f_1: $X \to Y$ are homotopic, then for every n the induced homomorphisms $H_n(f_0)$, $H_n(f_1)$: $H_n(X) \to F_n(Y)$ are equal.

The key step in the proof is the following.

LEMMA. (Construction of a chain homotopy) Under the above hypotheses, there exists a family of linear maps K_n: $C_n(X) \to C_{n+1}(Y)$ such that

$$\partial_{n+1} \circ K_n + K_{n-1} \circ \partial_n = C_n(f_1) - C_n(f_0).$$

PROOF. Associate to the singular n–simplex σ: $\Delta_n \to X$ the chain

$$K_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma_i$$

where the $(n+1)$–simplices σ_i: $\Delta_{n+1} \to Y$ are defined by

$$\sigma_i(x_0, x_1, \ldots, x_{n+1}) = H\big(\sigma(x_0, \ldots, x_{i-1}, x_i{+}x_{i+1}, x_{i+2}, \ldots, x_{n+1}), x_{i+1}{+}\cdots{+}x_{n+1}\big).$$

□

After this geometric step, the rest of the proof is a simple algebraic manipulation.

COROLLARY. If f: $X \to Y$ is a homotopy equivalence, then the induced homomorphism $H_n(f)$: $H_n(X) \to H_n(Y)$ is an isomorphism.

COROLLARY. If $X \subset \mathbb{R}^n$ is star-shaped,

$$H_n(X; R) \cong \begin{cases} 0 & \text{if } n \neq 0 \\ R & \text{if } n = 0. \end{cases}$$

CHAPTER 7

Basic homological algebra

1. Chain complexes

DEFINITION. A *chain complex* is a bi-infinite sequence $C = \{C_n, \partial_n\}_{n\in\mathbb{Z}}$ of R–modules $C_n$and linear maps $\partial_n \colon C_n \to C_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$ for every n.

In particular, in C_n, the kernel $Z_n(C) = \operatorname{Ker}\partial_n$ of ∂_n contains the image $B_n(C) = \operatorname{Im}\partial_{n+1}$ of ∂_{n+1}, and we can consider the quotient R–module $H_n(C) = Z_n(C)/B_n(C)$.

DEFINITION. The quotient $H_n(C) = Z_n(C)/B_n(C)$ is the *n–th homology module* of the chain complex $C = \{C_n, \partial_n\}_{n\in\mathbb{Z}}$.

The chain complex $C = \{C_n, \partial_n\}_{n\in\mathbb{Z}}$ is an *exact sequence* if $H_n(C) = 0$, namely if $\operatorname{Ker}\partial_n = \operatorname{Im}\partial_{n+1}$ for every n.

FUNDAMENTAL EXAMPLE. $C_n = C_n(X)$ for some topological space X, with the convention that $C_n(C) = 0$ if $n < 0$ and with the usual boundary homomorphisms $\partial_n \colon C_n(X) \to C_{-n}(X)$. The homology module $H_n(C)$ of the chain complex C is then the singular homology module $H_n(X)$.

2. Another example: reduced homology

Consider the homomorphism $\widetilde{\partial}_0 \colon C_0(X) \to R$ defined by the property that $\widetilde{\partial}_0(\sum_{i=1}^k a_i x_i) = \sum_{i=1}^k a_i$. For $n \in \mathbb{Z}$, define

$$\widetilde{C}_n(X) = \begin{cases} C_n(X) & \text{if } n \geqslant 0 \\ R & \text{if } n = -1 \\ 0 & \text{if } n \leqslant -2 \end{cases}$$

and let $\widetilde{\partial}_n$: $\widetilde{C}_n(X) \to \widetilde{C}_{n-1}(X)$ by the property that

$$\widetilde{\partial}_n = \begin{cases} \partial_n & \text{if } n > 0 \\ \text{the above } \widetilde{\partial}_0 & \text{if } n = 0 \\ 0 & \text{if } n \leqslant -1. \end{cases}$$

One can show that $\widetilde{C}(X) = \{\widetilde{C}_n(X), \widetilde{\partial}_n\}_{n\in\mathbb{N}}$ is a chain complex.

DEFINITION. The homology modules $H_n(\widetilde{C}(X))$ of the above chain complex are the *reduced homology modules* $\widetilde{H}_n(X)$ of X.

THEOREM.

1. $\widetilde{H}_n(X) = H_n(X)$ if $n \neq 0$;
2. $\widetilde{H}_0(X) = 0$ if X is path connected;
3. $\widetilde{H}_0(X) \cong R^{k-1}$ if X has k path connected components.

3. Chain maps

DEFINITION. A *chain map* $f\colon C \to C'$ from the chain complex $C = \{C_n, \partial_n\}_{n\in\mathbb{Z}}$ to the chain complex $C = \{C_n, \partial_n\}_{n\in\mathbb{Z}}$ is a family of linear maps $f_n\colon C_n \to C'_n$ such that $\partial'_n \circ f_n = f_{n-1} \circ \partial_n$ for every n. Namely, the infinite ladder of arrows provided by the data is a commutative diagram.

It is immediate from definitions that f_n sends $Z_n(C)$ to $Z_n(C')$ and $B_n(C)$ to $B_n(C')$. We can therefore consider:

DEFINITION. The *homomorphism induced in n–dimensional homology* induced by the chain map $f\colon C \to C'$ is the linear map $H_n(f)\colon H_n(C) \to H_n(C')$ induced by f_n.

DEFINITION. A *chain homotopy* from the chain map $f\colon C \to C'$ to the chain map $g\colon C \to C'$ is a family of linear maps $K_n\colon C_n \to C'_{n+1}$ such that

$$\partial'_{n+1} \circ K_n + K_{n-1} \circ \partial_n = g_n - f_n$$

for every n. The chain maps $f\colon C \to C'$ and $g\colon C \to C'$ are *chain homotopic* when there exists such a chain homotopy.

PROPOSITION. If the chain maps $f\colon C \to C'$ and $g\colon C \to C'$ are chain homotopic, the induced homomorphisms $H_n(f)$, $H_n(g)\colon H_n(C) \to H_n(C')$ are equal.

COROLLARY. A map $f\colon X \to Y$ induces a homomorphism $\widetilde{H}_n(f)\colon \widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ between the reduced homology modules of X and Y. If f and $g\colon X \to Y$ are homotopic, then $\widetilde{H}_n(f) = \widetilde{H}_n(g)$.

4. Exact sequences of chain complexes

Recall that a short exact sequence of R–modules is a sequence of modules and maps

$$0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} 0$$

where $\operatorname{Ker} f = \operatorname{Im} 0$ (so that f is injective), $\operatorname{Ker} g = \operatorname{Im} f$ and $\operatorname{Ker} 0 = \operatorname{Im} g$ (so that g is surjective).

DEFINITION. A *short exact sequence of chain complexes* is a seqence of chain complexes and chain maps

$$0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} 0$$

such that each $0 \xrightarrow{0} A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n \xrightarrow{0} 0$ is a short exact sequence of R–modules.

In particular, we have a commutative grid[1]

$$
\begin{array}{ccccccccc}
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} \\
0 & \xrightarrow{0} & A_{n+1} & \xrightarrow{f_{n+1}} & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} \\
0 & \xrightarrow{0} & A_{n} & \xrightarrow{f_{n}} & B_{n} & \xrightarrow{g_{n}} & C_{n} & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0} \\
0 & \xrightarrow{0} & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle 0}
\end{array}
$$

where each vertical line is a chain complex, and where each horizontal line is a short exact sequence of R–modules.

THEOREM. (Snake Lemma) Let $0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{0} 0$ be a short exact sequence of chain complexes. Then there exists a long exact sequence

$$
\begin{array}{l}
\cdots \xrightarrow{\delta_{n+2}} H_{n+1}(A) \xrightarrow[H_{n+1}(f)]{} H_{n+1}(B) \xrightarrow{H_{n+1}(f)} H_{n+1}(C) \\
\xrightarrow{\delta_{n+1}} H_{n}(A) \xrightarrow[H_{n}(f)]{} H_{n}(B) \xrightarrow{H_{n}(f)} H_{n}(C) \\
\xrightarrow{\delta_{n}} H_{n-1}(A) \xrightarrow[H_{n-1}(f)]{} H_{n-1}(B) \xrightarrow{H_{n-1}(f)} H_{n-1}(C) \xrightarrow{\delta_{n-1}} \cdots
\end{array}
$$

between the corresponding homology modules.

COMPLEMENT. The homomorphism $\delta_n \colon H_n(C) \to H_{n-1}(A)$ is defined as follows. Let $c_n \in C_n$ represent $[c_n] \in H_n(C)$. Then $\delta_n([c_n]) = [a_{n-1}]$ where $f_{n-1}(a_{n-1}) = \partial b_n$ for an arbitrary $b_n \in B_n$ such that $g_n(b_n) = c_n$.

The homomorphism δ_n is called the *connecting homomorphism* of the long exact sequence.

---

[1]I hope you are impressed by my commutative diagrams. It's a first time I am using this set of TEX macros, and I am really proud of myself.

The proof of the Snake Lemma is by "diagram chasing". It uses all nodes and maps of the grid

$$
\begin{array}{ccccccccc}
 & & & & B_{n+1} & \xrightarrow{g_{n+1}} & C_{n+1} & \xrightarrow{0} & 0 \\
 & & & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
 & & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{0} & 0 \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & \\
0 & \xrightarrow{0} & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & & \\
 & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle \partial} & & & & \\
 & & A_{n-2} & \xrightarrow{f_{n-2}} & B_{n-2} & & & &
\end{array}
$$

5. The Five Lemma

The same kind of diagram chasing proof gives:

THEOREM. (Five Lemma) Suppose that we have a commutative diagram of modules and linear maps

$$
\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{k} & E \\
{\scriptstyle \alpha}\downarrow{\scriptstyle \cong} & & {\scriptstyle \beta}\downarrow{\scriptstyle \cong} & & {\scriptstyle \gamma}\downarrow & & {\scriptstyle \delta}\downarrow{\scriptstyle \cong} & & {\scriptstyle \varepsilon}\downarrow{\scriptstyle \cong} \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{k'} & E'
\end{array}
$$

where the horizontal rows are exact sequences. If α, β, δ and ε are isomorphisms, then the middle map γ is also an isomorphism.

(Not all the hypotheses are needed here, but the statement is easier to remember in this way.)

CHAPTER 8

Relative homology

1. Relative homology

If $A \subset X$, any singular simplex valued in A can also be considered as a simplex valued in X. We can therefore consider $\mathcal{S}_n(A)$ as a subset of $\mathcal{S}_n(X)$, and $C_n(A)$ as a sub-module of $C_n(X)$.

DEFINITION. The module of *relative chains* in X modulo A is the quotient module $C_n(X, A) = C_n(X)/C_n(A)$.

The boundary map $\partial_n \colon C_n(X) \to C_{n-1}(X)$ clearly sends $C_n(A)$ to $C_{n-1}(A)$, and consequently induces a quotient map $\bar{\partial}_n \colon C_n(X, A) \to C_{n-1}(X, A)$. It is immediate that $\bar{\partial}_n \circ \bar{\partial}_{n-1} = 0$ since $\partial_n \circ \partial_{n-1} = 0$, so that $C(X, A) = \{C_n(X, A), \bar{\partial}_n\}_{n \in \mathbb{N}}$ is a chain complex.

DEFINITION. The *n–th relative homology module of X modulo A* is the homology module $H_n(X, A)$ of the above chain complex $C(X, A)$.

In particular, $H_n(X, A)$ is a quotient of quotients, which is not very convenient. The following is more convenient to understand what we are doing.

DEFINITION. The *module of relative n–cycles of X modulo A* is

$$Z_n(X, A) = \{c \in C_n(X); \partial c \in C_n(A)\}.$$

The *module of relative n–boundaries of X modulo A* is

$$B_n(X, A) = \{c \in C_n(X); \exists c' \in C_{n+1}(X), c - \partial c' \in C_n(A)\}.$$

THEOREM.

$$H_n(X, A) \cong Z_n(X, A)/B_n(X, A).$$

REMARK. If $A = \varnothing$, $H_n(X, \varnothing) = H_n(X)$.

PROPOSITION. If $\{X_i\}_{i \in I}$ is the family of path connected components of X, and if $A_i = A \cap X_i$, then

$$H_n(X, A) = \bigoplus_{i \in I} H_n(X_i, A_i).$$

2. Homomorphism induced by maps

Suppose that $A \subset X$ and $B \subset Y$.

Definition. A *map* $f\colon (X,A) \to (Y,B)$ from the pair (X,A) to the pair $Y,B)$ is a (continuous) map $f\colon X \to Y$ such that $f(A) \subset B$.

It is immediate that the homomorphism $C_n(f) : C_n(X) \to C_n(Y)$ induces homomorphisms $C_n(f)\colon C_n(X,A) \to C_n(Y,B)$ and $H_n(f)\colon H_n(X,A) \to H_n(Y,B)$.

Definition. $H_n(f)\colon H_n(X,A) \to H_n(Y,B)$ is the *homomorphism induced in n–th relative homology* by the map $f\colon (X,A) \to (Y,B)$.

3. Relative homology in dimension 0

Theorem. If X is path connected

$$H_0(X,A) \cong \begin{cases} 0 & \text{if } A \neq \varnothing \\ R & \text{if } A = \varnothing \end{cases}$$

4. The long exact sequence in relative homology

By definition, there is a short exact sequence of chain complexes

$$0 \longrightarrow C(A) \longrightarrow C(X) \longrightarrow C(X,A) \longrightarrow 0$$

Applying the Snake Lemma then automatically gives the following long exact sequence

Theorem. (Long Exact Sequence in Relative Homology) If $A \subset X$, there is a long exact sequence

$$\cdots \xrightarrow{\delta_{n+1}} H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow H_{n-1}(X) \longrightarrow H_{n-1}(X,A) \xrightarrow{\delta_{n-1}} \cdots$$

Complement. In the above long exact sequence, the connecting homomorphism δ_n associates to the homology class $[c] \in H_n(X,A)$, represented by $c \in Z_n(X,A)$, the homology class $[\partial c] \in H_{n-1}(A)$.

Note that ∂c is the boundary of something in X (so that $[\partial c] = 0 \in H_{n-1}(X)$), but is not necessarily the boundary of something in A so that $[\partial c] \in H_{n-1}(A)$ is not necessarily 0.

This kind of phenomenon indicates that extreme care should be used when writing equalities between homology classes. It is a good idea to systematically write the homology modules where such equalities take place.

5. Reduced relative homology

One can define reduced relative homology by the convention that

$$\widetilde{C}_n(X,A) = \begin{cases} C_n(X,A) & \text{if } n \neq -1 \\ 0 & \text{if } n = -1 \text{ and } A \neq \varnothing \\ R & \text{if } n = -1 \text{ and } A = \varnothing \end{cases}$$

The homology modules of the chain complex so defined are the *reduced relative homology modules* $\widetilde{H}_n(X,A)$ of X modulo A. Note that $\widetilde{H}_n(X,A)$ is just the standard homology module $H_n(X,A)$ when $A \neq \varnothing$, and the reduced homology module $\widetilde{H}_n(X)$ when $A = \varnothing$.

The definitions are designed so that we still have a short exact sequence of chain complexes, and a Long Exact Sequence in Reduced Relative Homology obtained from the Long Exact Sequence in Relative Homology above by putting tildes everywhere. This long exact sequence tends to have more zero modules when $n = 0$, which makes it more convenient for computations.

6. Examples

Recall that $B^n = \{x \in \mathbb{R}^n; \|x\| \leqslant 1\}$ is the (closed) unit ball in $\mathbb{R}^n$, bounded by the $(n-1)$-dimensional sphere $S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$.

PROPOSITION.

$$H_p(B^n, S^{n-1}) \cong H_{p-1}(S^{n-1}) \text{ if } p > 1$$

$$H_1(B^n, S^{n-1}) \cong \begin{cases} 0 & \text{if } n > 1 \\ R & \text{if } n = 1. \end{cases}$$

REMARK. We can combine the two cases (and their proofs) by using reduced homology. Then, $H_p(B^n, S^{n-1}) \cong \widetilde{H}_{p-1}(S^{n-1})$ for every $p > 0$ and every $n \geqslant 1$.

In the sphere S^n, we can also consider the lower hemisphere B^n_-, consisting of those $x \in S^n \subset \mathbb{R}^{n+1}$ whose last coordinate is $\leqslant 0$ (and homeomorphic to the ball B^n by projection to $\mathbb{R}^n$).

PROPOSITION.

$$H_p(S^n, B^n_-) \cong H_p(S^n) \text{ if } p > 0$$

$$H_0(S^n, B^n_-) \cong \begin{cases} 0 & \text{if } n > 0 \\ R & \text{if } n = 0. \end{cases}$$

These two propositions are still pretty useless right now. However, when combined with the Excision Theorem of the next chapter, they will enable us to compute all homology modules of spheres.

CHAPTER 9

The Excision Theorem

1. Excisions

Suppose that $B \subset A \subset X$. We now have an inclusion map $i\colon (X-B, A-B) \to (X, A)$.

DEFINITION. The subset B can be *excised* from the pair (X, A), or the inclusion map $i\colon (X-B, A-B) \to (X, A)$ is an *excision*, if the induced homomorphism

$$H_n(i)\colon H_n(X-B, A-B) \to H_n(X, A)$$

is an isomorphism for every n.

THEOREM. (Excision Theorem) Suppose that B is "deep inside of A", in the sense that the closure of B is contained in the interior of A. Then B can be excised from (X, A).

The key lemma in the proof, which is of independent interest, is the following. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X, meaning that $X = \bigcup_{i\in I} U_i$ and each U_i is open in X. Let $C_n^{\mathcal{U}}(X) \subset C_n(X)$ be the submodule generated by the $C_n(U_i)$. Namely, $C_n^{\mathcal{U}}(X)$ consists of linear combinations of singular n–simplices in X whose image is completely contained in one of the U_i. The $C_n^{\mathcal{U}}(X)$ are clearly well-behaved under the boundary maps, and therefore form a chain complex with homology modules $H_n^{\mathcal{U}}(X)$. The inclusion maps $C_n^{\mathcal{U}}(X) \to C_n(X)$ form a chain map, inducing homomorphisms $H_n^{\mathcal{U}}(X) \to H_n(X)$.

THEOREM. The above homomorphisms $H_n^{\mathcal{U}}(X) \to H_n(X)$ are isomorphisms.

2. Subdivisions

Let $\mathfrak{S}_n$ be the symmetric group of order n, considered as the group of bijections of the set $\{0, 1, \dots, n-1\}$. For every $s \in \mathfrak{S}_{n+1}$, consider the map $\delta_s\colon \Delta_n \to \Delta_n$ defined by

$$\delta_s(x_0, x_1, \dots, x_i, \dots, x_n) = \left(\tfrac{x_{s(0)}}{1}, \tfrac{x_{s(0)}+x_{s(1)}}{2}, \dots, \tfrac{\sum_{j=0}^{i} x_{s(j)}}{i+1}, \dots, \tfrac{\sum_{j=0}^{n} x_{s(j)}}{n+1}\right).$$

In particular, the $\delta_s(\Delta_n)$ are n–simplices in $\mathbb{R}^{n+1}$, whose vertices are barycenters of subsets of the set of vertices of the standard simplex Δ_n. By induction, one easily sees that, as s ranges over all permutations of $\mathfrak{S}_n$, the $\delta_s(\Delta_n)$ cover all of Δ_n, and have disjoint interiors in Δ_n.

Define a linear map $\Sigma_n : C_n(X) \to C_n(X)$ by the property that, for every singular simplex $\sigma\colon \Delta_n \to X$,

$$\Sigma_n(\sigma) = \sum_{s\in\mathfrak{S}_{n+1}} \operatorname{sign}(s)\,\sigma\circ\delta_s$$

where $\operatorname{sign}(s) = \pm 1$ is the signature of s.

LEMMA. The Σ_n form a chain map $\Sigma\colon C(X) \to C(X)$.

LEMMA. The chain map $\Sigma\colon C(X) \to C(X)$ is chain homotopic to the identity.

For every $k \geqslant 0$, let $\Sigma^k = \Sigma\circ\Sigma\circ\cdots\circ\Sigma$ be the chain map $C(X) \to C(X)$ be obtained by composing Σ with itself k times.

LEMMA. For every $k \geqslant 0$, the chain map $\Sigma^k\colon C(X) \to C(X)$ is chain homotopic to the identity.

For every singular simplex $\sigma\colon \Delta_n \to X$, the chain $\Sigma^k(\sigma)$ is a linear combination of simplices $\sigma\circ\delta_{s_1}\circ\delta_{s_2}\circ\cdots\circ\delta_{s_k}$ where $s_1, s_2, \ldots, s_k \in \mathfrak{S}_n$.

LEMMA. Endow $\mathbb{R}^{n+1}$ with the distance $d(x,y) = \max_i |x_i - y_i|$. For every s_1, s_2, $\ldots$, $s_k \in \mathfrak{S}_n$, the image of $\delta_{s_1}\circ\delta_{s_2}\circ\cdots\circ\delta_{s_k}\colon \Delta_n \to \Delta_n$ has diameter $\leqslant (\frac{n-1}{n})^k$.

The proof of the Excision Theorem follows from the following result, applied to $U = \operatorname{int}(A)$ and $V = X - \operatorname{cl}(B)$.

PROPOSITION. Let U and V be two open subsets of X such that $X = U \cup V$. Let $C_n^{UV}(X) \subset C_n(X)$ be the submodule generated by all singular simplices $\sigma\colon \Delta_n \to X$ whose image is, either completely contained in U, or completely contained in V. Let $H_n^{UV}(X)$ be the homology module of the chain complex formed by the $C_n^{UV}(X)$. Then, the homomorphism $H_n^{UV}(X) \to H_n(X)$ induced by the inclusion map $C_n^{UV}(X) \to C_n(X)$ is an isomorphism.

3. Application: homology modules of spheres

As an application of the Excision Theorem, consider the upper and lower hemispheres $B^n_\pm$ of the sphere S^n.

LEMMA. $H_p(S^n, B^n_-) \cong H_p(B^n_+, S^{n-1})$.

Combining this with the isomorphisms $H_p(B^n, S_{n-1}) \cong H_{p-1}(S^{-1})$ and $H_p(S^n, B^n_-) \cong H_p(S^n)$ provided by the Long Exact Sequence in Relative Homology gives the following computation.

THEOREM.

$$H_p(S^n) \cong \begin{cases} 0 & \text{if } p \neq n, 0 \\ R & \text{if } p = n, 0 \text{ and } n > 0 \\ R\oplus R & \text{if } p = n = 0 \end{cases}$$

COROLLARY. There is no retraction $r\colon B^n \to S^{n-1}$.

COROLLARY. (Brouwer Fixed Point Theorem) Every continuous map $f\colon B^n \to B^n$ has a fixed point, namely a point $x \in X$ with $f(x) = x$.

A *n–dimensional topological manifold* is a space M such that every $x \in M$ admits a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. For instance, an open subset of $\mathbb{R}^n$ is an n–dimensional topological manifold.

LEM. If M is an n–dimensional topological manifold and if $x \in M$,

$$H_p(M, M - \{x\}) \cong \begin{cases} 0 & \text{if } p \neq n \\ 0 & \text{if } p = n \end{cases}$$

CHAPTER 10

The Mayer-Vietoris exact sequence

1. The Mayer-Vietoris long exact sequence

THEOREM. (Mayer-Vietoris Exact Sequence) For X_1, $X_2 \subset X$, consider the inclusion maps $i_1 \colon X_1 \cap X_2 \to X_1$, $i_2 \colon X_1 \cap X_2 \to X_2$, $j_1 \colon X_1 \to X_1 \cup X_2$, $j_2 \colon X_2 \to X_1 \cup X_2$, and suppose that the inclusion map $(X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$ is an excision. Then there exists a long exact sequence

$$\cdots \xrightarrow{\delta_{n+2}} H_{n+1}(X_1 \cap X_2) \xrightarrow{f_{n+1}} H_{n+1}(X_1) \oplus H_{n+1}(X_2) \xrightarrow{g_{n+1}} H_{n+1}(X_1 \cup X_2)$$

$$\xrightarrow{\delta_{n+1}} H_n(X_1 \cap X_2) \xrightarrow{f_n} H_n(X_1) \oplus H_n(X_2) \xrightarrow{g_n} H_n(X_1 \cup X_2)$$

$$\xrightarrow{\delta_n} H_{n-1}(X_1 \cap X_2) \xrightarrow{f_{n-1}} H_{n-1}(X_1) \oplus H_{n-1}(X_2) \xrightarrow{g_{n-1}} H_{n-1}(X_1 \cup X_2) \xrightarrow{\delta_{n-1}} \cdots$$

where $f_n = H_n(i_1) \oplus H_n(i_2)$, $g_n = H_n(j_1) - H_n(j_2)$ and δ_n is a connecting homomorphism provided by the Snake Lemma.

The key step in the proof is the following result.

LEMMA. Let $C_n^{X_1X_2}(X_1 \cup X_2) = C_n(X_1) + C_n(X_2) \subset C_n(X)$ consist of linear combinations of n-simplicies that are, either completely contained in X_1, or completely contained in X_2. Let $H_n^{X_1X_2}(X_1 \cup X_2)$ be the homology module of the corresponding chain complex. If the inclusion map $(X_1, X_1 \cap X_2) \to (X_1 \cup X_2, X_2)$ is an excision, the homomorphism $H_n^{X_1X_2}(X_1 \cup X_2) \to H_n(X_1 \cup X_2)$ induced by the inclusion $C_n^{X_1X_2}(X_1 \cup X_2) \subset C_n(X)$ is an isomorphism.

COMPLEMENT. The connecting homomorphism $\delta_n \colon H_n(X_1 \cup X_2) \to H_{n-1}(X_1 \cap X_2)$ is defined as follows. By the Lemma, each homology class $[c] \in H_n(X_1 \cup X_2)$ can be written as $[c] = [c_1 + c_2] \in H_n(X_1 \cup X_2)$ with $c_1 \in C_n(X_1)$ and $c_2 \in C_n(X_2)$. Then $\delta_n([c_1 + c_2]) = [\partial c_1] = [-\partial c_2] \in H_{n-1}(X_1 \cap X_2)$.

2. Some computations

PROPOSITION. Let $T^n = S^1 \times S^1 \times \cdots \times S^1$ be the n-dimensional torus. Then

$$H_p(T^n) \cong R^{\binom{n}{p}}$$

where $\binom{n}{p}$ is the binomial coefficient.

PROPOSITION. Let $\tau\colon S^n \to S^n$ be the reflection across $0\times\mathbb{R}^n$ defined by $\tau(x_0, x_1, \ldots, x_n) = (-x_0, x_1, \ldots, x_n)$. Then $H_n(\tau)$ acts by multiplication by -1 on $H_n(S^n) \cong R$.

PROPOSITION. Let K be the Klein bottle. In the coefficient ring R, let $\psi_2\colon R \to R$ be the multiplication by 2 defined by $\psi_2(a) = 2a$. Then,

$$H_n(K) \cong \begin{cases} \ker \psi_2 & \text{if } n = 2 \\ R \oplus (R/\operatorname{im} \psi_2) & \text{if } n = 1 \\ R & \text{if } n = 0 \\ 0 & \text{if } n \neq 0, 1, 2. \end{cases}$$

For instance, $H_1(K;\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z}_2$ and $H_2(K;\mathbb{Z}) = 0$, $H_1(K;\mathbb{R}) \cong \mathbb{R}$ and $H_2(K;\mathbb{R}) = 0$, and $H_1(K;\mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $H_2(K;\mathbb{Z}_2) = \mathbb{Z}_2$.

CHAPTER 11

Homology of manifolds

1. Manifolds

DEFINITION. A *(topological) manifold* of dimension n is a topological space X such that every $x \in X$ admits a neighborhood U_x which is homeomorphic to an open subset V_x of $\mathbb{R}^n$.

In particular, there exists a family $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ where:

1. each U_i is open in X and $X = \bigcup_{i \in I} U_i$
2. each $\varphi_i \colon U_i \to V_i$ is a homeomorphism from U_i to an open subset $V_i \subset \mathbb{R}^n$.

Such a family $\mathcal{A}$ is a *topological atlas* for X.

PROPOSITION. If X is an n–dimensional manifold, $H_n(X, X - \{x\}) \cong R$ for every $x \in X$.

PROOF. If $y \in V_x \subset \mathbb{R}^n$ corresponds to x,

$$\begin{aligned} H_n(X, X-\{x\}) &\cong H_n(U_x, U_x - \{x\}) \cong H_n(V_x, V_x - \{y\}) \\ &\cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{y\}) \cong \widetilde{H}_{n-1}(S^{n-1}) \cong R \end{aligned}$$

by the Excision Theorem (twice) and the Long Exact Sequence in Relative Homology. □

2. Orientation-preserving homeomorphisms

DEFINITION. A homeomorphism $\varphi \colon U \to V$ between two open subsets U, $V \subset \mathbb{R}^n$ is *R–orientation preserving at $x \in U$* if the composition

$$\begin{aligned} &\widetilde{H}_{n-1}(S^{n-1}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_n(U, U - \{y\}) \\ &\xrightarrow{H_n(\varphi)} H_n(V, V - \{\varphi(x)\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - \{\varphi(x)\}) \cong \widetilde{H}_{n-1}(S^{n-1}) \end{aligned}$$

is the identity map of $\widetilde{H}_{n-1}(S^{n-1})$. The homeomorphism φ is *R–orientation preserving* if it is R–orientation preserving at each $x \in U$.

EXAMPLES.

1. If $R = \mathbb{Z}_2$, every homeomorphism $\varphi \colon U \to V$ is $\mathbb{Z}_2$–orientation preserving (since the identity is the only isomorphism of $H_{n-1}(S^{n-1}) = \mathbb{Z}_2$).

2. Let $\varphi\colon U \to V$ be the restriction of a linear map $\mathbb{R}^n \to \mathbb{R}^n$. If there exists an $a \in R$ with $2a \neq 0$ (for instance $R = \mathbb{Z}$, $\mathbb{R}$ or $\mathbb{Z}_3$), then φ is R–orientation preserving at $x \in U$ if and only if $\det(\varphi) > 0$.
3. Let $\varphi\colon U \to V$ be differentiable at $x \in U$, and assume in addition that the jacobian matrix $T_x\varphi = \left(\frac{\partial\varphi_j(x)}{\partial x_i}\right)$ is invertible. If, again, there exists an $a \in R$ with $2a \neq 0$, then φ is R–orientation preserving at $x \in U$ if and only if $\det\left(\frac{\partial\varphi_j(x)}{\partial x_i}\right) > 0$.

PROPOSITION. Let $\varphi : U \to V$ be a homeomorphism between *connected* open subsets U, $V \subset \mathbb{R}^n$. Then φ is R–orientation preserving at *some* $x \in U$ if and only if it is R–orientation preserving at *every* $x \in U$.

3. Oriented manifolds

DEFINITION. An atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ for a manifold X is *R–oriented* if, for every i, $j \in I$, the coordinate change homeomoprhism

$$\varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is R–orientation preserving.

DEFINITION. The manifold X is *R–orientable* if it admits an R–oriented atlas. An *R–orientation* for X is an equivalence class of atlases of R–oriented atlases, for the equivalence relation $\sim$ defined by the property that $\mathcal{A} \sim \mathcal{A}'$ if and only if $\mathcal{A} \cup \mathcal{A}'$ is R–oriented.

In particular, every manifold X is $\mathbb{Z}_2$–orientable, and admits a unique $\mathbb{Z}_2$–orientation.

LEMMA. A connected manifold X admits exactly zero or two $\mathbb{Z}$–orientations.

If you know what this words mean, you will notice that a differentiable manifold is $\mathbb{Z}$–orientable if and only if it is orientable in the differentiable sense.

4. Homology of manifolds

THM. If X is a manifold of dimension n, then $H_p(X) = 0$ for every $p > n$.

We now consider the n–dimensional homology. Suppose that X is *oriented*, with oriented atlas $\{(U_i, \varphi_i)\}_{i\in I}$. For every $x \in X$, we now have a composition

$$H_n$$