Notes for Math 535
Differential Geometry
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CHAPTER 1

A crash course in pointset topology

Topological spaces.

Definition 1.1. A *topological space* consists of a set X and of a family $\mathcal{T}$ of subsets of X such that:

(1) the empty set $\emptyset$ and the whole space X are both elements of $\mathcal{T}$;
(2) if $\{U_i\}_{i\in I}$ is a (possibly infinite) family of elements of $\mathcal{T}$, their union $\bigcup_{i\in I} U_i$ is also in $\mathcal{T}$.
(3) if $\{U_i\}_{i=1,2,\dots,n}$ is a *finite* family of elements of $\mathcal{T}$, their intersection $\bigcap_{i=1,2,\dots,n} U_i$ is also in $\mathcal{T}$.

The family $\mathcal{T}$ is then called a *topology* on the set X. The elements of $\mathcal{T}$ are called *open subsets* of X for this topology.

Recall that, given a family $\{X_i\}_{i\in I}$ of sets X_i indexed by a set I, their union $\bigcup_{i\in I}$ is the set of all x for which there exists an $i \in I$ with $x \in X_i$. Their intersection $\bigcap_{i\in I}$ is the set of all x such that $x \in X_i$ for every $i \in I$. Note that I does not need to be finite.

Example 1.2 (Fundamental example). Let (X, d) be a *metric space*, namely a set X endowed with a function d: $X \times X \to \mathbb{R}$ such that

(1) $d(x,y) \geqslant 0$ for every x, y;
(2) $d(x,y) = 0$ if and only if $x = y$;
(3) $d(y,x) = d(x,y)$ for every x, y;
(4) $d(x,z) \leqslant d(x,y) + d(y,z)$ for every x, y, z.

Given x in such a metric space X and given $\varepsilon > 0$, let the open ball of radius ε centered at x be $B(x,\varepsilon) = \{y \in X; d(x,y) < \varepsilon\}$.

Define $\mathcal{T}$ by the property that $U \subset X$ is an element of $\mathcal{T}$ if and only if, for every $x \in U$, there exists a ball $B(x,\varepsilon)$ which is contained in U. Then, $\mathcal{T}$ is a topology for X.

Definition 1.3. If x is an element of a topological space X, a *neighborhood* of x is a subset $W \subset X$ for which there exists an open subset U of X with $x \in U \subset W$.

Note that the terminology is not standard. This one is used in most research articles. Many textbook require a neighborhood to be open, namely just an open subset containing x.

Definition 1.4. A topological space X is *Hausdorff* if, for every x, $y \in X$ with $x \neq y$, there exists a neighborhood U of x and a neighborhood V of y such that $U \cap V = \emptyset$.

For instance, the topology of a metric space is always Hausdorff.

DEFINITION 1.5. A subset A of a topological space X is *dense* if every non-empty open subset of X contains an element of A. The topological space X is *separable* if it contains a dense subset which is countable.

For instance, $\mathbb{R}^n$ with the topology associated to the metric $d(x,y) = \|x-y\|$ is separable, since it contains the dense set $\mathbb{Q}^n$.

Continuous functions.

DEFINITION 1.6. Let $f\colon X \to Y$ be a map between two topological spaces. The map f is *continuous* if, for every open subset U of Y, its pre-image $f^{-1}(U)$ is an open subset of X.

Recall that the pre-image of U is $f^{-1}(U) = \{x \in X; f(x) \in U\}$.

For metric spaces, this definition of continuity turns out to be equivalent to a more familiar one:

PROPOSITION 1.7 ($\varepsilon - \delta$ definition of continuity). *Let $f : X \to Y$ be a map between metric spaces (X, d_X) and (Y, d_Y). Then f is continuous in the above sense if and only if, for every $x \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ for every $x' \in X$ with $d_X(x, x') < \delta$.*

DEFINITION 1.8. A *homeomorphism* between two topological spaces X and Y is a bijection $f\colon X \to Y$ such that f and f^{-1} are both continuous.

If $f\colon X \to Y$ is a homeomorphism and U is a subset of X, it follows from definitions that U is open in X if and only if $f(U)$ is open in Y. Therefore, f establishes a 'dictionary', which enables us to translate any topological property of X to a similar property of Y. A homeomorphism is therefore an 'isomorphism of topological spaces'.

The subspace topology. Let X be a topological space, with topology $\mathcal{T}$. If X' is a subset of X, it is immediate that

$$\mathcal{T}' = \{U' \subset X'; \exists U \in \mathcal{T}, U' = X' \cap U\}$$

is a topology for X'.

DEFINITION 1.9. The topology $\mathcal{T}'$ is the *subspace topology* induced on X' by the topology of X.

The quotient topology. Let $p\colon X \to Y$ be a surjective map from a topological space X to a set Y. This situation typically arises when (and exactly when) Y is the quotient of X under an equivalence relation $\sim$, namely when Y is the set of equivalence classes of $\sim$, and when p is the natural projection.

The topology of X induces a topology $\mathcal{T}$ on Y, defined by

$$\mathcal{T} = \{U \subset Y; p^{-1}(U) \text{ open in } X\}.$$

DEFINITION 1.10. The topology $\mathcal{T}$ is the *quotient topology* induced on Y by the topology of X.

The quotient topology is specially designed for the following property.

PROPOSITION 1.11. *Let $f\colon X \to Z$ be a map inducing a map $g\colon Y \to Z$ in the sense that $f = g \circ p$. If Z is a topological space and if f is continuous, then g is continuous for the quotient topology on Y.*

CHAPTER 2

Review of differentiable functions in $\mathbb{R}^n$

The real n–space $\mathbb{R}^n$ is the set of all n–uples $(x_1, x_2, \ldots, x_n)$ where each x_i is a real number.

The *norm* of a vector $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ is

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

DEFINITION 2.1. The *open ball* of radius ε centered at $x \in \mathbb{R}^n$ is

$$B(x, \varepsilon) = \{y \in \mathbb{R}^n; \|x - y\| < \varepsilon\}$$

DEFINITION 2.2. A subset $U \subset \mathbb{R}^n$ is *open* if, for every $x \in U$, there is an $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$.

Now, consider U open in $\mathbb{R}^n$, and a function

$$\begin{aligned} f\colon U &\longrightarrow \mathbb{R}^p \\ (x_1, \ldots, x_n) &\longmapsto (f_1(x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n)) \end{aligned}$$

DEFINITION 2.3. The function $f\colon U \to \mathbb{R}^p$ is *(infinitely) differentiable* at $x \in U$ if all the partial derivatives $\frac{\partial}{\partial x_{i_1}} \frac{\partial}{\partial x_{i_2}} \cdots \frac{\partial}{\partial x_{i_k}} f_j(x)$ exist. A *differentiable* function $f\colon U \to \mathbb{R}^p$ is one that is differentiable at each $x \in U$.

Note that one needs U to be open for the partial derivatives to make sense.

1. The tangent map

DEFINITION 2.4. The *derivative*, or *differential map*, or *tangent map* (preferred terminology) of $f\colon U \to \mathbb{R}^p$ at $x \in U$ is the linear map $T_x f\colon \mathbb{R}^n \to \mathbb{R}^p$ whose matrix is

$$\left(\frac{\partial f_j}{\partial x_i}(x)\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_p}{\partial x_1}(x) & \frac{\partial f_p}{\partial x_2}(x) & \cdots & \frac{\partial f_p}{\partial x_n}(x) \end{pmatrix}$$

PROPOSITION 2.5 (Linear Approximation). *The tangent map $T_x f$ is the only linear map $L\colon \mathbb{R}^n \to \mathbb{R}^p$ such that*

$$\lim_{h \to 0} \frac{f(x + h) - f(x) - L(h)}{\|h\|} = 0$$

Geometric interpretation of the tangent map. Given x and $v \in \mathbb{R}^n$, there is always a small parametrized curve $\alpha\colon (-\varepsilon, \varepsilon) \to U$, namely a differentiable map defined over the open interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$, such that $\alpha(0) = x$ and $\alpha'(0) = v$. For instance, one can take $\alpha(t) = x + tv$.

If, in addition, we have a function $f\colon U \to \mathbb{R}^p$, the parametrized curve α in U gives a parametrized curve $f \circ \alpha\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^p$ in $\mathbb{R}^p$, defined by $f \circ \alpha(t) = f(\alpha(t))$. Note that $f \circ \alpha(0) = f(x)$. Let us consider its tangent vector.

PROPOSITION 2.6. $T_x f(v) = (f \circ \alpha)'(0)$.

This provides an interpretation of the tangent map $T_x f\colon \mathbb{R}^n \to \mathbb{R}^p$ in terms of parametrized curves: To determine $T_x f(v)$, pick any parametrized curve α in U with $\alpha(0) = x$ and $\alpha'(0) = v$ and consider the parametrized curve $f \circ \alpha$ in $\mathbb{R}^p$; then $T_x f(v)$ is the tangent vector $(f \circ \alpha)'(0)$ to this curve.

An immediate application is the following. Let U be open in $\mathbb{R}^m$, let V be open in $\mathbb{R}^n$, let W be open in $\mathbb{R}^p$ and consider two functions $f : U \to V$ and $g\colon V \to W$. For $x \in U$, we have tangent maps $T_x f\colon \mathbb{R}^m \to \mathbb{R}^n$, $T_{f(x)} g\colon \mathbb{R}^n \to \mathbb{R}^p$ and $T_x(g \circ f)\colon \mathbb{R}^m \to \mathbb{R}^p$.

PROPOSITION 2.7 (Generalized Chain Rule). $T_x(g \circ f) = T_{f(x)} g \circ T_x f$.

2. Diffeomorphims and the Inverse Function Theorem

DEFINITION 2.8. A *diffeomorphism* between two open subsets U and V of $\mathbb{R}^n$ is a bijection $f\colon U \to V$ such that both f and f^{-1} are differentiable.

PROPOSITION 2.9. *If $f : U \to V$ is a diffeomorphism, then the tangent map $T_x f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism.*

This result has a converse statement, which will play a very important role in the course.

THEOREM 2.10 (Inverse Function Theorem). *Let $f\colon U \to V$ be a differentiable map between two open subsets U, V of $\mathbb{R}^n$. Suppose that, at the point $x \in U$, the tangent map $T_x f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Then there is an open neighborhood $U' \subset U$ of x such that $f(U')$ is open in $\mathbb{R}^n$ such that the restriction $f_{|U'}\colon U' \to f(U')$ is a diffeomorphism.*

The proof can be found in any good textbook in (second semester) real analysis or in advanced calculus. We will assume this result in the class.

CHAPTER 3

Submanifolds of $\mathbb{R}^n$

1. Definitions

DEFINITION 3.1. An m–dimensional *submanifold* of $\mathbb{R}^n$ is a subset $M \subset \mathbb{R}^n$ such that, for every $x \in M$, there exists an open neighborhood U of x in $\mathbb{R}^n$ and a diffeomorphism $\varphi\colon U \to V$ from U to an open subset $V \subset \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ such that $\varphi(U \cap M) = V \cap (\mathbb{R}^m \times \{0\})$

Recall that a *neighborhood* of x is a subset containing an open subset containing x.

EXAMPLE 3.2. An n–dimensional submanifold of $\mathbb{R}^n$ is exactly an open subset of $\mathbb{R}^n$.

EXAMPLE 3.3. A 0–dimensional submanifold of $\mathbb{R}^n$ is a discrete subset, namely such that every point $x \in M$ has a neighborhood U such that $U \cap M = \{x\}$. (Recall that $\mathbb{R}^0$ is a vector space of dimension 0, namely $\mathbb{R}^0 = \{0\}$.)

EXAMPLE 3.4. Let $f : U \to \mathbb{R}^p$ be a differentiable function defined on an open subset U of $\mathbb{R}^n$. Then the *graph* $G_f = \{(x, y) \in U \times \mathbb{R}^p; y = f(x)\}$ is an n–dimensional submanifold M of $\mathbb{R}^{n+p}$.

We will see in §4 that every submanifold of $\mathbb{R}^{n+p}$ is locally of this type, in a sense which will be precised there.

EXAMPLE 3.5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \sqrt{x^2 + y^2}$. The graph G_f is *not* a submanifold of $\mathbb{R}^3$. (Note that f is not differentiable at $(0, 0)$.)

EXAMPLE 3.6. The *n–dimensional sphere*

$$S^n = \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}; \sum_{i=0}^{n} x_i^2 = 1 \right\}$$

is an n–dimensional submanifold of $\mathbb{R}^{n+1}$.

2. Tangent vectors

DEFINITION 3.7. Let M be an m–dimensional submanifold of $\mathbb{R}^n$. A vector $v \in \mathbb{R}^n$ is *tangent to M at x* if there exists a curve $\alpha\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ with image contained in M such that $\alpha(0) = x$ and $\alpha'(0) = v$.

The set $T_xM\colon \mathbb{R}^n \to \mathbb{R}^p$ of tangent vectors of M at x is the *tangent space* of M at x.

PROPOSITION 3.8. *The tangent space T_xM is an m–dimensional linear subspace of $\mathbb{R}^m$.*

3. Submanifolds defined by equations

THEOREM 3.9. *Let $f\colon U \to \mathbb{R}^p$ be a function defined on an open subset $U \subset \mathbb{R}^n$. For $y_0 \in \mathbb{R}^p$, consider*

$$M = \{x \in U; f(x) = y_0\}.$$

Suppose that, for every $x \in M$, the tangent map $T_x f\colon \mathbb{R}^n \to \mathbb{R}^p$ is surjective. Then M is a submanifold of $\mathbb{R}^n$ of dimension $n-p$.

In other words, if $M \subset U$ open $\subset \mathbb{R}^n$ is defined by p equations, and it these equations are well-behaved, then M is a submanifold of dimension $n-p$.

PROPOSITION 3.10. *For data as above, the tangent space $T_x M$ is the kernel of the tangent map $T_x f\colon \mathbb{R}^n \to \mathbb{R}^p$.*

EXAMPLE 3.11. The *n–dimensional sphere*

$$S^n = \left\{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}; \sum_{i=0}^{n} x_i^2 = 1\right\}$$

is an n–dimensional submanifold of $\mathbb{R}^{n+1}$.

EXAMPLE 3.12. Let $\mathrm{M}_n(\mathbb{R}) = \mathbb{R}^{n^2}$ be the set of n by n matrices. The subsets

$$\begin{aligned}
\mathrm{GL}_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}); \det A \neq 0\}\\
\mathrm{SL}_n(\mathbb{R}) &= \{A \in M_n(\mathbb{R}); \det A = 1\}\\
\mathrm{O}_n &= \{A \in M_n(\mathbb{R}); AA^{\mathrm{t}} = \mathrm{Id}\}\\
\mathrm{SO}_n &= \{A \in M_n(\mathbb{R}); AA^{\mathrm{t}} = \mathrm{Id} \text{ and } \det A = 1\} = \{A \in \mathrm{O}_n; \det A > 0\}
\end{aligned}$$

are submanifolds of $\mathrm{M}_n(\mathbb{R}) = \mathbb{R}^{n^2}$, of respective dimensions n^2, n^2-1, $\frac{1}{2}n(n-1)$ and $\frac{1}{2}n(n-1)$.

4. Submanifolds as graphs

An example of submanifold of $\mathbb{R}^n$ was the graph $G_f = \{(x,y) \in U \times \mathbb{R}^{n-m}; y = f(x)\}$ of a function $f\colon U \to \mathbb{R}^{n-m}$, where U is an open subset of $\mathbb{R}^m$. We show that this example is actually quite general, because every submanifold of $\mathbb{R}^n$ is *locally* of this type.

THEOREM 3.13. *Let M be an m–dimensional submanifold of $\mathbb{R}^n$. For every $x \in M$, after possible reshuffling of the coordinates of $\mathbb{R}^n$, there exists a neighborhood U of x in $\mathbb{R}^n$ and a differentiable function $f\colon V \to \mathbb{R}^{n-m}$, defined on an open subset V of $\mathbb{R}^m$, such that $U \cap M$ is equal to the graph G_f of the function f.*

5. Coordinate charts and atlases

The following is a preparation for the definition of abstract manifolds in the next chapter.

PROPOSITION 3.14. *Let M be an m–dimensional submanifold of $\mathbb{R}^n$. Then there exists a family $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ where:*

(1) *U_i is an open subset of M (for the subspace topology) for every $i \in I$, and $M = \bigcup_{i\in I} U_i$;*
(2) *for every $i \in I$, $\varphi_i\colon U_i \to V_i$ is a homeomorphism from U_i to an open subset V_i of $\mathbb{R}^m$;*

(3) *for every* i, $j \in I$, *the composition* $\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_i \cap U_j)} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ *is a diffeomorphism.*

For the last condition, note that $\varphi_i(U_i \cap U_j)$ is open in V_i for the subspace topology, and therefore is open in $\mathbb{R}^m$. The same holds for $\varphi_j(U_i \cap U_j)$, so that the requirement that the restriction $\varphi_j \circ \varphi_i^{-1}|_{\varphi_i(U_i \cap U_j)}$ be a diffeomorphism really makes sense.

The pairs (U_i, φ_i) are *coordinate charts* for M, and the collection $\mathcal{A}$ of charts covering M is an *atlas*.

CHAPTER 4

Differentiable manifolds

1. Manifolds

DEFINITION 4.1. A *topological manifold* of dimension n is a separable Hausdorff topological space M such that, for every $x \in M$, there is an open neighborhood U of x and a homeomorphism $\varphi\colon U \to V$ from U to an open subset V of $\mathbb{R}^n$.

DEFINITION 4.2. A *differentiable atlas* for the topological manifold M is a family $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ where:

(1) U_i is an open subset of M for every $i \in I$, and $M = \bigcup_{i \in I} U_i$;
(2) for every $i \in I$, $\varphi_i \colon U_i \to V_i$ is a homeomorphism from U_i to an open subset V_i of $\mathbb{R}^m$;
(3) for every i, $j \in I$, the composition $\varphi_j \circ \varphi_i^{-1}{}_{|\varphi_i(U_i \cap U_j)} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is a diffeomorphism.

The (U_i, φ_i) are the *charts*, or *coordinate patches* of the atlas $\mathcal{A}$, and the diffeomorphisms $\varphi_j \circ \varphi_i^{-1}{}_{|\varphi_i(U_i \cap U_j)}$ are its *changes of charts* or *coordinate changes.*

Very often, we will just write $\varphi_j \circ \varphi_i^{-1}$ for $\varphi_j \circ \varphi_i^{-1}{}_{|\varphi_i(U_i \cap U_j)}$, since this map is only defined on $\varphi_i(U_i \cap U_j)$.

DEFINITION 4.3. A *differentiable structure* on the topological manifold M is a differentiable atlas $\mathcal{A}$ which is maximal, namely for which there is no other differentiable atlas $\mathcal{A}' \neq \mathcal{A}$ with $\mathcal{A} \subset \mathcal{A}'$. A *differentiable manifold* of dimension n is an n–dimensional topological manifold M endowed with a differentiable structure $\mathcal{A}$.

To explain the use of maximal atlases, consider the following easy lemma.

LEMMA 4.4. *Any differentiable atlas $\mathcal{A}$ is contained in a unique maximal atlas, which is equal to the set $\mathcal{A}_{max}$ of all pairs $\{(U', \varphi')$ where U' is open in M, $\varphi'\colon U' \to V'$ is a homeomorphism from U' to an open subset V' of $\mathbb{R}^n$, and where $\varphi' \circ \varphi^{-1}$ is differentiable for every $(U, \varphi) \in \mathcal{A}$.*

An immediate consequence of Lemma 4.4 is that every (not necessarily maximal) atlas on the topological manifold M defines a unique differentiable structure for M. This is more useful in practice, since a maximal atlas is really huge whereas a manifold usually admits atlases of a more reasonable size.

A more natural approach to defining differentiable structures would have been to say that two atlases $\mathcal{A}$ and $\mathcal{A}'$ are *compatible* if their union $\mathcal{A} \cup \mathcal{A}'$ is also a differentiable atlas, namely if any change of chart $\varphi' \circ \varphi^{-1}$ between a chart $(U, \varphi) \in \mathcal{A}$ and a chart $(U', \varphi') \in \mathcal{A}'$ is a diffeomorphism. We could then define a differentiable structure as an equivalence class of the relation "is compatible with" over the set of all possible atlases for M. Lemma 4.4 shows that this definition is equivalent to the first one. Maximal atlases were actually introduced as an elegant but impractical way of avoiding the use of this compatibility condition.

2. Some examples

EXAMPLE 4.5. Every open subset U of $\mathbb{R}^n$ is a differentiable manifold, with atlas $\mathcal{A} = \{(U, \mathrm{Id}_U)\}$, where $\mathrm{Id}_U\colon U \to U$ is the identity map.

EXAMPLE 4.6. More generally, every submanifold of $\mathbb{R}^n$ admits a natural differentiable structure by (the proof of) Proposition 3.14.

EXAMPLE 4.7. Consider the *real projective n-space* $\mathbb{RP}^n$, defined as

$$\begin{aligned}\mathbb{RP}^n &= \{\text{lines through the origin in } \mathbb{R}^{n+1}\}\\ &= (\mathbb{R}^{n+1} - \{0\})/\sim\end{aligned}$$

where the equivalence relation $\sim$ on $\mathbb{R}^{n+1} - \{0\}$ is defined by the property that $x \sim y \Leftrightarrow \exists\lambda \in \mathbb{R}, y = \lambda x$. The set $\mathbb{RP}^n = (\mathbb{R}^{n+1} - \{0\})/\sim$ is endowed with the quotient topology. For $i = 0, 1, \ldots$, n, let U_i denote the image in $\mathbb{RP}^n$ of $\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}; x_i \neq 0\} \subset \mathbb{R}^{n+1} - \{0\}$. Let $\varphi_i\colon U_i \to \mathbb{R}^n$ associate to the class of $(x_0, x_1, \ldots, x_n)$ the element $(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}) \in \mathbb{R}^n$. Then, $\mathcal{A} = \{(U_0, \varphi_0), (U_1, \varphi_1), \ldots, (U_n, \varphi_n)\}$ is a differentiable atlas for $\mathbb{RP}^n$. This turns $\mathbb{RP}^n$ into a differentiable manifold of dimension n.

EXAMPLE 4.8. The *complex projective plane* $\mathbb{CP}^n$ is the space of complex lines through the origin in $\mathbb{C}^{n+1}$. Namely, $\mathbb{CP}^n$ is equal to the quotient space $(\mathbb{C}^{n+1} - \{0\})/\sim$ where $x \sim y \Leftrightarrow \exists\lambda \in \mathbb{C}, y = \lambda x$, and is endowed with the quotient topology. The same formulas as for the real projective space provide charts $\varphi_i\colon U_i \to \mathbb{C}^n = \mathbb{R}^{2n}$, which endows $\mathbb{CP}^n$ with the structure of a differentiable manifold of dimension $2n$.

EXAMPLE 4.9 (Products of manifolds). If $\{(U_i, \varphi_i\}_{i\in I}$ is a differentiable atlas for M and $\{(V_j, \psi_j\}_{j\in J}$ is a differentiable atlas for N, then $\{(U_i \times V_j, \varphi_i \times \psi_j\}_{(i,j)\in I\times J}$is a differentiable atlas for the product $M \times N$ (with the product topology). In particular, if M and N are differentiable manifolds of dimensions m and n, respectively, then $M \times N$ is a differentiable manifold of dimension $m + n$.

3. Differentiable functions between manifolds

DEFINITION 4.10. Let M be a differentiable manifold of dimension m with atlas $\{(U_i, \varphi_i)\}_{i\in I}$, and let N be a differentiable manifold of dimension n with atlas $\{(V_j, \psi_j)\}_{j\in J}$. A function $f\colon M \to N$ is *differentiable at* x if f is continuous and if, for any $i \in I$ and $j \in J$ with $x \in U_i$ and $f(x) \in V_j$, the map $\psi_j \circ f \circ \varphi_i^{-1}\colon \varphi_i(U_i \cap f^{-1}(V_j) \to \psi_j(V_j)$ is differentiable at x in the sense of Chapter 2, which makes sense since it goes from an open subset of $\mathbb{R}^m$ to an open subset of $\mathbb{R}^n$. The map $f\colon M \to N$ is *differentiable* if it is differentiable at every $x \in M$.

LEMMA 4.11. *This does not depend on the choice of i, j such that $x \in U_i$ and $f(x) \in V_j$.*

To a large extent, the notion of differentiable atlas was introduced precisely to make sense of differentiable maps $f\colon M \to N$.

DEFINITION 4.12. A *diffeomorphism* from the manifold M to the manifold N is a bijection $f\colon M \to N$ such that both f and f^{-1} are differentiable.

The manifolds M and N are *diffeomorphic* if there exists a diffeomorphism $f\colon M \to N$.

EXAMPLE 4.13. Let M be a differentiable manifold of dimension m with atlas $\{(U_i, \varphi_i)\}_{i\in I}$. Then, each chart $\varphi_i \colon U_i \to \varphi_i(U_i)$ is a diffeomorphism between the manifold U_i, endowed with the single chart differentiable atlas $\{(U_i, \varphi_i)\}$ and the open subset $\varphi_i(U_i)$ of $\mathbb{R}^m$.

Diffeomorphic manifolds are undistinguishable for any property involving differentiable maps. This is a stronger property than being homeomorphic, which involves only continuity properties. For instance Milnor showed around 1960 that there are exactly 28 differentiable manifolds that are homeomorphic to the 7–dimensional sphere S^7 and not diffeomorphic to each other.

4. Constructing manifolds as quotient spaces

DEFINITION 4.14. An *action* of the group Γ on the manifold M is a group homomorphism ρ from Γ to the group $\mathrm{Diff}(M)$ of diffeomorphisms of M.

In general, if $\gamma \in \Gamma$ and $x \in M$, we write $\gamma x = \rho(\gamma)(x)$ for the image of x under the diffeomorphism $\rho(\gamma)$.

DEFINITION 4.15. The *quotient space* M/Γ of the action of Γ on the manifold M is the set of equivalence classes of the equivalence relation $\sim$ defined by $x \sim y \Leftrightarrow \exists\gamma \in \Gamma, y = \gamma x$, endowed with the quotient topology.

For an arbitrary action, the quotient space M/Γ may not even be Hausdorff.

DEFINITION 4.16. The action of Γ on M is *discontinuous* if, for every compact subset K of M, the set $\{\gamma \in \Gamma; K \cap \gamma K \neq \emptyset\}$ is finite.

DEFINITION 4.17. The action of Γ on M is *free* if $\gamma x \neq x$ for every $x \in M$ and every $\gamma \in \Gamma - \{\mathrm{Id}\}$.

LEMMA 4.18. *Let X be a Hausdorff locally compact topological space, and let the group Γ act freely and discontinuously on X. Then, every $x \in X$ admits a neighborhood U such that the restriction $p_{|U} \colon U \to p(U)$ of the canonical projection $p \colon X \to X/\Gamma$ is a homeomorphism. In addition, X/Γ is Hausdorff.*

THEOREM 4.19. *Suppose that the action of Γ on he manifold M is free and discontinuous. Then the quotient space M/Γ is a topological manifold. In addition, the maximal atlas of M contains an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i\in I}$ such that, for every $i \in I$, the canonical projection $p \colon M \to M/\Gamma$ restricts to a homeomorphism $p_{|U_i} \colon U_i \to p(U_i)$ in such a way that $\mathcal{A}' = \{(p(U_i), \varphi_i \circ (p_{|U_i})^{-1})\}_{i\in I}$ is a differentiable atlas for M/Γ.*

In particular, if Γ acts freely and discontinuously on the manifold M, then M/Γ inherits a natural differentiable structure.

EXAMPLE 4.20. Let $\mathbb{Z}^n$ act on M by $(k_1, k_2, \dots, k_n)(x_1, x_2, \dots x_n) = (x_1 + k_1, x_2 + k_2, \dots, x_n + k_n)$ for every $(k_1, k_2, \dots, k_n) \in \mathbb{Z}^n$ and $(x_1, x_2, \dots x_n) \in \mathbb{R}^n$. The quotient manifold $\mathbb{R}^n/\mathbb{Z}^n$ is the *n–dimensional torus*, and is diffeomorphic to the product $S^1 \times S^1 \times \dots \times S^1$ of n copies of the circle S^1.

EXAMPLE 4.21. Let $\mathbb{Z}$ act on the plane $\mathbb{R}^2$ by $k(x, y) = (x + k, y)$. The quotient manifold $\mathbb{R}^2/\mathbb{Z}$ is the cylinder, diffeomorphic to $S^1 \times \mathbb{R}$.

EXAMPLE 4.22. Let $\mathbb{Z}$ act on the plane $\mathbb{R}^2$ by $k(x, y) = (x + k, (-1)^k y)$. The quotient manifold $\mathbb{R}^2/\mathbb{Z}$ is the (open) *Möbius strip*.

EXAMPLE 4.23. Let $\mathbb{Z}^2$ act on the plane $\mathbb{R}^2$ by $(k_1, k_2)(x_1, x_2) = (x_1+k_1, (-1)^{k_1}x_2+k_2)$. The quotient manifold $\mathbb{R}^2/\mathbb{Z}^2$ is the *Klein bottle.*

EXAMPLE 4.24. Let $\mathbb{Z}_2$ act on the n–dimensional sphere $S^n \subset \mathbb{R}^{n+1}$ by sending its generator to the diffeomorphism $x \mapsto -x$. The quotient manifold $S_n/\mathbb{Z}_2$ is diffeomorphic to the projective space $\mathbb{RP}^n$.

CHAPTER 5

Tangent vectors

1. Tangent vectors for manifolds

We now extend the notion of tangent vector to the abstract framework of manifolds.

1.1. Tangent vectors as equivalence class of curves.

DEFINITION 5.1. A *parametrized curve* in the manifold M is a differentiable map $\alpha\colon I \to M$ where I is an open interval in $\mathbb{R}$.

Let M be a differentiable manifold of dimension n, with atlas $\mathcal{A} = \{(U_i, \varphi); i \in I\}$, and let $x \in M$. Consider the set C_x of all parametrized curves $\alpha\colon I \to M$ such that $0 \in I$ and $\alpha(0) = x$.

Choose a chart (U_i, φ_i) of $\mathcal{A}$ such that $x \in U_i$. Because U_i is open, a parametrized curve $\alpha \in C_x$ defines a parametrized curve $\varphi_i \circ \alpha$ in $\mathbb{R}^n$ defined on a small neighborhood of 0. Define an equivalence relation $\sim$ on C_x by the property that $\alpha \sim \beta$ exactly when $(\varphi_i \circ \alpha)'(0) = (\varphi_i \circ \beta)'(0)$.

LEMMA 5.2. *The equivalence relation $\sim$ on C_x is independent of the choice of the chart (U_i, φ_i) containing x.*

DEFINITION 5.3. The *tangent space* T_xM of the manifold M at x is the set of equivalence classes of the relation $\sim$ on C_x. An element of T_xM is a *tangent vector* of M at x.

LEMMA 5.4. *If the chart (U_i, φ_i) contains x, the map $\Phi_i\colon T_xM \to \mathbb{R}^n$ defined by $\alpha \mapsto (\varphi_i \circ \alpha)'(0)$ is bijective.*

In addition, for any other chart (U_j, φ_j) containing x, the map $\Phi_j \circ \Phi_i^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$ coincides with the tangent map $T_{\varphi_i(x)}\varphi_j \circ \varphi_i^{-1}$ of the coordinate change diffeomorphism $\varphi_j \circ \varphi^{-1}$, and in particular is linear.

COROLLARY 5.5. *The tangent space T_xM has a unique structure as a vector space such that, for every chart (U_i, φ_i) containing x, the above map $\Phi_i\colon T_xM \to \mathbb{R}^n$ is a linear isomorphism.*

COROLLARY 5.6. *If M is a submanifold of $\mathbb{R}^n$, there is a natural one-to-one correspondence between the tangent space T_xM defined here and the tangent space $T_xM \subset \mathbb{R}^n$ defined earlier.*

In practice, we will say that the tangent vector $v \in T_xM$ represented by the curve $\alpha\colon I \to M$ with $\alpha(0) = x$ is the vector *tangent to* α at $t = 0$ (or at x), and we will write that $v = \alpha'(0)$.

1.2. Tangent vectors as derivations.

DEFINITION 5.7. Let $\mathcal{C}(M)$ denote the vector space of all differentiable functions $f\colon M \to \mathbb{R}$. A *derivation* at the point $x \in M$ is a linear map $D\colon \mathcal{C}(M) \to \mathbb{R}$ that satisfies the following product rule:

$$D(fg) = D(f)\, g(x) + f(x)\, D(g)$$

for every $f,\ g \in \mathcal{C}(M)$.

A typical example is provided by the *directional derivative* D_v along a tangent vector $v \in T_xM$, defined by

$$D_v(f) = (f \circ \alpha)'(0)$$

for an arbitrary curve $\alpha\colon I \to M$ such that $\alpha(0) = x$ and $\alpha'(0) = v$.

THEOREM 5.8. *The map $v \mapsto D_v$ defines a linear isomorphism between the tangent space T_xM and the vector space of all derivations at x.*

EXAMPLE 5.9. If U is open in $\mathbb{R}^n$, the derivation $\frac{\partial}{\partial x_i}$ corresponds to the basis vector $(0, \dots, 0, 1, 0 \dots, 0)$ where the 1 is in the i–th position.

2. The tangent map to a map between manifolds

PROPOSITION 5.10. *Let $f\colon M \to N$ be a differentiable map between manifolds. For every $x \in M$, there is a unique linear map $T_xf\colon T_xM \to T_{f(x)}N$ such that, if $v \in T_xM$ is represented by the parametrized curve $\alpha\colon (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = x$, then $T_xf(v) \in T_{f(x)}N$ is represented by the curve $f \circ \alpha\colon (-\varepsilon, \varepsilon) \to N$.*

PROPOSITION 5.11 (Chain rule). *Given three manifolds M, N, P and two differentiable maps $f\colon M \to N$ and $g\colon N \to P$, then $T_x(g \circ f) = T_{f(x)}g \circ T_xf$.*

COROLLARY 5.12. *If $f\colon M \to N$ is a diffeomorphism between manifolds, the tangent map $T_xf\colon T_xM \to T_{f(x)}N$ is a linear isomorphism.*

As in the case of open subsets of $\mathbb{R}^n$, this fact has a local converse.

THEOREM 5.13 (Inverse Function Theorem). *Let $f\colon M \to N$ be a differentiable map between manifolds M and N of the same dimension. If the tangent map $T_xf\colon T_xM \to T_{f(x)}N$ is an isomorphism, then there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ in N such that f restricts to a diffeomorphism $U \to V$.*

3. Submanifolds of manifolds

DEFINITION 5.14. Let M be a manifold of dimension m. A *submanifold* of M of dimension n is a subset $N \subset M$ such that, for every $x \in N$, there exists an open neighborhood U of x in M and a diffeomorphism $\varphi\colon U \to V$ from U to an open subset V of $\mathbb{R}^n$ such that

$$\varphi(U \cap M) = V \cap (\mathbb{R}^n \times \{0\})$$

in $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$.

As in the case of submanifolds of $\mathbb{R}^m$, the differentiable structure of M induces a differentiable structure on N.

PROPOSITION 5.15. *If N is a submanifold of the manifold M, the inclusion map $i\colon N \to M$ defined by $i(x) = x$ is differentiable.*

COMPLEMENT 5.16. *The tangent map $T_x i\colon T_x N \to T_x M$ of the inclusion map $i\colon N \to M$ is injective, and identifies $T_x N$ to a linear subspace of $T_x M$.*

3.1. Immersions and embeddings.

DEFINITION 5.17. A differentiable map $f\colon M \to N$ between two manifolds is an *immersion* if the tangent map $T_x f\colon T_x M \to T_{f(x)} N$ is injective at each $x \in M$.

DEFINITION 5.18. An *embedding* of the manifold M into the manifold N is an immersion $f\colon M \to N$ for which the range restriction $M \to f(N)$ is a homeomorphism.

THEOREM 5.19. *Let $f\colon M \to N$ be an embedding between differentiable manifolds. Then the image $f(M)$ is a submanifold of N, and the restriction $M \to f(M)$ of f is a diffeomorphism.*

COMPLEMENT 5.20. *If $x \in M$ and $y = f(x)$, the tangent space $T_y f(M) \subset T_y N$ of the submanifold $f(M) \subset N$ is the image of $T_x M$ under the tangent map $T_x f$*

3.2. Critical points, regular values.

DEFINITION 5.21. A *critical point* of the differentiable map $f\colon M \to N$ is a point $x \in M$ such that the tangent map $T_x f\colon T_x M \to T_{f(x)} N$ is not surjective.

A *regular point* is a point $x \in M$ that is not critical.

DEFINITION 5.22. A *critical value* of the differentiable map $f\colon M \to N$ is a point $y \in N$ that is the image of a critical point.

A *regular value* is a point $y \in N$ that is not critical. In other words, $y \in N$ is regular if every $x \in f^{-1}(y)$ is a regular point.

REMARK 5.23. If the dimension of M is smaller than the dimension of N, the regular values of $f\colon M \to N$ are just the points of N which are not in the image $f(M)$.

THEOREM 5.24. *Let $f\colon M \to N$ be a differentiable map from a manifold M of dimension m to a manifold N of dimension n. If y is a regular value of f, then its preimage $f^{-1}(y)$ is a submanifold of M of dimension $m - n$ (possibly empty if $y \notin f(M)$).*

COMPLEMENT 5.25. *If y is a regular value of $f\colon M \to N$, and if $P = f^{-1}(y)$, the tangent space $T_x P \subset T_x M$ of the submanifold $P \subset M$ is equal to the kernel of the tangent map $T_x f\colon T_x M \to T_y N$.*

4. The tangent bundle

DEFINITION 5.26. The *tangent bundle* of the m–dimensional manifold M is the set

$$TM = \{(x, v); x \in M, v \in T_x M\}.$$

The tangent bundle comes endowed with a natural projection $p\colon TM \to M$, defined by $p(x, v) = x$.

Let $\{(U_i, \varphi_i)\}_{i\in I}$ be an atlas for M. Set $\widetilde{U}_i = p^{-1}(U_i) \subset TM$, and define $\widetilde{\varphi}_i\colon \widetilde{U}_i \to V_i \times \mathbb{R}^m \subset \mathbb{R}^{2m}$ by the property that

$$\widetilde{\varphi}_i(x, v) = (\varphi(x), T_x\varphi(v)).$$

We endow TM with the topology for which $U \subset TM$ is open if and only if $\varphi_i(U \cap U_i)$ is open in $\mathbb{R}^m$ for every $i \in I$.

LEMMA 5.27. *The family* $\{(\widetilde{U}_i, \widetilde{\varphi}_i)\}_{i\in I}$ *is a differentiable atlas for* TM.

As a consequence, the tangent bundle TM is a differentiable manifold of dimension $2m$.

5. Vector fields

DEFINITION 5.28. A *vector field* on the manifold M is a differentiable map $X\colon M \to TM$ such that $p \circ X = \mathrm{Id}_M$.

In other words, a vector field associates a tangent vector $X_x \in T_xM$ to each $x \in M$, in such a way that the map $x \to X_x$ is differentiable. The tangent bundle non-sense is here to make sense of the differentiability property.

5.1. Integrating vector fields.

DEFINITION 5.29. A *flow* on the manifold M is a differentiable map $\varphi\colon M\times\mathbb{R} \to M$ such that $\varphi(x, t+t') = \varphi\big(\varphi(x,t), t'\big)$ for every $x \in M$ and t, $t' \in \mathbb{R}$.

This is better expressed in terms of the map $\varphi_t\colon M \to M$ defined by $\varphi_t(x) = \varphi(x,t)$; then the above property is equivalent to the equality $\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$. In particular, each φ_t is a diffeomorphism, and $\varphi_0 = \mathrm{Id}_M$ and $\varphi_t^{-1} = \varphi_{-t}$. Sometimes, a flow is also called a 1*-parameter group of diffeomorphisms* $(\varphi_t)_{t\in\mathbb{R}}$ of M.

A flow defines a vector field X by the property that $X_x = \frac{d}{dt}\varphi_t(x)_{|t=0}$ for every $x \in X$.

PROPOSITION 5.30. *Let M be a compact manifold, and let X be a vector field on M. Then, there exists a unique flow $(\varphi_t)_{t\in\mathbb{R}}$ such that $X_x = \frac{d}{dt}\varphi_t(x)_{|t=0}$ for every $x \in X$.*

There exists local version of this result when M is not compact.

5.2. The Lie bracket of two vector fields. Let X and Y be two vector fields. Given a function $f \in \mathcal{C}(M)$ and a point $x \in M$, we can interpret the vector $Y_x \in T_xM$ defined by Y as a derivation, and consider the directional derivative $Y_x(f) \in \mathbb{R}$. The map $x \mapsto Y_x(f)$ is now a new function $Y(f)$ of x, and we can consider its directional derivative $X_x(Y(f)) = (XY)_x(f)$. In general, the map $f \mapsto (XY)_x(f)$ is not a derivation, because it does not satisfy the product rule.

LEMMA 5.31. *Let X and Y be two vector fields on M. Then, there exists another vector field $[X,Y]$ such that*

$$[X,Y]_x(f) = X_x(Y(f)) - Y_x(X(f))$$

for every function $f\colon M \to \mathbb{R}$ and every point $x \in M$.

The vector field $[X,Y]$ is the *Lie bracket* of X and Y.

PROPOSITION 5.32. *The Lie bracket satisfies the following properties:*

(1) *$[X,Y]$ is a linear function of X and Y, in the sense that the maps $X \mapsto [X,Y]$ and $Y \mapsto [X,Y]$ are linear;*
(2) *$[Y,X] = -[X,Y]$;*
(3) *$[X,fY] = X(f)Y + f[X,Y]$ for every function f;*
(4) *(Jacobi identity)*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

5.3. The Frobenius theorem.

THEOREM 5.33. *Let X_1, X_2, ..., X_p be p vector fields in the m–dimensional manifold M, such that at every $x \in M$ the vectors X_{1x}, X_{2x}, ..., X_{px} are linearly independent in the tangent space T_xM. (In particular, $p \leqslant m$.) Let $V_x \subset T_xM$ denote the p–dimensional linear subspace generated by X_{1x}, X_{2x}, ..., X_{px}. The following two properties are equivalent:*

(i) *Every $x \in M$ is contained in a p–dimensional submanifold $N \subset M$ such that $T_yN = V_y$ for every $y \in N$;*

(ii) *For every i, j and every $x \in M$, the Lie bracket $[X_i, X_j]_x \in T_xM$ is contained in the subspace V_x generated by the vectors X_{kx}.*

EXAMPLE 5.34. In $\mathbb{R}^3$, there is no 2–dimensional submanifold N that is tangent to the vectors $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ at every $(x, y, z) \in N$.

CHAPTER 6

Differentiable partitions of unity

1. Partitions of unity

DEFINITION 6.1. Let $\{U_i; i \in I\}$ be an open covering of the manifold M. A *(differentiable) partition of unity* subordinate to this open covering is a family of differentiable functions $\psi_j \colon M \to \mathbb{R}$, defined for j ranging over an index set J, such that:

(1) $\psi_j(x)$ for every $j \in J$ and $x \in X$;
(2) for every $j \in J$ the *support* $\mathrm{Supp}(\psi_j)$ of ψ_j, namely the closure of $\{x \in M; \psi_j(x) \neq 0\}$, is compact and contained in some U_i;
(3) for every compact subset K of M, the set $\{j \in J; K \cap \mathrm{Supp}(\psi_j) \neq \emptyset\}$ is finite;
(4) for every $x \in M$, $\sum_{j \in J} \psi_j(x) = 1$.

Note that the sum in Condition 4 is finite by Condition 3.

THEOREM 6.2. *Let $\{U_i; i \in I\}$ be an open covering of the manifold M. Then, there exists a partition of unity $\{\psi_j \colon M \to \mathbb{R}; j \in J\}$ subordinate to $\{U_i; i \in I\}$.*

A key ingredient in the proof, using the fact that M is separable, is the following.

LEMMA 6.3. *Let M be a locally compact, separable topological space. Then there is a countable family of subsets $\{K_n; n \in \mathbb{N}\}$ such that:*

(1) *for every $n \in \mathbb{N}$, K_n is compact and contained in the interior of K_{n+1};*
(2) $\bigcup_{n \in \mathbb{N}} K_n = M$.

Such a family $\{K_n; n \in \mathbb{N}\}$ is called an *exhaustion of M by compact subsets.*

Partitions of unity are a very useful technical tool to construct functions on manifolds. The following section illustrates an application.

2. Manifolds as submanifolds of $\mathbb{R}^n$

THEOREM 6.4. *If M is a compact manifold of dimension m, there exists a number n and an embedding $f \colon M \to \mathbb{R}^n$.*

The proof goes as follows. Let $\{(U_i, \varphi_i)\}_{i \in I}$ be an atlas for M, and let $\psi_j \colon M \to \mathbb{R}\}_{j \in J}$ be a partition of unity subordinate to the covering by the U_i. Since M is compact and since the collection of the supports of the ψ_j is locally finite, the index set J is finite. Assume that $J = \{1, 2, \dots, p\}$ with loss of generality. By definition of partitions of unity, for every $j \in J$, the support of ψ_j is contained in some U_{i_j}. Then we can take $n = p + pm$ and define f by

$$f(x) = (\psi_1(x), \psi_2(x), \dots, \psi_p(x), \psi_1(x)\varphi_{i_1}(x), \psi_2(x)\varphi_{i_2}(x), \dots, \psi_p(x)\varphi_{i_p}(x))$$

with image in $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m$.

The result is actually true even if M is not compact, but more unpleasant to prove.

COROLLARY 6.5. *Every compact manifold is diffeomorphic to a submanifold of some* $\mathbb{R}^n$.

CHAPTER 7

Sard's theorem and applications

1. Sard's Theorem

DEFINITION 7.1. Let M be an m–dimensional manifold with atlas $\{(U_i, \varphi_i)\}_{i\in I}$. A subset $A \subset M$ has *measure* 0 in M if, for every $i \in I$, $\varphi_i(A \cap U_i)$ has Lebesgue measure 0 in $\mathbb{R}^m$. A subset $B \subset M$ has *full measure* if $A = M - B$ has measure 0 in M.

Because every differentiable map from R^m to $\mathbb{R}^m$ sends a set of measure 0 to a subset of measure 0, this definition is independent of the atlas used.

THEOREM 7.2 (Sard's theorem). *Let $f\colon M \to N$ be a differentiable map between differentiable manifolds. Then the set of regular values of f has full measure in N.*

COROLLARY 7.3. *Let $f\colon M \to N$ be a differentiable map between differentiable manifolds, such that the dimension of M is strictly less than the dimension of M. Then the image $f(M)$ has measure 0 in N.*

2. Application: Embedding manifolds in small $\mathbb{R}^n$

THEOREM 7.4. *Let M be a compact manifold of dimension m. Then there exists an embedding of M in $\mathbb{R}^{2m+1}$.*

The proof is by downward induction, starting from Theorem 6.4. Suppose that we have an embedding $f\colon M \to \mathbb{R}^n$ with $n > 2n + 1$. For every $v \in S^{n-1}$, let $L_v \cong \mathbb{R}^{n-1}$ be the hyperplane orthogonal to v in $\mathbb{R}^n$, and let $\pi_v\colon \mathbb{R}^n \to L_v$ be the orthogonal projection to L_v.

LEMMA 7.5. *If $n > 2m+1$, then the set of those v for which $\pi_v \circ f\colon M \to L_v \cong \mathbb{R}^{n-1}$ is an embedding has full measure in S^{n-1}.*

Lemma 7.5 immediately proves Theorem 7.4 by induction. Lemma 7.5 follows from an application of Sard's Theorem to two functions $F\colon M \times M - \Delta \to S^{n-1}$ and $G\colon TM^* \to S^{n-1}$ defined as follows. The function $F\colon M \times M - \Delta \to S^{n-1}$ is defined outside of the diagonal $\Delta = \{(x, y) \in M \times M; x = y\}$ by

$$F(x, y) = \frac{f(x) - f(y)}{\|f(x) - f(y)\|}.$$

The function $G\colon TM^* \to S^{n-1}$ is defined on the open subset $TM^* = \{(x, u); x \in M, u \in T_xM, u \neq 0\}$ of the unit tangent bundle TM by

$$G(x, u) = \frac{T_xf(u)}{\|T_xf(u)\|}.$$

Note that one needs f to be an embedding for F and G to be defined. By construction, $\pi_v \circ f\colon M \to L_v$ is injective exactly when $v \in S^{n-1}$ is not in the image of F, and it is an immersion exactly when v is not in the image of G. Since the domains of F and G are manifolds of dimension $2m < n-1$, Lemma 7.5 immediately follows from these observations and from Corollary 7.3.

A slight variation of these arguments (using the restriction of G to the $(2m-1)$–manifold $T^1M = \{(x,u); x \in M, u \in T_xM, \|T_xf(u)\| = 1\})$ gives:

THEOREM 7.6. *If M is a compact manifold of dimension m, there exists an immersion g: $M \to \mathbb{R}^{2m-1}$.*

CHAPTER 8

The alternating algebra

This is pure linear algebra, in preparation for the introduction of differential forms.

Let V be a vector space over $\mathbb{R}$, usually finite-dimensional. Let $V^p = V \times V \times \cdots \times V$ denote the product of k copies of V.

DEFINITION 8.1. A *p–linear* form, or a *multilinear* form of degree p, on V is a map $\omega : V^p \to \mathbb{R}$ such that the map $V \to \mathbb{R}$ defined by $v_i \mapsto \omega(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_p)$ is linear, for every $i = 1, \dots, p$ and every $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_p \in V$. The p–linear form $\omega : V^p \to \mathbb{R}$ is *alternating* if $\omega(v_1, \dots, v_p) = 0$ whenever there exists $i \neq j$ such that $v_i = v_j$.

The definition of alternating forms (also called *antisymmetric*) is made somewhat clearer by the following statement. Let $\mathfrak{S}_p$ denote that symmetric group, consisting of all bijections of $\{1, 2, \dots, p\}$. Recall that the *signature homomorphism* is the unique group homomorphism $\mathrm{sign}\colon \mathfrak{S}_p \to \mathbb{Z}_2 = \{\pm 1\}$ sending any transposition to -1.

PROPOSITION 8.2. *The p–linear form $\omega : V^p \to \mathbb{R}$ is alternating if and only*

$$\omega(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(p)}) = \mathrm{sign}(\sigma)\, \omega(v_1, v_2, \dots, v_p)$$

for every $v_1, v_2, \dots, v_p \in V$ and every $\sigma \in \mathfrak{S}_p$.

DEFINITION 8.3. Let $\mathrm{Alt}^p(V)$ denote the vector space of all alternating p–linear forms $\omega\colon V^p \to \mathbb{R}$

In the special case where $p = 1$, note that $\mathrm{Alt}^1(V)$ is just the dual V^* of V. When $p = 0$, we decide that $\mathrm{Alt}^0(V) = \mathbb{R}$ by convention.

1. The wedge product

DEFINITION 8.4. The *wedge product* of the forms $\alpha \in \mathrm{Alt}^p(V)$ and $\beta \in \mathrm{Alt}^q(V)$ is the $(p+q)$–form $\alpha \wedge \beta \in \mathrm{Alt}^{p+q}(V)$ defined by

$$\alpha \wedge \beta(v_1, v_2, \dots, v_{p+q})$$
$$= \frac{1}{p!q!} \sum_{\sigma \in \mathfrak{S}_{p+q}} \mathrm{sign}(\sigma)\, \alpha(v_{\sigma(1)}, v_{\sigma(2)}, \dots, v_{\sigma(p)}) \beta(v_{\sigma(p+1)}, v_{\sigma(p+2)}, \dots, v_{\sigma(p+q)})$$

for every $v_1, v_2, \dots, v_{p+q} \in V$.

In the above definition, the sum over all $\sigma \in \mathfrak{S}_{p+q}$ is designed to make sure that the right hand term is really alternating. The $\frac{1}{p!q!}$ factor is introduced to guarantee the following:

PROPOSITION 8.5. *If* $\alpha \in \mathrm{Alt}^p(V)$, $\beta \in \mathrm{Alt}^q(V)$, *and* $\gamma \in \mathrm{Alt}^r(V)$, *then*

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$$

in $\mathrm{Alt}^{p+q+r}(V)$

PROPOSITION 8.6. *If* $\alpha \in \mathrm{Alt}^p(V)$ *and* $\beta \in \mathrm{Alt}^q(V)$, *then*

$$\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta.$$

All of this makes sense even when $p = 0$ (or $q = 0$, etc...), in which case the wedge product with the constant $\alpha \in \mathrm{Alt}^0(V) = \mathbb{R}$ is just the multiplication by the corresponding constant.

One often combines all $\mathrm{Alt}^p(V)$ into a single vector space

$$\mathrm{Alt}^\bullet(V) = \bigoplus_{p=0}^{\infty} \mathrm{Alt}^p(V).$$

Then, the wedge product $\wedge$ endows $\mathrm{Alt}^\bullet(V)$ with the structure of an algebra, called the *alternating algebra* of the vector space V.

If you already know what the exterior algebra $\Lambda^\bullet(V)$ of a vector space V is, the alternating algebra $\mathrm{Alt}^\bullet(V)$ has a natural identification with the exterior algebra $\Lambda^\bullet(V^*)$ of the dual space V^*.

2. A basis for $\mathrm{Alt}^p(V)$

We now assume that V is finite dimensional, with basis $\{e_1, e_2, \dots, e_n\}$. Let $\{e_1^*, e_2^*, \dots, e_n^*\}$ be the dual basis for the dual space $V^* = \mathrm{Alt}^1(V)$. Then, for every $i_1, i_2, \dots, i_p$, we can consider the element $e_{i_1}^* \wedge e_{i_2}^* \wedge \dots \wedge e_{i_p}^* \in \mathrm{Alt}^p(V)$.

THEOREM 8.7. *If* $\{e_1, e_2, \dots, e_n\}$ *is a basis for* V, *the elements* $\{e_{i_1}^* \wedge e_{i_2}^* \wedge \dots \wedge e_{i_p}^*; 1 \leqslant i_1 < i_2 < \dots < i_p \leqslant p\}$ *form a basis for* $\mathrm{Alt}^p(V)$.

COROLLARY 8.8. *If* V *is a vector space of dimension* n, *then* $\mathrm{Alt}^p(V)$ *has dimension* $\binom{n}{p}$.

Note that this holds even when $p = 0$. Also, $\mathrm{Alt}^p(V) = 0$ if $p > n$ or $p < 0$.

3. Linear maps induced by linear maps

Let $f \colon V \to M$ be a linear map between two vector spaces V and W. It induces a linear map

$$\mathrm{Alt}^p(f) \colon \mathrm{Alt}^p(W) \to \mathrm{Alt}^p(V)$$

where, for every $\omega \in \mathrm{Alt}^p(W)$, $\mathrm{Alt}^p(f)(\omega) \in \mathrm{Alt}^p(V)$ is defined by the natural formula

$$\mathrm{Alt}^p(f)(\omega)(v_1, v_2, \dots, v_p) = \omega(f(v_1), f(v_2), \dots, f(v_p))$$

for every $v_1, v_2, \dots, v_p \in V$.

The functorial notation where $\mathrm{Alt}^p(f)$ denotes a map between the vector spaces $\mathrm{Alt}^p(W)$ and $\mathrm{Alt}^p(V)$ takes a while to get used to, but eventually turns out to be very convenient when used systematically.

Note that, when $p = 1$, $\mathrm{Alt}^1(f) \colon \mathrm{Alt}^1(W) \to \mathrm{Alt}^1(V)$ is just the dual map $f^* \colon W^* \to V^*$ of the linear map $f \colon V \to W$. When $p = 0$, $\mathrm{Alt}^0(f) \colon \mathrm{Alt}^0(W) \to \mathrm{Alt}^0(V)$ is by convention the identity map $\mathbb{R} \to \mathbb{R}$.

PROPOSITION 8.9. *If we have a composition of linear maps* $U \xrightarrow{f} V \xrightarrow{g} W$, *inducing a composition* $\mathrm{Alt}^p(W) \xrightarrow{\mathrm{Alt}^p(g)} \mathrm{Alt}^p(V) \xrightarrow{\mathrm{Alt}^p(f)} \mathrm{Alt}^p(U)$, *then*

$$\mathrm{Alt}^p(g \circ f) = \mathrm{Alt}^p(f) \circ \mathrm{Alt}^p(g).$$

PROPOSITION 8.10. *If* $f\colon V \to W$, $\alpha \in \mathrm{Alt}^p(W)$ *and* $\beta \in \mathrm{Alt}^q(W)$,

$$\mathrm{Alt}^{p+q}(f)(\alpha \wedge \beta) = \mathrm{Alt}^p(f)(\alpha) \wedge \mathrm{Alt}^q(f)(\beta).$$

4. Connection with determinants

Consider the special case where f is a linear map from the n–dimensional vector space V to itself. Then f induces a linear map $\mathrm{Alt}^n(f)$ from $\mathrm{Alt}^n(V) \cong \mathbb{R}$ to itself, which must be the multiplication by a certain constant.

THEOREM 8.11. *If* $f\colon V \to V$ *is represented by the* $n \times n$ *matrix* F, *then the induced map*

$$\mathrm{Alt}^n(f)\colon \mathrm{Alt}^n(V) \to \mathrm{Alt}^n(V)$$

is just the multiplication by the determinant $\det(F)$.

The machinery of alternating forms therefore provides a way to define determinants in a manner which is independent of the choice of a basis in V. This property is the crucial motivation behind the definition of differential forms.

CHAPTER 9

Differential forms in $\mathbb{R}^m$

1. Differential forms in $\mathbb{R}^m$

DEFINITION 9.1. A *differential form of degree* p on the open subset U of $\mathbb{R}^m$ is a differentiable map $\omega\colon U \to \mathrm{Alt}^p(\mathbb{R}^m)$.

Let $\Omega^p(U)$ denote the vector space of all differential forms of degree p on U.

Let $e_i = (0,\dots,0,1,0,\dots,0)$ denote the standard basis element of $\mathbb{R}^m$, where the 1 occurs in i–th position, and let e_i^* be the corresponding element of the dual basis of $(\mathbb{R}^m)^*$. For every $x \in M$, we can therefore write

$$\omega(x) = \sum_{1\leqslant i_1<\dots<i_p\leqslant m} \omega_{i_1\dots i_p}(x)\, e_{i_1}^* \wedge \dots e_{i_p}^*$$

where the $\binom{n}{p}$ functions $\omega_{i_1\dots i_p}\colon U \to \mathbb{R}$ are differentiable.

EXAMPLE 9.2 (Degree 0). A differential form of degree 0 is just a differentiable function $\omega\colon U \to \mathrm{Alt}^0(\mathbb{R}^m) = \mathbb{R}$.

EXAMPLE 9.3 (A fundamental degree 1 example). The *differential* of a function $f\colon U \to \mathbb{R}$ is the differential form $df \in \Omega^1(U)$ defined by the property that $df(x) = T_x f \in \mathrm{Alt}^1(\mathbb{R}^m)$ for every $x \in U$.

In practice, the differential of the function $f\colon U \to \mathbb{R}$ is equal to

$$df = \sum_{i=1}^m \frac{\partial f}{\partial x_i}\, e_i^*.$$

In particular, if f is the function "x_i", namely the function that to $x \in U$ associates the coordinate x_i, then $df = e_i^*$. For this reason, we will henceforth write that $e_i^* = dx_i$.

The vector spaces $\Omega^p(U)$ also come with a multiplication, coming from the wedge product.

DEFINITION 9.4. The *wedge product* or *exterior product* of the differential forms $\alpha \in \Omega^p(U)$ and $\beta \in \Omega^q(U)$ is the differential form $\alpha\wedge\beta \in \Omega^{p+q}(U)$ defined by the property that $(\alpha\wedge\beta)(x) = \alpha(x)\wedge\beta(x) \in \mathrm{Alt}^{p+q}(\mathbb{R}^m)$.

PROPOSITION 9.5. *If* $\alpha \in \Omega^p(U)$, $\beta \in \Omega^q(U)$ *and* $\gamma \in \Omega^r(U)$, *then*

$$\alpha\wedge(\beta\wedge\gamma) = (\alpha\wedge\beta)\wedge\gamma$$

and

$$\beta\wedge\alpha = (-1)^{pq}\alpha\wedge\beta.$$

2. Homomorphisms induced by maps

Let $f\colon U \to V$ be a differentiable map between open subsets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$. For every $x \in U$, the tangent map $T_x f\colon \mathbb{R}^m \to \mathbb{R}^n$ induces a map $\mathrm{Alt}^p(T_x f)\colon \mathrm{Alt}^p(\mathbb{R}^n) \to \mathrm{Alt}^p(\mathbb{R}^m)$. We can therefore define a map

$$\Omega^p(f)\colon \Omega^p(V) \to \Omega^p(U)$$

by the formula

$$\Omega^p(f)(\omega)(x) = \mathrm{Alt}^p(T_x f)(\omega(f(x))$$

for every $\omega \in \Omega^p(V)$ and every $x \in U$.

By definition, $\Omega^p(f)(\omega) \in \Omega^p(U)$ is the *pull back* of $\omega \in \Omega^p(V)$ under the map $f\colon U \to V$, and $\Omega^p(f)$ is the *linear homomorphism* $\Omega^p(V) \to \Omega^p(U)$ *induced* by f.

PROPOSITION 9.6. *If* $f\colon U \to V$, $\alpha \in \Omega^p(V)$ *and* $\beta \in \Omega^p(V)$, *then*

$$\Omega^{p+q}(f)(\alpha \wedge \beta) = \Omega^p(\alpha) \wedge \Omega^q(\beta).$$

PROPOSITION 9.7. *If we have a composition of differentiable maps* $U \xrightarrow{f} V \xrightarrow{g} W$, *then* $\Omega^p(g \circ f) = \Omega^p(f) \circ \Omega^p(g)$.

Let us explicitly compute the homomorphism $\Omega^p(f)\colon \Omega^p(V) \to \Omega^p(U)$ induced by the map $f\colon U \to V$ with coordinate functions $f_j\colon U \to \mathbb{R}$, $j = 1, \ldots, n$

If $\omega \in \Omega^p(V)$ is given by

$$\omega = \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} \omega_{j_1 \ldots j_p}\, dy_{j_1} \wedge \cdots \wedge dy_{j_p}$$

then

$$\begin{aligned}
\Omega^p(f)(\omega) &= \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} \Omega^p(f)\left(\omega_{j_1 \ldots j_p}\, dy_{j_1} \wedge \ldots dy_{j_p}\right) \\
&= \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} \Omega^0(f)(\omega_{j_1 \ldots j_p})\, \Omega^1(f)(dy_{j_1}) \wedge \cdots \wedge \Omega^1(f)(dy_{j_p}) \\
&= \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} (\omega_{j_1 \ldots j_p} \circ f) \left(\sum_{i_1=1}^{m} \frac{\partial f_{j_1}}{\partial x_{i_1}} dx_{i_1}\right) \wedge \cdots \wedge \left(\sum_{i_p=1}^{m} \frac{\partial f_{j_p}}{\partial x_{i_p}} dx_{i_p}\right) \\
&= \sum_{1 \leqslant j_1 < \cdots < j_p \leqslant n} \sum_{i_1=1}^{m} \cdots \sum_{i_p=1}^{m} (\omega_{j_1 \ldots j_p} \circ f) \frac{\partial f_{j_1}}{\partial x_{i_1}} \cdots \frac{\partial f_{j_p}}{\partial x_{i_p}}\, dx_{i_1} \wedge \cdots \wedge dx_{i_p}.
\end{aligned}$$

Note the analogy between the expression $\Omega^1(f)(dy_j) = \sum_{i=1}^m \frac{\partial f_j}{\partial x_i} dx_i$ and the standard calculus formalism $dy_j = \sum_{i=1}^m \frac{\partial f_j}{\partial x_i} dx_i$ associated to $y = f(x)$.

The following special case, where $m = n = p$, will be fundamental when we integrate differential forms.

LEMMA 9.8. *Let* U *and* V *be open in* $\mathbb{R}^n$ *and consider a differentiable function* $f\colon U \to V$ *with coordinate functions* $f_j\colon U \to \mathbb{R}$. *Consider the form* $\omega \in \Omega^n(V)$ *defined by*

$$\omega = g\, dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n$$

for some function $g\colon V \to \mathbb{R}$. *Then*

$$\Omega^n(\varphi)(\omega) = g \circ f \, \det\left(\frac{\partial f_j}{\partial x_i}\right) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

3. The exterior derivative

DEFINITION 9.9. The *exterior derivative* is the linear map $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$ defined by the property that, if

$$\omega = \sum_{i_1<i_2<\cdots<i_p} \omega_{i_1 i_2 \ldots i_p} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

then

$$d\omega = \sum_{i_1<i_2<\cdots<i_p} \left(\sum_{i=1}^{m} \frac{\partial \omega_{i_1 i_2 \ldots i_p}}{\partial x_i} dx_i \right) \wedge dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_p}$$

PROPOSITION 9.10. *The exterior derivative has the following properties.*

(1) *d is linear;*
(2) *if $\alpha \in \Omega^p(U)$ and $\beta \in \Omega^q(U)$,*
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta;$$
(3) *if $f\colon M \to N$ is differentiable, $d \circ \Omega^p(f) = \Omega^{p+1}(f) \circ d$;*
(4) *$d \circ d = 0$;*
(5) *if $f \in \Omega^0(U)$ is interpreted as a function $f\colon U \to \mathbb{R}$, then $df \in \Omega^1(U)$ is the differential of f defined by the property that $df(x) = T_x f$ at each $x \in U$.*

CHAPTER 10

Differential forms on manifolds

1. Differential forms in manifolds

We now consider the case where M is a differentiable manifold of dimension m, with atlas $\{(U_i, \varphi_i)\}_{i \in I}$.

At every $x \in U_i$, corresponding to $y = \varphi_i(x) \in \varphi_i(U_i) \subset \mathbb{R}^m$, the tangent map $T_y\varphi_i^{-1}\colon \mathbb{R}^m \to T_xM$ induces an isomorphism $\mathrm{Alt}^p(T_y\varphi_i^{-1})\colon \mathrm{Alt}^p(T_xM) \to \mathrm{Alt}^p(\mathbb{R}^m)$.

DEFINITION 10.1. A *differential form of degree p* on the manifold M is the assignment of an element $\omega(x) \in \mathrm{Alt}^p(T_xM)$ to each $x \in M$, in such a way that $\omega(x)$ depends differentiably on x in the following sense: for every chart (U_i, φ_i), the map $\omega_i\colon \varphi_i(U_i) \to \mathrm{Alt}^p(\mathbb{R}^m)$ defined by $\omega_i(y) = \mathrm{Alt}^p(T_y\varphi_i^{-1})(\omega_{\varphi_i^{-1}(y)})$ is differentiable.

LEMMA 10.2. *To check the above differentiability property at $x \in M$, it suffices to verify for one U_i containing x. In particular, the differentiability condition depends only on the differentiable structure of M, and not on the specific atlas $\{(U_i, \varphi_i)\}_{i \in I}$ used.*

DEFINITION 10.3. The vector space of all differential forms of degree p on M is denoted by $\Omega^p(M)$.

EXAMPLE 10.4 (Degree 0). A differential form $\omega \in \Omega^0(M)$ is just a differentiable function $\omega\colon M \to \mathbb{R}$ since $\mathrm{Alt}^0(T_xM) = \mathbb{R}$.

EXAMPLE 10.5 (Degree 1). If $f\colon M \to \mathbb{R}$ is a differentiable function, its *differential* is the form $df \in \Omega^1(M)$ defined by the property that $df(x) = T_xf\colon T_xM \to \mathbb{R}$ for every $x \in M$. (Recall that $\mathrm{Alt}^1(T_xM)$ is just the dual of T_xM.)

2. The wedge product

DEFINITION 10.6. The *wedge product* or *exterior product* of the differential forms $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$ is the differential form $\alpha \wedge \beta \in \Omega^{p+q}(M)$ defined at each $x \in M$ by $(\alpha \wedge \beta)(x) = \alpha(x) \wedge \beta(x) \in \mathrm{Alt}^{p+q}(T_xM)$.

PROPOSITION 10.7. *If $\alpha \in \Omega^p(M)$, $\beta \in \Omega^q(M)$ and $\gamma \in \Omega^r(M)$, then*

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$$

and

$$\beta \wedge \alpha = (-1)^{pq} \alpha \wedge \beta.$$

3. Homomorphisms induced by maps

Let $f\colon M \to N$ be a differentiable between manifolds. For every $x \in M$, the tangent map $T_xf : T_xM \to T_{f(x)}N$ induces a map $\mathrm{Alt}^p(T_xf) : \mathrm{Alt}^p(T_{f(x)}N) \to \mathrm{Alt}^p(T_xM)$. We can therefore define a map

$$\Omega^p(f)\colon \Omega^p(N) \to \Omega^p(M)$$

by the formula

$$\Omega^p(f)(\omega)(x) = \mathrm{Alt}^p(T_x f)(\omega(f(x)))$$

for every $\omega \in \Omega^p(N)$ and every $x \in M$.

The differential form $\Omega^p(f)(\omega)$ is often called the *pull back* of ω under the map f.

PROPOSITION 10.8. *If $f\colon M \to N$, $\alpha \in \Omega^p(N)$ and $\beta \in \Omega^p(N)$, then*

$$\Omega^{p+q}(f)(\alpha \wedge \beta) = \Omega^p(\alpha) \wedge \Omega^q(\beta).$$

PROPOSITION 10.9. *If we have a composition of differentiable maps $M \xrightarrow{f} N \xrightarrow{g} P$, then $\Omega^p(g \circ f) = \Omega^p(f) \circ \Omega^p(g)$.*

4. Differential forms in local coordinates

Let $\{(U_i, \varphi_i)\}_{i\in I}$ be an atlas for the manifold M. If $\omega \in \Omega^p(M)$ is a p-form in M, the pull back $\omega_i = \Omega^p(\varphi_i^{-1}) \in \Omega^p(V_i)$ is a differential form on the open subset $V_i = \varphi_i(U_i)$ of $\mathbb{R}^m$.

PROPOSITION 10.10. *There is a one-to-one correspondence between differential forms $\omega \in \Omega^p(M)$ and families of differential forms $\{\omega_i \in \Omega^p(V_i)\}_{i\in I}$ that are compatible under coordinate changes $\varphi_j \circ \varphi_i^{-1}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$, in the sense that*

$$\omega_{i|\varphi_i(U_i \cap U_j} = \Omega^p(\varphi_j \circ \varphi_i^{-1})(\omega_{j|\varphi_j(U_i\cap U_j)})$$

for every i, $j \in I$.

5. The exterior derivative

THEOREM 10.11. *As M ranges over all manifolds and p over all integers, there is a unique family of maps $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ such that:*

1. *d is linear;*
2. *if $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$,*
$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta;$$
3. *if $f\colon M \to N$ is differentiable, $d \circ \Omega^p(f) = \Omega^{p+1}(f) \circ d$;*
4. *$d \circ d = 0$;*
5. *if $f \in \Omega^0(M)$ is interpreted as a function $f\colon M \to \mathbb{R}$, then $df \in \Omega^1(M)$ is defined by the property that $df(x) = T_x M$ at each $x \in M$.*

DEFINITION 10.12. The form $d\omega \in \Omega^{p+1}(M)$ is the *exterior derivative* of ω.

In practice, if $\{(U_i, \varphi_i)\}_{i\in I}$ is an atlas for M and if $\omega \in \Omega^p(M)$ is determined by the family $\{\omega_i \in \Omega^p(V_i)\}_{i\in I}$, then $d\omega \in \Omega^{p+1}(M)$ is determined by the family $\{d\omega_i \in \Omega^{p+1}(V_i)\}_{i\in I}$. The fact that the $d\omega_i$ are compatible with the coordinate changes $\varphi_j \circ \varphi_i^{-1}$ comes from the fact that d commutes (up to degree shift) with the induced homomorphisms $\Omega^p(\varphi_j \circ \varphi_i^{-1})$.

CHAPTER 11

de Rham cohomology

1. De Rham cohomology

DEFINITION 11.1. A differential form $\omega \in \Omega^p(M)$ is *closed* if $d\omega = 0$. It is *exact* if there exists a form $\alpha \in \Omega^{p-1}(M)$ such that $\omega = d\alpha$.

EXAMPLE 11.2. If U is an open subset of $\mathbb{R}^2$, every differential form $\omega \in \Omega^1(U)$ can be written as $\omega = P\,dx + Q\,dy$ for some functions P, $Q\colon U \to \mathbb{R}$. Then ω is closed if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. If is exact if and only if there exists a function $f\colon U \to \mathbb{R}$ such that $\frac{\partial f}{\partial x} = P$ and $\frac{\partial f}{\partial y} = Q$.

Since $d \circ d = 0$, every exact form is closed. In other words, in $\Omega^p(M)$, the kernel $Z^p(M)$ of $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ contains the image $B^p(M)$ of $d\colon \Omega^{p-1}(M) \to \Omega^p(M)$.

DEFINITION 11.3. The *p–th de Rham cohomology space* of M is the vector space $H^p(M)$ quotient of $Z^p(M)$ by $B^p(M)$.

In other words, the size of $H^p(M)$ measures the difference between "closed" and "exact" for p–forms in M.

PROPOSITION 11.4. *Let $\{M_i\}_{i\in I}$ be the (finite or infinite) set of connected components of the manifold M. Then, for every p,*

$$H^p(M) = \prod_{i\in I} H^p(M_i).$$

Recall that a form of degree 0 on a manifold M is the same as a function $f\colon M \to \mathbb{R}$. This form is closed if and only if f is locally constant. In particular, considering the value of the constant:

PROPOSITION 11.5. *If the manifold is connected, then $H^0(M)$ is canonically isomorphic to* 0.

2. Homomorphisms induced by differentiable maps

Consider a differentiable map $f\colon M \to N$. Because $d \circ \Omega^q(f) = \Omega^{q+1}(f) \circ d$ for $q = p - 1$ and p, the induced map $\Omega^p(f)\colon \Omega^p(N) \to \Omega^p(M)$ sends $Z^p(N)$ to $Z^p(M)$ and $B^p(N)$ to $B^p(M)$. As a consequence, it induces a linear map

$$H^p(f)\colon H^p(N) \to H^p(M)$$

PROPOSITION 11.6. *Given two maps $M \xrightarrow{f} N \xrightarrow{g} P$, $H^p(g\circ f) = H^p(f)\circ H^p(g)$.*

COROLLARY 11.7. *If $f\colon M \to N$ is a diffeomorphism, then $H^p(f)\colon H^p(N) \to H^p(M)$ for every p.*

3. Invariance under homotopy

DEFINITION 11.8. Two differentiable maps $f_0\colon M \to N$ and $f_1\colon M \to N$ are *(differentiably) homotopic* if there exists a differentiable map $H\colon M \times \mathbb{R} \to N$ such that

(1) $H(x,0) = f_0(x)$ for every $x \in M$;
(2) $H(x,1) = f_1(x)$ for every $x \in M$.

The definition is perhaps easier to understand if, for every $t \in \mathbb{R}$, one considers the map $f_t\colon M \to N$ defined by $f_t(x) = H(x,t)$.
The map H is called a *homotopy* from f_0 to f_1.

THEOREM 11.9. *If $f_0 : M \to N$ and $f_1 : M \to N$ are homotopic, then the induced homomorphisms $H^p(f_0)\colon H^p(N) \to H^p(M)$ and $H^p(f_1)\colon H^p(N) \to H^p(M)$ are equal.*

The key step in the proof is the following lemma.

LEMMA 11.10. *There exists for every p a homomorphism $K_p\colon \Omega^p(N) \to \Omega^{p-1}(M)$ such that*

$$d \circ K_p + K_{p+1} \circ d = \Omega^p(f_1) - \Omega^p(f_0)$$

DEFINITION 11.11. Two manifolds M and N are *homotopically equivalent* if there exists two maps $f\colon M \to N$ and $g\colon N \to M$ such that $g \circ f$ is homotopic to the identity map of M, and $f \circ g$ is homotopic to the identity of N.

Note that M and N may have different dimensions.

COROLLARY 11.12. *If M and N are homotopically equivalent, then $H^p(M)$ is isomorphic to $H^p(N)$ for every p.*

EXAMPLE 11.13. An open subset U of $\mathbb{R}^n$ is *star-shaped* if there exists $x_0 \in U$ such that, for every $x \in U$, the line segment joining x_0 to x is completely contained in U. For instance, $\mathbb{R}^n$ and any open ball in $\mathbb{R}^n$ are star-shaped. If U is star-shaped, it is homotopically equivalent to the 0–dimensional manifold $\{x_0\}$. As a consequence, $H^p(U) = 0$ if $p \neq 0$, and $H_0(U) = \mathbb{R}$.

EXAMPLE 11.14. The space $\mathbb{R}^n - \{0\}$ is homotopically equivalent to the sphere S^{n-1} by consideration of the inclusion map $f : S^{n-1} \to R^n - \{0\}$ and of the map $g : R^n - \{0\} \to S^{n-1}$ defined by $g(x) = x/\|x\|$. As a consequence, $H^p(R^n - \{0\})$ is isomorphic to $H^p(S^{n-1})$ for every p. At this point, the only corollary we get out of this is that $H^n(\mathbb{R}^n - \{0\}) = 0$.

4. De Rham cohomology and wedge product

THEOREM 11.15. *There exists a well-defined bilinear homomoprhism*

$$H^p(M) \times H^q(M) \to H^{p+q}(M)$$

which to $[\alpha] \in H^p(M)$ and $[\beta] \in H^q(M)$ associates $[\alpha \wedge \beta] \in H^{p+q}(M)$.

5. De Rham cohomology with compact support

Let $\Omega^p_{\mathrm{c}}(M)$ denote the space of forms $\omega \in \Omega^p(M)$ whose support is compact. If M is compact, $\Omega^p_{\mathrm{c}}(M)$ is of course the same as $\Omega^p(M)$.

The exterior derivative $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ restricts to $d\colon \Omega^p_{\mathrm{c}}(M) \to \Omega^{p+1}_{\mathrm{c}}(M)$. In particular, we can consider the kernel $Z^p_{\mathrm{c}}(M)$ of $d\colon \Omega^p_{\mathrm{c}}(M) \to \Omega^{p+1}_{\mathrm{c}}(M)$ and the image $B^p_{\mathrm{c}}(M)$ of $d\colon \Omega^{p-1}_{\mathrm{c}}(M) \to \Omega^p_{\mathrm{c}}(M)$.

Definition 11.16. The *p-th de Rham cohomology space with compact support* of M is the vector space $H^p_{\mathrm{c}}(M)$ quotient of $Z^p_{\mathrm{c}}(M)$ by $B^p_{\mathrm{c}}(M)$.

If $f\colon M \to N$, the image of $\omega \in \Omega^p_{\mathrm{c}}(N)$ under the induced $\Omega^p(f)\colon \Omega^p(N) \to \Omega^p(M)$ does not necessarily have compact support. We need an additional hypothesis on f for this.

Definition 11.17. A map $f\colon M \to N$ between topological spaces is *proper* if the preimage under f of every compact subset of N is a compact subset of M.

If $f\colon M \to N$ is proper, then its induced map $\Omega^p(f)$ sends $\Omega^p_{\mathrm{c}}(N)$ to $\Omega^p_{\mathrm{c}}(M)$. By the above considerations, it induces a map

$$H^p_{\mathrm{c}}(f)\colon H^p_{\mathrm{c}}(N) \to H^p_{\mathrm{c}}(M).$$

Definition 11.18. Two proper maps $f_0, f_1\colon M \to N$ are *properly homotopic* if there exists a differentiable map $H\colon M \times \mathbb{R} \to N$ such that:

(1) $H(x, 0) = f_0(x)$ for every $x \in M$;
(2) $H(x, 1) = f_1(x)$ for every $x \in M$;
(3) for every compact subset $K \subset N$, there exists a compact subset $L \subset M$ such that $H^{-1}(K) \subset L \times \mathbb{R}$.

The third condition essentially says that H is "proper in the M-direction."

Theorem 11.19. *If $f_0\colon M \to N$ and $f_1\colon M \to N$ are properly homotopic, then the induced homomorphisms $H^p_{\mathrm{c}}(f_0)\colon H^p_{\mathrm{c}}(N) \to H^p_{\mathrm{c}}(M)$ and $H^p_{\mathrm{c}}(f_1)\colon H^p_{\mathrm{c}}(N) \to H^p_{\mathrm{c}}(M)$ are equal.*

CHAPTER 12

Orientation and integration

1. Orientation of manifolds

DEFINITION 12.1. An atlas $\{(U_i, \varphi_i)\}_{i\in I}$ for the manifold M is *oriented* if, for every i, $j \in I$ and for every $x \in \varphi_i(U_i \cap U_j)$, the tangent map $T_x\varphi_j \circ \varphi_i^{-1}$ of the coordinate change map has positive determinant.

DEFINITION 12.2. The manifold M is *orientable* if it admits an oriented atlas (compatible with the differentiable structure of M). An *orientation* for M is an oriented atlas which is maximal among all oriented atlases. An *oriented* manifold consists of the data of a manifold M together with an orientation for M.

DEFINITION 12.3. In the manifold M of dimension m, a *volume form* is a form $\omega \in \Omega^m(M)$ of degree m such that, for every $x \in M$, $\omega(x) \neq 0$ in $\mathrm{Alt}^m(T_xM)$.

THEOREM 12.4. *There exists a volume form in M if and only if M is orientable. In addition, a volume form ω specifies an orientation for M.*

COROLLARY 12.5. *A connected manifold admits 0 or 2 orientations.*

EXAMPLE 12.6. Every open subset of $\mathbb{R}^n$ is oriented. The sphere S^n and the torus T^n are orientable. The Möbius strip and the Klein bottle are not orientable.

2. Integration of differential forms

The starting point is the Change of Variables Formula for multiple integration in calculus:

THEOREM 12.7. *Let U and V be open in $\mathbb{R}^n$, let $f: V \to \mathbb{R}^n$ be a differentiable function, and let $\varphi: U \to V$ be a diffeomorphism. Then*

$$\iint_V f(y_1,\dots,y_n)\,dy_1\dots dy_n = \iint_U f\circ\varphi(x_1,\dots,x_n)\left|\det\left(\frac{\partial\varphi_i}{\partial x_j}(x_1,\dots,x_n)\right)\right| dx_1\dots dx_n$$

See a standard real analysis textbook for a proof.

THEOREM 12.8. *Let M be an oriented manifold of dimension m, with oriented atlas $\{(U_i,\varphi_i)\}_{i\in I}$. There exists a unique linear map*

$$I\colon \Omega^m_c(M) \longrightarrow \mathbb{R}$$

such that, when the support of ω is contained in U_i and $\Omega^m(\varphi_i^{-1})(\omega) = f_i\,dx_1 \wedge \dots \wedge dx_n$ in $\Omega^m_c(\varphi_i(U_i))$,

$$I(\omega) = \int_M \omega = \iint_{\varphi_i(U_i)} f_i(x_1,\dots,x_n)\,dx_1\dots dx_n.$$

By definition, $I(\omega)$ is *the integral of* ω *over the manifold* M, and is denoted by

$$I(\omega) = \int_M \omega$$

The uniqueness and definition of I are immediately proved by using a suitable partition of unity.

To prove that the definition makes sense, the key property here is that, when the support of ω is contained in both U_i and U_j, the above multiple integrals over $\varphi_i(U_i)$ and $\varphi_j(U_j)$ give the same result. This immediately follows from Lemma 9.8 and the above Change of Variables Formula, together with the fact that the atlas $\{(U_i, \varphi_i)\}_{i \in I}$ is oriented to remove absolute values.

To a large extent, the abstract definition of differential forms was specially designed so that the above integral makes sense. Which begs for another question: Why did we bother with differential forms of degree $p < m$ on an m–dimensional manifold?

DEFINITION 12.9. Let $\omega \in \Omega^p(M)$ be a differential form of degree p on the m–dimensional manifold M, and let P be an oriented p–dimensional submanifold of M. If, in addition, $P \cap \mathrm{supp}(\omega)$ is compact, the *integral of* ω *over the submanifold* P is

$$\int_P \Omega^p(i)(\omega)$$

where $i\colon P \to M$ is the inclusion map. (Note that $\Omega^p(i)(\omega)$ has degree p and compact support in P.) This integral is denoted by $\int_P \omega$.

CHAPTER 13

Manifolds with boundary Stokes Theorem

1. Manifolds-with-boundary

DEFINITION 13.1. The standard *n-dimensional lower half-space* is

$$H^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n; x_1 \leqslant 0\}.$$

Its *boundary* is the hyperplane

$$\partial H^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n; x_1 = 0\} = 0 \times \mathbb{R}^{n-1}.$$

DEFINITION 13.2. A *topological manifold-with-boundary* (in one word) of dimension m is a separable Hausdorff topological space M such that, for every $x \in M$, there is an open neighborhood U of x and a homeomorphism $\varphi\colon U \to V$ from U to an open subset V of the half-space H^m.

DEFINITION 13.3. A *differentiable atlas* for the topological manifold-with-boundary M is a family $\mathcal{A} = \{(U_i, \varphi_i); i \in I\}$ where:

(1) U_i is an open subset of M for every $i \in I$, and $M = \bigcup_{i \in I} U_i$;
(2) for every $i \in I$, $\varphi_i\colon U_i \to V_i$ is a homeomorphism from U_i to an open subset V_i of H^m;
(3) for every i, $j \in I$, the composition $\varphi_j \circ \varphi_i^{-1}{}_{|\varphi_i(U_i \cap U_j)}\colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is differentiable.

Here, a function φ defined on an open subset U of H^m is differentiable if it is the restriction to U of a differentiable (in the usual sense) function defined on an open subset V of $\mathbb{R}^m$ with $U = V \cap H^m$.

Everything we have done so far for manifolds (without boundary) straightforwardly extends to manifolds with boundary. In particular, a differentiable structure is defined by a maximal atlas, etc....

DEFINITION 13.4. The *boundary* of the manifold-with-boundary M is the subset ∂M consisting of those points which are sent to points of the boundary ∂H^n by the coordinate charts (U_i, φ_i).

PROPOSITION 13.5. *If M is a manifold-with-boundary of dimension m, the restriction of its maximal atlas makes ∂M a differentiable manifold (with no boundary) often dimension $m - 1$. Its complement $M - \partial M$ is a manifold (with no boundary) of dimension m.*

2. Stokes's theorem

Suppose now that the manifold-with-boundary M is oriented. By restriction of the oriented atlas, this defines an orientation on the boundary ∂M.

The inclusion map $i\colon \partial M \to M$ is easily seen to be differentiable, and therefore induces a map $\Omega^p(i)\colon \Omega^p(M) \to \Omega^p(\partial M)$.

THEOREM 13.6 (Stokes Theorem). *Let M be an oriented manifold-with-boundary of dimension m. Let $\alpha \in \Omega_c^{m-1}(M)$ be a differential form with compact support of degree $m-1$. Then*

$$\int_M d\alpha = \int_{\partial M} \Omega^{m-1}(i)(\alpha)$$

COROLLARY 13.7. *Let M be an oriented m–dimensional manifold without boundary. Then,*

$$\int_M d\alpha = 0$$

for every $\alpha \in \Omega_c^{m-1}(M)$.

EXAMPLE 13.8. On $\mathbb{R}^2 - \{0\}$, the 1–form $\frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$ is closed but not exact.

CHAPTER 14

Top-dimensional de Rham cohomology

A corollary of the Stokes Theorem is the following:

COROLLARY 14.1. *Let M be an oriented manifold of dimension m, with no boundary. There is a well-defined map*

$$I\colon H^m_{\mathrm{c}}(M) \to \mathbb{R}$$

wich to the class of $\omega \in \Omega^m_{\mathrm{c}}(M)$ associates its integral $\int_M \omega$.

THEOREM 14.2. *Let M be an oriented manifold of dimension m, with no boundary. Supposed in addition that M is connected. Then the above map $I\colon H^m_{\mathrm{c}}(M) \to \mathbb{R}$ is an isomorphism.*

EXAMPLE 14.3. $H^{n-1}(\mathbb{R}^n - \{0\}) \cong H^{n-1}(S^{n-1}) \cong \mathbb{R}$.

1. A topological application

Let $B^n = \{x \in \mathbb{R}^n; \|x\| \leqslant 1\}$ be the unit closed ball in $\mathbb{R}^n$. This is a manifold-with-boundary, with boundary the $(n-1)$–sphere S^{n-1}.

PROPOSITION 14.4. *There is no differentiable map $r\colon B^n \to S^{n-1}$ such that $r(x) = x$ for every $x \in S^{n-1}$.*

COROLLARY 14.5 (Brouwer Fixed Point Theorem). *For every differentiable map $f\colon B^n \to B^n$, there exists at least one point $x \in M$ with $f(x) = x$.*

CHAPTER 15

The degree of a map

Let M and N be two oriented connected manifolds of the same dimensions m, and let $f\colon M \to N$ be a proper map. Remember that f will be automatically proper if M is compact.

1. The cohomological degree of f

From the diagram

$$\begin{array}{ccc} H_c^m(N) & \xrightarrow{H_c^m(f)} & H_c^m(M) \\ {\scriptstyle I}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle I} \\ \mathbb{R} & \dashrightarrow & \mathbb{R} \end{array}$$

we obtain a linear map $I \circ H_c^m(f) \circ I^{-1}\colon \mathbb{R} \to \mathbb{R}$, which must be multiplication by a real number $\deg(f)$.

By definition, the number $\deg(f)$ is the *degree* of the proper map $f\colon M \to N$. Note that the definition of this degree is independent of any choice.

2. The geometric degree of f

Let x be a regular point of f. Let $\{(U_i, \varphi_i)\}_{i\in I}$ be an oriented atlas for the orientation of M, and let $\{(V_j, \psi_j)\}_{j\in J}$ be an oriented atlas for the orientation of N. If $x \in U_i$ and $f(x) \in V_j$, we can consider the translation $\psi_j \circ f \circ \varphi_i^{-1}$ of f in these charts, and its (invertible) tangent map $T_{\varphi_i(x)}\psi_j \circ f \circ \varphi_i^{-1}\colon \mathbb{R}^m \to \mathbb{R}^m$.

We say that $f\colon M \to N$ is *orientation-preserving* at the regular point $x \in M$ if $T_{\varphi_i(x)}\psi_j \circ f \circ \varphi_i^{-1}$ has positive determinant, and *orientation-reversing* otherwise. Because the atlases $\{(U_i, \varphi_i)\}_{i\in I}$ and $\{(V_j, \psi_j)\}_{j\in J}$ are oriented, this is independent of the choice of charts.

We define the *geometric degree of f at the regular point x* as

$$\deg_x(f) = \begin{cases} +1 & \text{if } f \text{ is orientation-preserving at } x \\ -1 & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Finally, if $y \in N$ is a regular value of f, the *geometric degree of f at the regular value y* is

$$\deg_y(f) = \sum_{x\in f^{-1}(y)} \deg_x(f) \in \mathbb{Z}.$$

Note that the sum is finite because f is proper.

3. The Degree Theorem

THEOREM 15.1. *The geometric degree* $\deg_y(f)$ *of* $f\colon M \to N$ *at each regular value* $y \in N$ *is equal to the cohomological degree* $\deg(f)$.

COROLLARY 15.2. *The cohomological degree* $\deg(f)$ *is an integer.*

COROLLARY 15.3. *The geometric degree* $\deg_y(f)$ *is independent of the regular value* $y \in N$.

COROLLARY 15.4. *If* M *is compact and* N *is non-compact, the degree* $\deg(f)$ *is* 0.

COROLLARY 15.5. *The parity* $\in \mathbb{Z}_2$ *of the number of points of* $f^{-1}(y)$ *is independent of the regular value* y.

4. Example: the Gauss integral

Consider two disjoint closed curves α, β in $\mathbb{R}^3$. We can choose the parametrizations $\alpha, \beta\colon \mathbb{R} \to \mathbb{R}^3$ so that $\alpha(t+1) = \alpha(t)$, $\beta(t+1) = \beta(t)$ for every $t \in \mathbb{R}$, and $\alpha(t) \neq \beta(u)$ for every t, $u \in \mathbb{R}$ since the curves are disjoint. Consider the integral

$$L(\alpha,\beta) = \frac{1}{4\pi}\int_0^1\int_0^1 \frac{\det\big[\alpha(t)-\beta(u), \alpha'(t), \beta'(u)\big]}{\|\alpha(t)-\beta(u)\|^3}\,dt\,du.$$

PROPOSITION 15.6. *The number* $L(\alpha,\beta)$ *is an integer.*

Indeed, let $\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy \in \Omega^2(\mathbb{R}^3)$, and consider the function $F\colon \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to S^2$ defined by

$$F(t,u) = \frac{\alpha(t)-\beta(u)}{\|\alpha(t)-\beta(u)\|}.$$

If $i\colon S^2 \to \mathbb{R}^3$ is the inclusion map, a computation shows that

$$\Omega^2(i \circ F)(\omega) = \frac{\det\big[\alpha(t)-\beta(u), \alpha'(t), \beta'(u)\big]}{\|\alpha(t)-\beta(u)\|^3}\,dt \wedge du \in \Omega^2(\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}),$$

so that $L(\alpha,\beta)$ is the degree of F.

We can give a geometric computation of $L(\alpha,\beta) \in \mathbb{Z}$. Let $v \in S^2$ be a regular value of f. In particular, there are finitely many parameter values (t_1,u_1), (t_2,u_2), $\dots$, $(t_n,u_n) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ such that $\alpha(t_i) - \beta(u_i)$ is parallel to and points in the same direction as v. In addition, the three vectors v, $\alpha'(t_i)$ and $\beta'(u_i)$ are linearly independent at these points. Let $\varepsilon_i = \pm 1$ be the sign of $\det\big[v, \alpha'(t_i), \beta'(u_i)\big]$. (See what this means when v points towards your eye.) Then

$$L(\alpha,\beta) = \sum_{i=1}^n \varepsilon_i.$$

CHAPTER 16

The de Rham theorem

1. Singular cohomology

This is a brief description of singular cohomology, as taught in a typical course in algebraic topology (for instance MATH 540 at USC).

DEFINITION 16.1. The *standard simplex* of dimension n is

$$\Delta_n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1}; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geqslant 0, \forall i\}.$$

Namely Δ_n is the convex hull of the $n+1$ points that are the tips of the coordinate vectors of $\mathbb{R}^{n+1}$.

We have to think of Δ_n as a "piece of n–dimensional manifold".

We can embed the $(n-1)$–dimensional simplex Δ_{n-1} in the n–dimensional simplex Δ_n, but there are $n+1$ natural ways to do so.

DEFINITION 16.2. For $i = 0, \dots, n$, the i*–face map* is the map $F_i\colon \Delta_{n-1} \to \Delta_n$ defined by

$$F_i(t_0, t_1, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}),$$

where the 0 is inserted in the $(i+1)$–th position.

Note that the boundary $\partial\Delta_n$ is the union of the images $F_i(\Delta_{n-1})$ of the $n+1$ face maps.

DEFINITION 16.3. A *(singular) n–simplex* in a topological space X is a continuous map $\sigma\colon \Delta_n \to X$.

The set of all singular n–simplices is denoted by $\mathcal{S}_n(X)$.

DEFINITION 16.4. A *(real valued) singular n–cochain* in the topological space X is a map $c\colon \mathcal{S}_n(X) \to \mathbb{R}$.

The singular cochains form a vector space $C^n(X; \mathbb{R})$.

DEFINITION 16.5. The *coboundary* of a cochain $c \in C^n(X; \mathbb{R})$ is the cochain $dc \in C^{n+1}(X; \mathbb{R})$ defined by the formula

$$dc(\sigma) = \sum_{i=0}^{n+1} (-1)^i c(\sigma \circ F_i) \in R$$

for every $(n+1)$–simplex σ. Note that each face map $F_i\colon \Delta_n \to \Delta_{n+1}$ defines an n–simplex $\sigma \circ F_i\colon \Delta_n \to X$.

We now have a sequence of vector spaces and linear maps

$$\cdots \xrightarrow{d} C^{n-1}(X; \mathbb{R}) \xrightarrow{d} C^n(X; \mathbb{R}) \xrightarrow{d} C^{n+1}(X; \mathbb{R}) \xrightarrow{d} \cdots$$

LEMMA 16.6.

$$d \circ d = 0$$

As a consequence, the kernel $Z^n(X;\mathbb{R})$ of $d\colon C^n(X;\mathbb{R}) \to C^{n+1}(X;\mathbb{R})$ contains the image $B^n(X;\mathbb{R})$ of $d\colon C^{n-1}(X;\mathbb{R}) \to C^n(X;\mathbb{R})$.

DEFINITION 16.7. The *n-th cohomology space (with coefficients in $\mathbb{R}$)* of X is the vector space

$$H^n(X;\mathbb{R}) = Z^n(X;\mathbb{R})/B^n(X;\mathbb{R}).$$

2. Differentiable cohomology

When the space X is a differentiable manifold M, it is natural to restrict attention to singular simplices $\sigma\colon \Delta_n \to M$ that are differentiable. To makes sense of this, consider the standard simplex Δ_n as contained in the hyperplane $V_n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1}; \sum_{i=0}^{n} t_i = 1\}$ in $\mathbb{R}^{n+1}$, and say that $\sigma\colon \Delta_n \to M$ is differentiable when it extends to a differentiable map from an open subset of V_n to M.

Let $\mathcal{S}_n^{\text{diff}}(M)$ be the set of differentiable singular simplices $\sigma\colon \Delta_n \to M$. We then define the vector spaces $C^n_{\text{diff}}(M;\mathbb{R})$ of cochains $c\colon \mathcal{S}_n^{\text{diff}}(M) \to \mathbb{R}$, and coboundary maps $d\colon C^n_{\text{diff}}(M;\mathbb{R}) \to C^{n+1}_{\text{diff}}(M;\mathbb{R})$ as before.

DEFINITION 16.8. The *n-th differential cohomology space (with coefficients in $\mathbb{R}$)* of the manifold M is the vector space

$$H^n_{\text{diff}}(M;\mathbb{R}) = Z^n_{\text{diff}}(M;\mathbb{R})/B^n_{\text{diff}}(M;\mathbb{R}),$$

quotient of the kernel $Z^n_{\text{diff}}(M;\mathbb{R})$ of $d\colon C^n_{\text{diff}}(M;\mathbb{R}) \to C^{n+1}_{\text{diff}}(M;\mathbb{R})$ by the image $B^n_{\text{diff}}(M;\mathbb{R})$ of $d\colon C^{n-1}_{\text{diff}}(M;\mathbb{R}) \to C^n_{\text{diff}}(M;\mathbb{R})$.

The inclusion map $\mathcal{S}_n^{\text{diff}}(M) \to \mathcal{S}_n(M)$ induces a linear map $H^n(M;\mathbb{R}) \to H^n_{\text{diff}}(M;\mathbb{R})$.

THEOREM 16.9. *The map $H^n(M;\mathbb{R}) \to H^n_{\text{diff}}(M;\mathbb{R})$ is a linear isomorphism.*

This is proved by the standard techniques of approximation of continuous maps by differentiable maps.

3. De Rham's Theorem

A differential form $\omega \in \Omega^p(M)$ defines a differentiable cochain $c_\omega \in C^p_{\text{diff}}(M;\mathbb{R})$ by the property that

$$c_\omega(\sigma) = \int_{\Delta_p} \Omega^p(\sigma)(\omega)$$

for every differentiable singular simplex $\sigma\colon \Delta_p \to M$.

This is the "historic example" of cochains, which justifies the definition of cohomology (and homology).

The following result is a consequence of Stokes's theorem.

PROPOSITION 16.10.

$$c_{d\omega} = dc_\omega$$

for every differential form $\omega \in \Omega^p(M)$, where the first d is the exterior derivative $d\colon \Omega^p(M) \to \Omega^{p+1}(M)$ and the second d is the coboundary map $d\colon C^p_{\text{diff}}(M;\mathbb{R}) \to C^{p+1}_{\text{diff}}(M;\mathbb{R})$.

This property provides a linear map $H^p(M) \to H^p_{\text{diff}}(M; \mathbb{R})$ from the de Rham cohomology to the differentiable cohomology.

THEOREM 16.11 (de Rham). *The map $H^p(M) \to H^p_{\text{diff}}(M; \mathbb{R})$ is a linear isomorphism.*

Consequently, for a manifold M, the de Rham cohomology space $H^p(M)$, the differentiable cohomology space $H^p_{\text{diff}}(M; \mathbb{R})$ and the singular cohomology space $H^p(M; \mathbb{R})$ are all isomorphic. Note that de Rham cohomology, based on differential forms, looks very different from the other cohomologies, based on simplices.