MATH 540, Fall 2015
Final Exam

Rules: You have 3 hours to complete the exam. You can use your personal notes, and the notes that I posted on the web, but you can only refer to topology results that we have seen in class (as documented by my notes). Good luck!

Problem 1. (12.5 %) In the 3-dimensional space $\mathbb{R}^3$, let L be the z–axis, let C_1 be the circle of radius 1 that is contained in the horizontal plane $z = 1$ and is centered at $(0, 0, 1)$, and let C_2 be the circle of radius 1 that is contained in the horizontal plane $z = 2$ and is centered at $(0, 0, 2)$. Finally, let $X = \mathbb{R}^3 - L \cup C_1 \cup C_2$ be the complement of these three curves in $\mathbb{R}^3$.

- **a.** (No credit) Draw a picture. It may also be useful to observe the nice rotational symmetry, for future use in Part b.
- **b.** Compute the fundamental group $\pi_1(X; x_0)$, for whatever choice of base point x_0 you may prefer. Try to be as convincing as possible, but a little bit of handwaving is OK if, for instance, you use deformation retracts or other geometric arguments.

Problem 2. (12.5 %) Let $p : \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ path connected. Suppose in addition that the fundamental group $\pi_1(X; x_0)$ has exactly n elements. Show that the fiber $p^{-1}(x_0)$ has at most n elements.

Problem 3. (25 %) Let $p \colon \widetilde{X} \to X$ be a covering space with X locally path connected. In addition, suppose that each fiber $p^{-1}(x)$ has exactly f elements.

- **a.** Show that a singular n–simplex $\sigma \colon \Delta_n \to X$ has exactly f lifts $\widetilde{\sigma} \colon \Delta_n \to \widetilde{X}$ with $p \circ \widetilde{\sigma} = \sigma$.
- **b.** Consider the module homomorphisms $\tau_n \colon C_n(X) \to C_n(\widetilde{X})$ defined by the property that

$$\tau_n(\sigma) = \sum_{\widetilde{\sigma} \text{ lifts } \sigma} \widetilde{\sigma}$$

  for every n–simplex $\sigma \colon \Delta_n \to X$, where the sum is over the f lifts $\widetilde{\sigma} \colon \Delta_n \to \widetilde{X}$ of σ as in Part a. Show that the τ_n form a chain map $\tau \colon C(X) \to C(\widetilde{X})$, and induce a homomorphism $H_n(\tau) \colon H_n(X) \to H_n(\widetilde{X})$.
- **c.** If $H_n(p) \colon H_n(\widetilde{X}) \to H_n(X)$ denotes the homomorphism induced by the covering map $p \colon \widetilde{X} \to X$, compute the composition $H_n(p) \circ H_n(\tau) \colon H_n(X) \to H_n(X)$.

Problem 4. (20 %) Let A be a subset of X.

- **a.** Suppose that A is homotopy equivalent to a point. Show that $H_n(X, A)$ is isomorphic to $H_n(X)$ for every $n \geqslant 1$. Hint: long exact sequence in relative homology; make sure that you carefully deal with the case $n = 1$. Is $H_0(X, A)$ is isomorphic to $H_0(X)$?

b. Suppose that X is homotopy equivalent to a point. Show that $H_n(X, A)$ is isomorphic to $H_{n-1}(A)$ for every $n \geqslant 2$. Show that this is in general false if $n = 1$.

Problem 5. (15 %) In this whole problem, we assume that $n \geqslant 2$ (to avoid having to worry about the non-connectedness of S^0.

a. Let B be an open ball in $\mathbb{R}^n$ centered at 0. Compute the relative homology modules $H_p(B, B - \{0\})$. Hint: long exact sequence.

b. Let U be an open subset of $\mathbb{R}^n$. For $x \in X$, compute the modules $H_p(U, U - \{x\})$. Hint: excision.

c. Use Part **b** to show that, if an open subset U of $\mathbb{R}^m$ (with $m \geqslant 2$) is homeomorphic to an open subset V of $\mathbb{R}^n$, then necessarily $m = n$.

Problem 6. (20 %) Let $X = S^1 \times S^n$ be the product of the circle S^1 with the sphere S^n with $n \geqslant 2$. Compute the homology modules $H_p(X; R)$. Hint: Write the circle S^1 as the union of two intervals, and remember the case of the n–dimensional torus.

Math 535, Spring 2016

Final Exam

Problem 1. (12%) For a differentiable map $f : S^5 \to S^3$ and a differential form $\alpha \in \Omega^3(S^3)$, show that the form $\Omega^3(f)(\alpha) \in \Omega^3(S^5)$ is closed.

Problem 2. (16%) Let M be an m–dimensional submanifold of $\mathbb{R}^n$. For $p \leqslant n$, let A be the set of those $a \in \mathbb{R}^p$ for which $M \cap (\{a\} \times \mathbb{R}^{n-p})$ is a manifold of dimension $m - p$. Show that $\mathbb{R}^p - A$ has measure 0 in $\mathbb{R}^p$. (Hint: $\mathbb{R}^n \to \mathbb{R}^p$).

Problem 3. (16%) Let M be an m–dimensional submanifold of $\mathbb{R}^n$. Consider the subset N of $\mathbb{R}^m \times \mathbb{R}^n$ consisting of all pairs (x, v) where $x \in M$ and where $v \in \mathbb{R}^n$ is orthogonal to the tangent space $T_xM \subset \mathbb{R}^n$. We want to show that N is a manifold of dimension n.

Recall that, by definition of submanifolds of $\mathbb{R}^n$, there exists a family $\{(U_i, \varphi_i)\}_{i\in I}$ such that: each $\varphi_i \colon U_i \to V_i$ is a diffeomorphism between open subsets U_i and V_i of $\mathbb{R}^n$; for each i, $\varphi_i(U_i \cap M) = V_i \cap (\mathbb{R}^m \times 0)$; and $M \subset \bigcup_{i\in I} U_i$. Define $W_i = \{(x, v) \in N; x \in U_i\}$ and $\psi_i \colon W_i \to \mathbb{R}^n$ by

$$\psi_i(x, v) = (\varphi_i(x), \pi \circ T_x\varphi_i(v)) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$$

where $\pi \colon \mathbb{R}^n \to \mathbb{R}^{n-m}$ denotes the projection to the last $n - m$ coordinates. Show that $\{(W_i, \psi_i)\}_{i\in I}$ is a differentiable atlas for N.

Problem 4. (20%) Let M be an oriented m–dimensional manifold. Let y_0 be a regular value of the differentiable map $f \colon M \to \mathbb{R}$, and let $N = f^{-1}(y_0)$.

We showed in class (when proving that N is a submanifold of M) that, for every $x \in N$, there exists an open subset $U_x \subset M$ and a diffeomorphism $\varphi_x \colon U_x \to V_x$ to an open subset $V_x \subset \mathbb{R}^m = \mathbb{R} \times \mathbb{R}^{m-1}$ such that $f_{|U_x} = p \circ \varphi_x$, where $p \colon \mathbb{R}^m \to \mathbb{R}$ is the projection to the first coordinate.

- **a.** Show that the homeomorphisms φ_x can be chosen so that (U_x, φ_x) is in the maximal oriented atlas $\mathcal{A}$ defining the orientation of M. Hint: Do not let the maximal oriented atlas non-sense scare you.
- **b.** Show that N is orientable.

Problem 5. (16%) An m–dimensional manifold M is *parallelizable* if its tangent bundle $TM = \{(x, v); x \in M, v \in T_xM\}$ admits a differentiable map $f \colon TM \to M \times \mathbb{R}^m$ such that $f(x, v) \in \{x\} \times \mathbb{R}^m$ for every $x \in M$ and $v \in T_xM$, and such that the map $v \mapsto f(x, v)$ induces a linear isomorphism $T_xM \to \{x\} \times \mathbb{R}^m \cong \mathbb{R}^m$. Show that M is parallelizable if and only if it admits m vector fields X_1, X_2, …, X_m such that the vectors $X_1(x)$, $X_2(x)$, …, $X_m(x)$ are linearly independent in T_xM for every $x \in M$.

Problem 6. (20%) For $p \geqslant 2$, let S^{p-1} be the unit sphere in $\mathbb{R}^p$.

- **a.** Consider the inclusion map $j \colon S^{p-1} \to (\mathbb{R}^p - \{0\}) \times \mathbb{R}^q$ defined by $j(x) = (x, 0)$. Show that the induced homomorphism $H^{p-1}(j) \colon H^{p-1}((\mathbb{R}^p - \{0\}) \times \mathbb{R}^q) \to H^{p-1}(S^{p-1})$ is an isomorphism.

b. Show that, for every differentiable map $f\colon \mathbb{R}^p \longrightarrow \mathbb{R}^p \times \mathbb{R}^q$ such that $f(x) = (x, 0)$ for every $x \in S^{p-1}$, the image $f(\mathbb{R}^p)$ has non-empty intersection with $0 \times \mathbb{R}^q$. Hint: Otherwise, consider the domain restriction $f_{(\mathbb{R}^p - \{0\}) \times \mathbb{R}^q}\colon \mathbb{R}^p \longrightarrow (\mathbb{R}^p - \{0\}) \times \mathbb{R}^q$, the inclusion map $i\colon S^{p-1} \longrightarrow (\mathbb{R}^p - \{0\}) \times \mathbb{R}^q$, the above inclusion map $j\colon S^{p-1} \longrightarrow (\mathbb{R}^p - \{0\}) \times \mathbb{R}^q$, and the homomorphisms between various $H^{p-1}(\;)$ induced by these maps.

Diff Geo

Math 535a
Practice Midterm

Problem 1. Let $M \subset \mathbb{R}^{2n}$ consist of those pairs $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ such that x and y are orthogonal in $\mathbb{R}^n$, namely such that $\sum_{i=1}^n x_i y_i = 0$ if $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Show that $M - \{0\}$ is a submanifold of $\mathbb{R}^{2n} - \{0\}$.

Problem 2. Let $M \subset \mathbb{R}^{2n}$ be the subset defined in the previous problem. Show that, if M was a submanifold of $\mathbb{R}^{2n}$, the tangent space $T_0 M \subset \mathbb{R}^{2n}$ would contain the subspaces $\mathbb{R}^n \times \{0\}$ and $\{0\} \times \mathbb{R}^n$. Conclude that M is not a submanifold of $\mathbb{R}^{2n}$.

Problem 3. Let M be an m–dimensional submanifold of $\mathbb{R}^n$ such that, at every $x \in M$, the tangent space $T_x M$ is never horizontal, namely is never contained in $\mathbb{R}^{n-1} \times \{0\}$. Show that, for every $a \in \mathbb{R}$, the intersection of M with the hyperplane $P_a = \{(x_1, \dots, x_n) \in \mathbb{R}^n; x_n = a\}$ is a submanifold of M of dimension $m - 1$.

Problem 4. Let the cyclic group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ act on the manifold $S^{n-1} \times \mathbb{R}$ by the property that $a \cdot (x, t) = \big((-1)^a x, (-1)^a t\big)$ for every $a \in \mathbb{Z}_2$ and $(x, t) \in S^{n-1} \times \mathbb{R}$.

- **a.** Show that the quotient space $M = (S^{n-1} \times \mathbb{R})/\mathbb{Z}_2$ admits a differentiable structure for which the quotient map $p\colon S^{n-1} \times \mathbb{R} \to M$ is a local diffeomorphism.
- **b.** Let $\varphi\colon S^{n-1} \times \mathbb{R} \to \mathbb{RP}^n$ be the function which to $(x, t) \in S^{n-1} \times \mathbb{R}$ associates the line in $\mathbb{R}^{n+1}$ joining the origin 0 to the point (x, t). Show that φ is differentiable for the standard differentiable structure of the projective space $\mathbb{RP}^n$.
- **c.** Show that there exists a differentiable map $\bar{\varphi}\colon M \to \mathbb{RP}^n$ such that $\bar{\varphi} \circ p = \varphi$. Make sure that you prove differentiability.
- **d.** Show that $\bar{\varphi}$ induces a diffeomorphism between M and the complement of a point in $\mathbb{RP}^n$.

MATH 535

Spring 2016

Homework 6, due on Monday March 21

Problem 1. Let $f\colon M \to \mathbb{R}^n$ be an immersion from the m–dimensional manifold M in $\mathbb{R}^n$.

a. In the tangent bundle TM, consider the subset

$$T^1M = \{(x, v) \in TM; \|T_xf(v)\| = 1\}$$

where $\|w\|$ denotes the length of the vector $w \in \mathbb{R}^n$. Show that T^1M is a manifold of dimension $2m-1$. Possible hint: Consider the map $\varphi\colon TM \to \mathbb{R}$ defined by $\varphi(x, v) = \|T_xf(v)\|$.

b. For every $v \in S^{n-1}$, let $\pi_v\colon \mathbb{R}^n \to v^\perp$ be the orthogonal projection from $\mathbb{R}^n$ to the orthogonal subspace $v^\perp \cong \mathbb{R}^{n-1}$. Show that, if $n > 2m$, there exists $v \in S^{n-1}$ such that $\pi_v \circ f\colon M \to v^\perp \cong \mathbb{R}^{n-1}$ is an immersion. Possible hint: Consider the map $F\colon T^1M \to S^{n-1}$ defined by $F(x, v) = T_xf(v)/\|T_xf(v)\|$.

c. Show that there exists an immersion $g\colon M \to \mathbb{R}^{2m}$.

Problem 2. Let $f\colon M \to \mathbb{R}^n$ be an embedding from an m–dimensional manifold M to $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$.

a. Show that every horizontal subspace $\mathbb{R}^p \times \{z_0\}$ is arbitrarily close to a subspace $\mathbb{R}^p \times \{z\}$ whose preimage $f^{-1}\big(\mathbb{R}^p \times \{z\}\big)$ is an $(m-n+p)$–dimensional submanifold of M.

b. Show that, for $z \in \mathbb{R}^{n-p}$ as in Part **b**, the intersection $f(M) \cap \mathbb{R}^p \times \{z\}$ is a submanifold of $\mathbb{R}^n$.

Diff Geo

Another practice midterm

Problem 1. Recall that $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ denotes the vector space of all square matrices of order n, and that the general linear group $\mathrm{GL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ is the open subset consisting of all invertible matrices. Let $\Sigma \in \mathrm{GL}_n(\mathbb{R})$ be an invertible matrix that is antisymmetric, in the sense that $\Sigma^{\mathrm{t}} = -\Sigma$ (where Σ^{t} denotes the transpose).

a. Let $f\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ be the map defined by $f(A) = A\Sigma A^{\mathrm{t}}$. Show that the image of $T_A f\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ consists of all antisymmetric matrices.

b. Use Part a to show that

$$S = \{A \in \mathrm{GL}_n(\mathbb{R}); A\Sigma A^{\mathrm{t}} = \Sigma\}$$

is a submanifold of $\mathrm{M}_n(\mathbb{R})$. What is its dimension?

Problem 2.

a. Let U be an open subset of $\mathbb{R}^m$, and let $x_0 \in U$ be a fixed point for the map $f\colon U \to \mathbb{R}^m$, namely a point such that $f(x_0) = x_0$. Suppose in addition that the eigenvalues of the tangent map $T_{x_0} f\colon \mathbb{R}^m \to \mathbb{R}^m$ are all different from 1, namely that $T_{x_0} f(v) \neq v$ for every $v \neq 0$. Show that there exists an open subset $V \subset U$ such that x is the only fixed point of f that is contained in V. Hint: Set $g(x) = f(x) - x$.

b. Let x_0 be a fixed point of the map $f\colon M \to M$, where M is a manifold. Suppose in addition that the eigenvalues of the tangent map $T_{x_0} f\colon T_{x_0} M \to T_{x_0} M$ are all different from 1, namely that $T_{x_0} f(v) \neq v$ for every $v \neq 0$ in $T_{x_0} M$. Show that there exists an open subset $V \subset M$ such that x is the only fixed point of f that is contained in V.

Problem 3.

Let M be an m–dimensional submanifold of $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$. Show that there exists a subset X of full measure in $\mathbb{R}^p$ such that, for every $x \in X$, the intersection of M with the subspace $\{x\} \times \mathbb{R}^{n-p}$ is a submanifold of M. What is the dimension of this intersection?

Math 535a
Spring 2016
Midterm exam

Problem 1. Consider

$$M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4; x_1x_2 + x_3x_4 = 0\}.$$

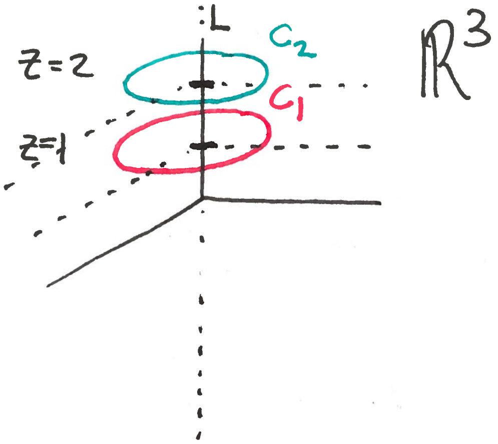
- **a.** Show that $M - \{(0,0,0,0)\}$ is a submanifold of $\mathbb{R}^4$.
- **b.** For each coordinate vector $v \in \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$, show that there exists a curve $\alpha_v\colon (\varepsilon, \varepsilon) \to M$ such that $\alpha_v(0) = (0,0,0,0)$ and $\alpha_v'(0) = v$.
- **c.** Show that M is not a submanifold of $\mathbb{R}^4$. Hint: Part b.

Problem 2. Let $f\colon M \to N$ be a differentiable map between manifolds M and N of the same dimension. Show that, for almost every $y_0 \in N$, the preimage $f^{-1}(y_0)$ is discrete in the sense that, for every $x \in f^{-1}(y_0)$, there exists an open subset $U \subset M$ such that $U \cap f^{-1}(y_0) = \{x\}$.

Problem 3. Let the group Γ act differentiably, freely and discontinuously on the manifold M, and let $p\colon M \to M/\Gamma$ is the natural projection to the quotient space. Let $f\colon M \to N$ be a differentiable map from M to another differentiable manifold N. Suppose in addition that $f(\gamma x) = f(x)$ for every $x \in M$ and every $\gamma \in \Gamma$, so that f induces a map $g\colon M/\Gamma \to N$ such that $f = g \circ p$. Show that g is differentiable for the quotient differentiable structure on M/Γ.

Fall 2015 Bonahon Topology Final

①



$$X = \mathbb{R}^3 - (L \cup C_1 \cup C_2)$$

Compute the fundamental group $\pi_1(X; x_0)$ for any desired $x_0 \in X$.

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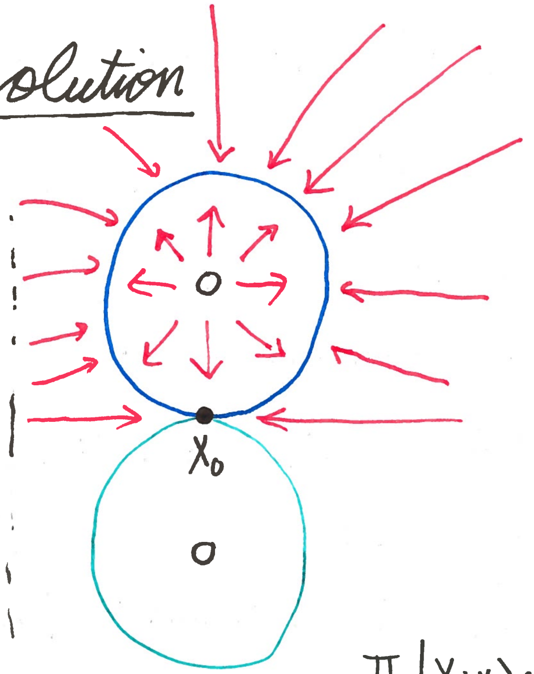
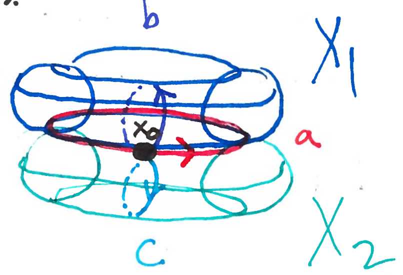
solution



The picture shows how X deformation retracts onto two tori connected along a common circle. So

$\pi_1(X; x_0) \cong \pi_1(X_1 \cup X_2; x_0)$.



$\pi_1(X_1 \cap X_2; x_0) = \langle [a] \rangle$

$\pi_1(X_1; x_0) = \langle [a], [b] \mid [a][b][\bar{a}][\bar{b}] = 1 \rangle$

$\pi_2(X_2; x_0) = \langle [a], [c] \mid [a][c][\bar{a}][\bar{c}] = 1 \rangle$

(*) next page.

By SvK II, $\pi_1(X; x_0) \cong \langle [a], [b], [c] \mid [a][b][\bar{a}][\bar{b}] = [a][c][\bar{a}][\bar{c}] = 1 \rangle$ V $\square$

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② Let $p : \tilde{X} \to X$ be a covering space w/ $\tilde{X}$ p.c. Suppose $\pi_1(X; x_0)$ has exactly n elts. Show that the fiber $p^{-1}(x_0)$ has at most n elts.

<u>Solution</u> Fix $\tilde{x}_0 \in p^{-1}(x_0)$. We define a function $f : \pi_1(X; x_0) \longrightarrow p^{-1}(x_0)$ and show it is a surjection. Given a loop α based at x_0, let $\tilde{\alpha}$ be the unique lift of α based at $\tilde{x}_0$. Set $f([\alpha]) = \tilde{\alpha}(1)$. We need to show this is well-defined. Let $\alpha \sim \beta$. So there is a path homotopy $F : I \times I \to X$, $F(s,0) = \alpha$, $F(s,1) = \beta$, $F(t,0) = F(t,1) = x_0$. By the homotopy lifting property, F lifts to a path homotopy $\tilde{F}$ between $\tilde{\alpha}$ and $\tilde{\beta}$. In particular $\tilde{\alpha}(1) = \tilde{\beta}(1)$, so f is well-defined. Since $\tilde{X}$ is p.c., f is a surjection, as $\tilde{x} \in p^{-1}(x_0)$ is mapped to under f by $[p \circ \tilde{\gamma}]$ where $\tilde{\gamma}$ is a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}$. We conclude $p^{-1}(x_0)$ has at most n elts. $\square$.

(*) More specifically, $\pi_1(X_1) = \langle a, b \mid ab\bar{a}\bar{b} = 1 \rangle$ and $\pi_2(X_2) = \langle a', c \mid a'c\bar{a'}\bar{c} = 1 \rangle$ and $\pi_1(X_1 \cap X_2) = \langle x \rangle$, and so $\pi_1(X_1) *_{\pi_1(X_1 \cap X_2)} \pi_2(X_2) = \langle a, b, a', c \mid ab\bar{a}\bar{b} = a'c\bar{a'}\bar{c} = i_1(x)\overline{i_2(x)} = 1 \rangle =$
$= \langle a, b, a', c \mid ab\bar{a}\bar{b} = a'c\bar{a'}\bar{c} = a\bar{a'} = 1 \rangle \cong \langle a, b, c \mid ab\bar{a}\bar{b} = ac\bar{a}\bar{c} = 1 \rangle$

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③ Let $p : \tilde{X} \to X$ be a covering space w/ X p.c., locally p.c. Assume every fiber has the same number f of elts.

(a) Show the singular n-simplex $\sigma : \Delta_n \to X$ has exactly f lifts $\tilde{\sigma} : \Delta^n \to \tilde{X}$ w/ $p \circ \tilde{\sigma} = \sigma$.

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solution to (a)

Since X is p.c., locally p.c., and since $\pi_1(\Delta^n_{;y_0}) = 1$, by the Lifting Criterion there is a unique lift $\tilde{\sigma} : \Delta^n \to \tilde{X}$ satisfying $y_0 \mapsto \tilde{x}_0$ and $p \circ \tilde{\sigma} = \sigma$ where $\tilde{x}_0$ is fixed in the fiber of $x_0 = \sigma(y_0)$.

Since $\tilde{x}_0$ is arbitrary and there are f elts in the fiber, we see there are exactly f lifts $\tilde{\sigma} : \Delta^n \to \tilde{X}$ s.t. $p \circ \tilde{\sigma} = \sigma$. $\square$.

(b) Define module homomorphisms $\tau_n : C_n(X) \to C_n(\tilde{X})$ by the property that

$$\tau_n(\sigma) = \sum_{\tilde{\sigma} \text{ lifts } \sigma} \tilde{\sigma} \quad \text{for } \sigma \text{ a singular } n\text{-simplex.}$$

Show that the τ_n form a chain map $\tau : C(X) \to C(\tilde{X})$ and thus induce a homomorphism $H_n(\tau) : H_n(X) \to H_n(\tilde{X})$.

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solution to (b)

We need to show $\tilde{\partial}\tau_n(\sigma) = \tau_n(\partial\sigma)$.

$$\tilde{\partial}\tau_n(\sigma) = \tilde{\partial}\Big(\sum_{\tilde{\sigma} \text{ lifts } \sigma} \tilde{\sigma}\Big) = \sum_{\tilde{\sigma}}\sum_{i=0}^{n}(-1)^i\,\tilde{\sigma}\circ f_i .$$

$$\tau_n(\partial\sigma) = \sum_{i=0}^{n}(-1)^i\,\tau_n(\sigma\circ f_i).$$

So we need to show $\tau_n(\sigma\circ f_i) = \sum_{\tilde{\sigma} \text{ lifts } \sigma} \tilde{\sigma}\circ f_i$.

This will be the case if
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \Delta^n \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & X \end{tikzcd}$$
implies
$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ \Delta^{n-1} \arrow[ur, "\tilde{\sigma}\circ f_i"] \arrow[r, "\sigma\circ f_i"'] & X \end{tikzcd}.$$
Indeed,
$$\begin{tikzcd} & & \tilde{X} \arrow[d, "p"] \\ \Delta^{n-1} \arrow[r, "f_i"] & \Delta^n \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & X \end{tikzcd}.$$

So τ is a chain map, as desired. $\square$.

(c) If $H_n(p) : H_n(\tilde{X}) \longrightarrow H_n(X)$ is the homomorphism induced by $\tilde{X} \xrightarrow{p} X$, compute $H_n(p)\circ H_n(\tau) : H_n(X) \longrightarrow H_n(X)$.

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solution to (c)

Let $[c] \in H_n(X)$, $c \in C_n(X)$, $\partial c = 0 \in C_{n-1}(X)$. Write $c = \sum_i a_i\sigma_i$ for $\sigma_i : \Delta^n \to X$.

$$H_n(p)\circ H_n(\tau)([c]) = H_n(p)\Big(\Big[\sum_i a_i \sum_{\tilde{\sigma}_i \text{ lifts } \sigma_i} \tilde{\sigma}_i\Big]\Big) \overset{i}{=} \Big[\sum_i a_i \sum_{\tilde{\sigma}_i \text{ lifts } \sigma_i} p\circ\tilde{\sigma}_i\Big] = \Big[\sum_i a_i \sum_{\tilde{\sigma}_i \text{ lifts } \sigma_i} \sigma_i\Big] =$$

$= [f \sum_i a_i \sigma_i] = f[\sum_i a_i \sigma_i] = f[c]. \quad \square$

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④ Let $A \subset X$.

(a) Suppose $A \simeq \{*\}$. Show $H_n(X, A) \cong H_n(X)$ for all $n \geq 1$. what about $n=0$?

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solution to (a)

$n>1$)

$$\to \underset{0}{H_n(A)} \to H_n(X) \xrightarrow{\sim} H_n(X,A) \to \underset{0}{H_{n-1}(A)} \to \qquad \checkmark.$$

$n=1$)

$$\to \underset{0}{H_1(A)} \to H_1(X) \xrightarrow{f} H_1(X,A) \xrightarrow{g} H_0(A) \xrightarrow{h} H_0(X) \to$$

$$\mathbb{Z} \to \mathbb{Z}^r$$

$H_1(X) \cong \operatorname{im} f \cong \ker g = H_1(X, A). \quad \checkmark$

$x \mapsto (x, 0, \dots, 0)$

(since $A \subset$ component of X)

$\operatorname{im} g \cong \ker h = 0$

$A \neq \emptyset$, so $H_0(X, A) = 0 \not\cong H_0(X) \cong \mathbb{Z}^r \; (r \geq 1)$.

$\square$

(b) Let $X \simeq \{*\}$. Show

$H_n(X, A) \cong H_{n-1}(A)$ for $n \geq 2$.

Show this is false in general for $n = 1$.

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solution to (b)

$n>1$) $\to H_n(A) \to \underset{\bigcirc}{H_n(X)} \to H_n(X,A) \overset{\sim}{\to} H_{n-1}(A) \to \underset{\bigcirc}{H_{n-1}(X)} \to$

$n=1$) $A = \{*\}$. $H_n(X,A) = 0$. $H_n(A) = \mathbb{Z}$. $\square$.

---

⑤ Let $n \geq 2$.

(a) Let B be an open ball in $\mathbb{R}^n$ centered at 0. Compute the relative homology modules $H_p(B, B\sim\{0\})$.

---

solution to (a)

$p \geq 2$) By part (b) of prev problem, $H_p(B, B-\{0\}) \cong H_{p-1}(B-\{0\}) \cong H_{p-1}(S^{n-1})$

$$\cong \begin{cases} \mathbb{Z}, & p = n \\ 0, & p \neq n \end{cases}.$$

$p=1$

$$\to H_1(\overset{S^{n-1}}{B-\{0\}}) \to H_1(B) \xrightarrow{f} H_1(B,B-\{0\}) \xrightarrow{g} H_0(\overset{S^{n-1}}{B-\{0\}}) \xrightarrow{h} H_0(B) \to$$

$$\begin{cases} \mathbb{Z}, & n=2 \\ 0, & n>2 \end{cases} \qquad 0 \qquad\qquad \mathbb{Z} \to \mathbb{Z}$$

$$x \mapsto x$$

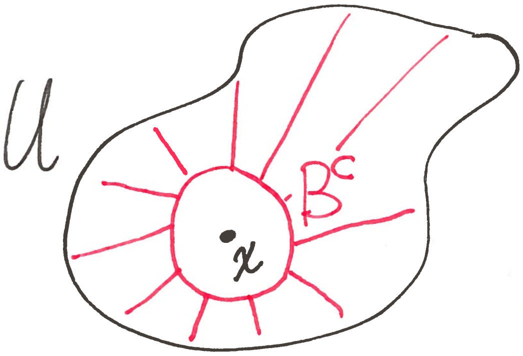
$$H_1(B,B-\{0\}) \cong \operatorname{img} \cong \ker h = 0 .$$

$$H_0(B,B-\{0\}) \cong 0 . \qquad \square .$$

(b) Let $U \subset \mathbb{R}^n$ open. For $x \in U$, compute the modules $H_p(U, U-\{x\})$.

Solution to (b)

Excision: $H_p(X-B^c, A-B^c) \cong H_p(X,A)$



$X \equiv U, \quad A \equiv U-\{x\},$

$\overline{B^c} = B^c \subset A .$

Here, B is a small ball centered at x.

So $H_p(U, U-\{x\}) \cong H_p(U-B^c, U-B^c-\{x\}) \cong H_p(B, B-\{x\})$

which is $\cong \begin{cases} \mathbb{Z}, & p=2=n \\ 0 & \text{else} \end{cases}$ by (a). $\square .$

(c) Use (b) to show if an open subset U of $\mathbb{R}^m$ is homeomorphic to an open subset V of $\mathbb{R}^n$, then $m=n$. (Here, $m,n \geq 2$).

solution to (c)

Remove a point x from U and a point y from V.

Since $U \cong V$, $H_p(U, U-\{x\}) \cong H_p(V, V-\{x\})$ for all p ✓

from which we see $m=n$ is forced. □.

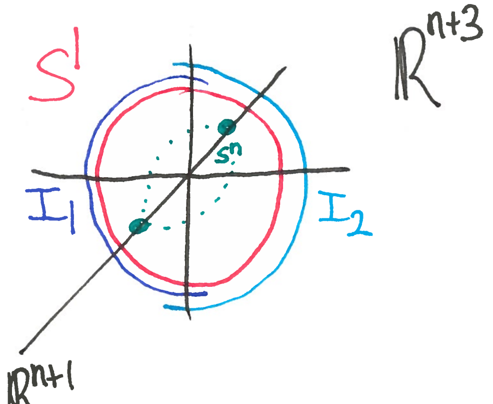
---

⑥ Let $X = S^1 \times S^n$, $n \geq 2$.

Compute $H_p(X)$.

---

solution



$$I_1 \times S^n \xrightarrow{j_1} S^1 \times S^n$$

$$I_2 \times S^n \xrightarrow{j_2} S^1 \times S^n$$

$$I_1 \cap I_2 \times S^n \xrightarrow{i_1} I_1 \times S^n$$

$$I_1 \cap I_2 \times S^n \xrightarrow{i_2} I_2 \times S^n$$

$$H_k(S^n)\oplus H_k(S^n) \to H_k(S^n) \oplus H_k(S^n) \to H_k(S^1\times S^n) \to$$

$$\to H_k(I_1\cap I_2\times S^n) \xrightarrow{(-H_k(i_1))\oplus H_k(i_2)} H_k(I_1\times S^n)\oplus H_k(I_2\times S^n) \xrightarrow{H_k(j_1)+H_k(j_2)} H_k(S^1\times S^n) \to$$

$$\to H_{k-1}(I_1\cap I_2\times S^n) \to H_{k-1}(I_1\times S^n)\oplus H_{k-1}(I_2\times S^n) \to$$

$$\to H_{k-1}(S^n)\oplus H_{k-1}(S^n) \longrightarrow H_{k-1}(S^n)\oplus H_{k-1}(S^n)$$

Recall
$n\geq 2$.

When $k\neq n, n+1$ and $k\neq 1$:

$$0\to H_k(S^1\times S^n)\to 0 \quad\Rightarrow\quad \boxed{H_k(S^1\times S^n)=0.}$$

When $k=n$:

$(x,y)\mapsto x+y$

$$\mathbb{Z}\oplus\mathbb{Z}\xrightarrow{f}\mathbb{Z}\oplus\mathbb{Z}\xrightarrow{g}H_n(S^1\times S^n)\xrightarrow{h}$$

$(x,y)\mapsto(-x-y,x+y)$

$$\to 0$$

$H_n(S^1\times S^n)\cong\ker h\cong \operatorname{im} g.$

$\ker f=\{(x,y):x=-y\}\cong\mathbb{Z}.$

$\operatorname{rank}(\operatorname{im} f)+\operatorname{rank}(\ker f)=\operatorname{rank}(\mathbb{Z}\oplus\mathbb{Z})=2.$

$\qquad\qquad\quad \|$

$\operatorname{rank}(\operatorname{im} f)+1 \Rightarrow \underline{\operatorname{rank}(\operatorname{im} f)=1}$.

we know kerg is free, we know it is at least $\{(x,-x)\}$ which has rank 1, and so we know this is the whole kernel.

$\therefore \ker g\cong \operatorname{im} f\cong\mathbb{Z}.$

next page (*)

$\Rightarrow \ker g=\{(x,y):x=-y\}.$

$\Rightarrow \operatorname{im} g\cong \mathbb{Z}\oplus\mathbb{Z}/\ker g.$

$=\mathbb{Z}\oplus\mathbb{Z}/\{(x,-x)\}$

$=\mathbb{Z}(x,-x)\oplus\mathbb{Z}(x,x)/\mathbb{Z}(x,-x)\cong\mathbb{Z}(x,x)\cong\boxed{\mathbb{Z}}$

When $k=n+1$:

$$0 \to 0 \to H_{n+1}(S^1\times S^n) \xrightarrow{f} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g} \mathbb{Z}\oplus\mathbb{Z} \to$$

$$(x,y)\mapsto(-x-y,\, x+y)$$

$$H_{n+1}(S^1\times S^n) \cong \operatorname{im} f \cong \ker g \cong \boxed{\mathbb{Z}.}$$

When $k=1$:

$$0 \to 0 \to H_1(S^1\times S^n) \xrightarrow{f} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g} \mathbb{Z}\oplus\mathbb{Z} \to$$

and the exact same calculation as for $k=n+1$ gives again $\boxed{H_1(S^1\times S^n) \cong \mathbb{Z}.}$

Conclusion:

$H_k(S^1\times S^n)$

0	1	2	--------	n-1	n	n+1	n+2	-----
$\mathbb{Z}$	$\mathbb{Z}$	0	--------	0	$\mathbb{Z}$	$\mathbb{Z}$	0	-----

∎

(*) What we used was a little short cut, exploiting that the rank-nullity theorem holds for maps $R^a \to R^b$ where R is any PID. If instead of $\mathbb{Z}$ we were working over an arbitrary commutative ring R (recall R-submodules of a free R-module need not be free in this case) then we see explicitly that $\operatorname{im} f = \{(x,-x) : x\in R\} \cong R$ is free of rank 1. In hindsight, this is a clearer way to see $\ker g = \operatorname{im} f = \{(x,-x): x\in R\}$ which is what we needed.

Spring 2016 Bonahon Differential Geometry Final

① $f : S^5 \to S^3$ smooth. $\alpha \in \Omega^3(S^3)$.

Show $\Omega^3(f)(\alpha) \in \Omega^3(S^5)$ is closed.

---

solution

We have the general formula $\partial \Omega^p(f)(\omega) = \Omega^p(f)(\partial\omega)$.

So $\partial \Omega^3(f)(\alpha) = \Omega^4(f)(\partial\alpha) = 0$

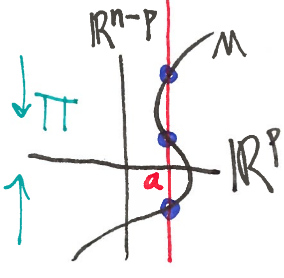
where we have used $\partial\alpha = 0$ since $\Omega^4(S^3) = 0$ and $\alpha \in \Omega^3(S^3)$. $\square$.

---

② M m-dim submfld of $\mathbb{R}^n$. For $p \leq n$ let A be the collection of $a \in \mathbb{R}^p$ s.t. $M \cap (\{a\} \times \mathbb{R}^{n-p})$ is a mfld of dim $m-p$. Show $\mathbb{R}^p - A$ has measure 0 in $\mathbb{R}^p$.

---

solution



Let $i : M \to \mathbb{R}^n$ be the natural embedding of M into $\mathbb{R}^n$.
Let π be the natural projection from $\mathbb{R}^n \cong \mathbb{R}^p \times \mathbb{R}^{n-p} \longrightarrow \mathbb{R}^p$. According to Sard's theorem, a.e. elt of $\mathbb{R}^p$ is a regular value of $\pi \circ i$. That is, a.e. elt a of $\mathbb{R}^p$ has the

property that $(\pi\circ i)^{-1}(y) = M\cap(\{a\}\times\mathbb{R}^{n-p})$ is a submanifold of M of dim $m-p$. That is, $\mathbb{R}^p - A$ has measure 0 in $\mathbb{R}^p$. □

---

③ Let M be an m-dim submfld of $\mathbb{R}^n$. Consider $N\subset\mathbb{R}^m\times\mathbb{R}^n$ consisting of all pairs (x,v) where $x\in M$ and $v\in(T_xM)^{\perp}\subset\mathbb{R}^n$. The goal is to show N is a manifold of dimension n.

By the definition of a submanifold of $\mathbb{R}^n$, there is a family $\{(U_i,\varphi_i)\}_{i\in I}$ s.t. for each $i\in I$, $\varphi_i: U_i\to V_i$ is a diffeomorphism between open subsets U_i and V_i of $\mathbb{R}^n$; for each $i\in I$, $\varphi_i(U_i\cap M) = V_i\cap(\mathbb{R}^m\times\{0\})$; and $M\subset\bigcup_{i\in I}U_i$.

Define $W_i = \{(x,v)\in N : x\in U_i\}$ and $\psi_i: W_i\to\mathbb{R}^n$ by $\psi_i(x,v) = (\varphi_i(x),\pi\circ T_x\varphi_i(v))\in\mathbb{R}^m\times\mathbb{R}^{n-m}$ where $\pi:\mathbb{R}^n\cong\mathbb{R}^m\times\mathbb{R}^{n-m}\to\mathbb{R}^{n-m}$ is the natural projection. Show $\{(W_i,\psi_i)\}_{i\in I}$ is a smooth atlas for M.

---

solution

Recall that for $\{(W_i, \psi_i)\}_{i \in I}$ to be a smooth atlas for N means (i) $N \subset \bigcup_{i \in I} W_i$; (ii) for each $i \in I$, ψ_i gives a homeomophism of W_i onto its image; and (iii) for every $i, j \in I$, the map $\psi_j \circ \psi_i^{-1} : \psi_i(W_i \cap W_j) \longrightarrow \psi_j(W_i \cap W_j)$ is a smooth map between open subsets of $\mathbb{R}^n$.

Since $M \subset \bigcup_{i \in I} U_i$, it is clear condition (i) is satisfied. Regarding (ii) certainly ψ_i is continuous. For injectivity, assume $\psi_i(X, v) = \psi_i(X', v')$. So $\varphi_i(X) = \varphi_i(X')$ and $\pi \circ T_X \varphi_i(v) = \pi \circ T_{X'} \varphi_i(v')$. Since φ_i is bijective, $X = X'$. Since φ_i is a diffeomorphism, $T_X \varphi_i : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. We need to show π is an isomorphism of $T_X\varphi_i((T_X M)^{\perp})$ onto its image $\subset \mathbb{R}^{n-m}$. By definition, $T_X M$ is the image of $\mathbb{R}^m \times 0$ under $T_{\varphi_i(X)} \varphi_i^{-1}$. Since $T_X \varphi_i$ is an isomorphism, this means $(T_x M)^\perp$ is mapped by $T_x \varphi_i$ to a complement of $\mathbb{R}^m \times 0$. Hence π acts isomorphically on $T_x \varphi_i((T_x M)^\perp)$. So as $\pi \circ T_x \varphi_i(v) = \pi \circ T_x \varphi_i(v')$, it follows that $T_x \varphi_i(v) = T_x \varphi_i(v')$, from which it follows that $v = v'$. Therefore, ψ_i is injective.

We also need that $\varphi_i^{-1} : V_i \to U_i$ is continuous. We have already established that $\pi : T_x \varphi_i((T_x M)^\perp) \to$ $\to \pi(T_x \varphi_i((T_x M)^\perp))$ is an iso. Therefore, we have an explicit formula for ψ_i^{-1}; it is given by $(y, w) = (\varphi_i^{-1}(y), (T_x \varphi_i)^{-1} \circ \pi^{-1}(w))$. Since φ_i is a homeo, the first coordinate is continuous, as is the second by linearity. We conclude ψ_i is a homeomorphism.

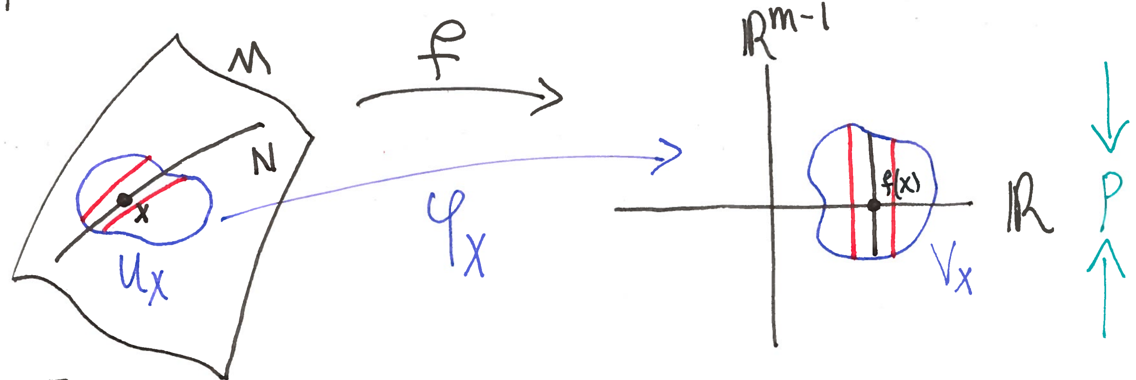
Lastly, we need to show (iii). We have an explicit formula for $\psi_j \circ \psi_i^{-1} : \psi_i(W_i \cap W_j) \to \psi_j(W_i \cap W_j)$ given by $(y, w) \mapsto (\varphi_j \circ \varphi_i^{-1}(y), \pi \circ T_x \varphi_j \circ (T_x \varphi_i)^{-1} \circ \pi^{-1}(w))$. Since φ_i and φ_j are diffeomorphisms, the left coordinate is smooth, as is the right coordinate by linearity.

We conclude N is a smooth mfld given the atlas $\{(U_i, \varphi_i)\}_{i \in I}$. $\square$

④ Let M be an oriented m-dim mfld, w/ maximal oriented atlas $\mathcal{A}$.
Let y_0 be a regular value of the smooth map $f: M \to \mathbb{R}$, and let $N = f^{-1}(y_0)$.

When proving N is a submanifold of M, it is shown that for each $x \in N$ there is an open set $U_x \subset M$ and a diffeomorphism$^{(*)}$ $\varphi_x : U_x \to V_x$ onto an open set $V_x \subset \mathbb{R}^m \cong \mathbb{R} \times \mathbb{R}^{m-1}$ s.t. $f|_{U_x} = p \circ \varphi_x$ where $p: \mathbb{R}^m \cong \mathbb{R} \times \mathbb{R}^{m-1} \to \mathbb{R}$ is the projection onto the first coordinate.

(*) we assume (U_x, φ_x) has already been shown to be compatible w/ the smooth structure on M as given by the oriented atlas $\mathcal{A}$.



(a) For $m>1$. Show that the φ_x can be chosen s.t. (U_x, φ_x) is in the maximal oriented atlas $\mathcal{A}$ defining the orientation of M.

(b) Show N is orientable.

solution to (a)

What does it mean to say $(\mathcal{U}_X, \varphi_X) \in \mathcal{A}$? It means for any $(\mathcal{U}_i, \varphi_i) \in \mathcal{A}$,

$$\det T_y(\varphi_X \circ \varphi_i^{-1}) > 0 \quad \text{and} \quad \det T_z(\varphi_i \circ \varphi_X^{-1}) > 0$$

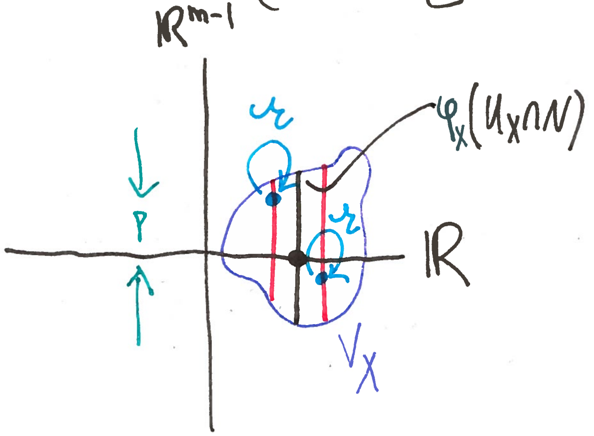
for all $y \in \varphi_i(\mathcal{U}_i \cap \mathcal{U}_X)$ and $z \in \varphi_X(\mathcal{U}_i \cap \mathcal{U}_X)$.

Since $T_z(\varphi_i \circ \varphi_X^{-1}) = \left(T_{\varphi_i \circ \varphi_X^{-1}(z)}(\varphi_X \circ \varphi_i^{-1})\right)^{-1}$

it suffices just to show the first of the two inequalities above. However, we will proceed in a different manner.

Assume for the time being that $\mathcal{M}$ is connected. The chart $(\mathcal{U}_X, \varphi_X)$ is contained in one of the two maximal oriented atlases, because the one-element collection $\{(\mathcal{U}_X, \varphi_X)\}$ is vacuously an oriented atlas.

Suppose $(U_x, \varphi_x) \notin \mathcal{A}$. Consider the reflection $r : \mathbb{R}^m \cong \mathbb{R} \times \mathbb{R}^{m-1} \to \mathbb{R} \times \mathbb{R}^{m-1} \cong \mathbb{R}^m$ along the second coordinate (recall $m>1$), i.e. $(x_1, x_2, \dots, x_m) \overset{r}{\mapsto} (x_1, -x_2, \dots, x_m)$. we claim $(U_x, r \circ \varphi_x) \in \mathcal{A}$. Note first that clearly $(U_x, r \circ \varphi_x)$ is compatible w/ the Smooth structure on M, and we still have the equality $f|_{U_x} = p \circ \varphi_x = p \circ r \circ \varphi_x$.



And $(U_x, r\circ\varphi_x)$ is in same maximal oriented atlas.

Then for any $y \in \varphi_x(U_x)$,

$$\det T_y((r \circ \varphi_x) \circ \varphi_x^{-1}) = \det r = -1.$$

Hence, $(U_x, r \circ \varphi_x)$ is not in the same maximal oriented atlas as (U_x, φ_x). Since there are only two maximal oriented atlases, we gather $(U_x, r \circ \varphi_x) \in \mathcal{A}$, as desired.

If M is not connected, we just do this procedure on each component, depending on whether φ_x needs to be reflected or not. $\square$

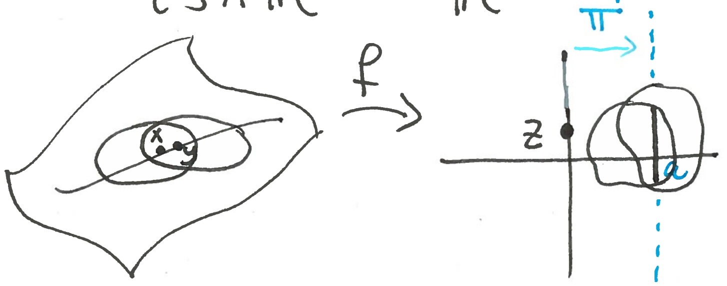
---

solution to (b)

If $m=1$, then N is 0-dim and is trivially orientable. So assume $m>1$. Let (U_x, φ_x) be as in part (a) and assume $\{(U_x,\varphi_x)\}_{x\in N} \subset \mathcal{A}$ the maximal oriented atlas defining the orientation on M.

The smooth atlas for N is $\{(U_x\cap N, \pi\circ\varphi_x)\}_{x\in N}$ where $\pi:\mathbb{R}^m\to\mathbb{R}^{m-1}$ is the projection onto the last $m-1$ coordinates.

We need to show $\det T_z(\pi\circ\varphi_x\circ\varphi_y^{-1}\circ\pi^{-1})>0$ for all $z\in\pi\circ\varphi_y(U_x\cap U_y\cap N)$. Here, we are viewing π^{-1} as an embedding $\mathbb{R}^{m-1}\cong\{0\}\times\mathbb{R}^{m-1}\to$ $\to \{a\}\times\mathbb{R}^{m-1}\subset\mathbb{R}^m$, where $\{a\}=p(f(U_x\cap U_y\cap N))$.



Now, $T_z(\pi \circ \varphi_x \circ \varphi_y^{-1} \circ \pi^{-1}) =$

$$= T_{\varphi_x \circ \varphi_y^{-1}(a,z)}(\pi)\, T_{(a,z)}(\varphi_x \circ \varphi_y^{-1})\, T_z(\pi^{-1})$$

$$= \underbrace{\Pi}_{\mathbb{R}^m \to \mathbb{R}^{m-1}}\ \underbrace{T_{(a,z)}(\varphi_x \circ \varphi_y^{-1})}_{\mathbb{R}^m \to \mathbb{R}^m}\ \underbrace{\pi^{-1}}_{\mathbb{R}^{m-1} \to \mathbb{R}^m}.$$

The ij-th entry of this matrix (*) is

$$\sum_{k=1}^{m} \Pi_{ik} \left(T_{(a,z)}(\varphi_x \circ \varphi_y^{-1})\, \pi^{-1} \right)_{kj} \qquad \begin{pmatrix} i = 2, \dots, m \\ j = 2, \dots, m \end{pmatrix}$$

(*) the first row is $i=2$, the first column is $j=2$, etc.

$$= \sum_{k=1}^{m} \sum_{\ell=1}^{m} \Pi_{ik} \left(T_{(a,z)}(\varphi_x \circ \varphi_y^{-1}) \right)_{k\ell} (\pi^{-1})_{\ell j}.$$

Since $\Pi_{ik} = \delta_{ik}$ and $(\pi^{-1})_{\ell j} = \delta_{\ell j}$

this becomes $\overset{i=k,\ell=j}{=} T_{(a,z)}(\varphi_x \circ \varphi_y^{-1})_{ij}$.

~~Observe as well that $T_{(a,z)}(\varphi_x \circ \varphi_y^{-1})_{11} = \partial_1 (\varphi_x \circ \varphi_y^{-1})^1 (a,z) = \partial_1$~~

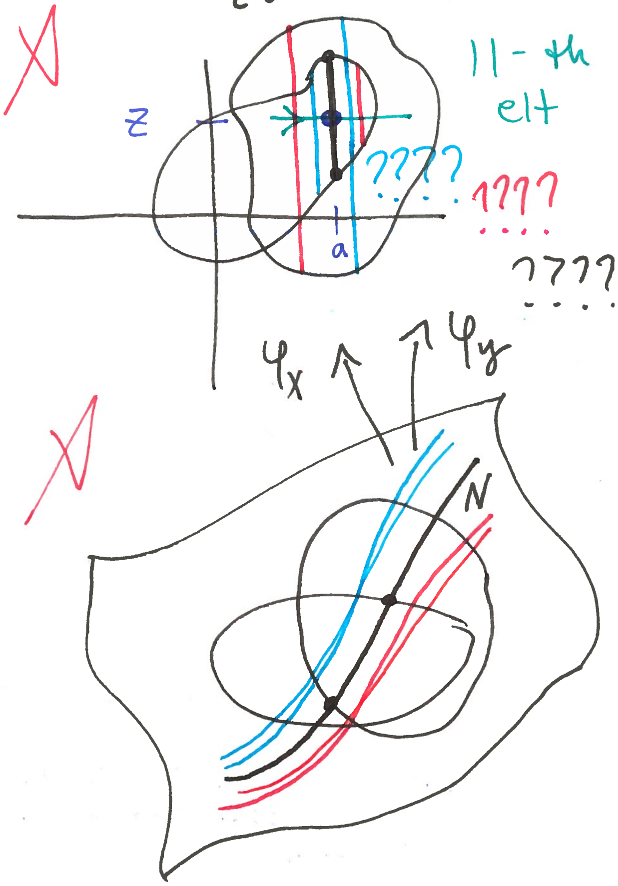
Now, we need to understand the 1st row and column.

$$\left(T_{(a,z)}(\varphi_x \circ \varphi_y^{-1})\right)_{\|} = \partial_1 (\varphi_x \circ \varphi_y^{-1})^1 (a,z) =$$

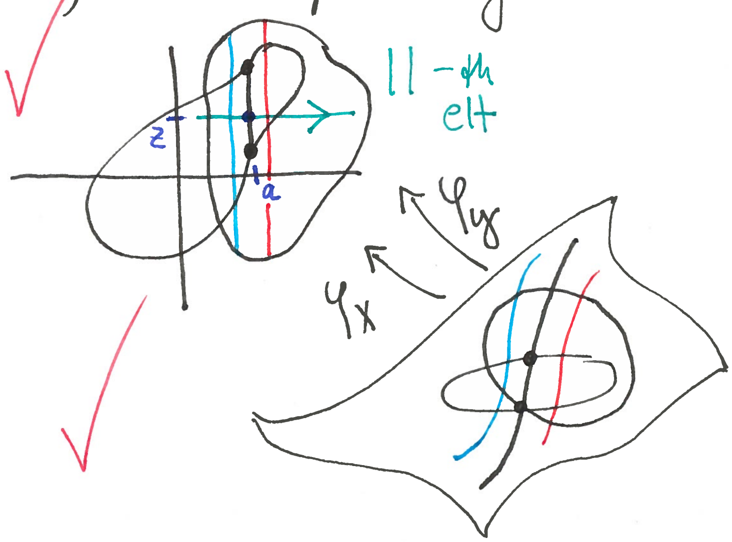
$$= \frac{d}{dt}\Big|_{t=0} (\varphi_x \circ \varphi_y^{-1})^1 (t+a, z).$$

Conceivably, this partial derivative could be $\geqslant 0$ or $\leqslant 0$.

We need to show this is not possible (the ≤ 0).



This concern was in vain, and the pictures above are absurd. There can only be one blue line and one red line, each representing a level set of f. Here is the correct picture:



Intuitively, φ_x and φ_y only differ on a level set in the amount of vertical stretching.

In equations, $(\varphi_X \circ \varphi_y^{-1})'(t+a, z) =$

$$= (p \circ \varphi_X) \circ \varphi_y^{-1}(t+a, z) = f \circ \varphi_y^{-1}(t+a, z) =$$

$$= (p \circ \varphi_y) \circ \varphi_y^{-1}(t+a, z) = t+a.$$

So $\left(T_{(a,z)}(\varphi_X \circ \varphi_y^{-1})\right)_{11} = \frac{d}{dt}\Big|_{t=0}(t+a) = 1.$

Similarly, $\left(T_{(a,z)}(\varphi_X \circ \varphi_y^{-1})\right)_{1j} = \frac{d}{dt}\Big|_{t=0} (\varphi_X \circ \varphi_y^{-1})'(a, z+te_j)$

$= \frac{d}{dt}\Big|_{t=0}(a) = 0.$ Consequently, the matrix

$$T_{(a,z)}(\varphi_X \circ \varphi_y^{-1}) = \left(\begin{array}{c|c} 1 & 0 \cdots\cdots 0 \\ \hline * & \\ | & T_z(\pi \circ \varphi_X \circ \varphi_y^{-1} \circ \pi^{-1}) \\ * & \end{array}\right).$$

By the assumption of part (a), we have

$\operatorname{Det} T_{(a,z)}(\varphi_X \circ \varphi_y^{-1}) > 0$ hence

$\operatorname{Det} T_z(\pi \circ \varphi_X \circ \varphi_y^{-1} \circ \pi^{-1}) > 0$ as desired.

We conclude N is orientable. $\square$

⑤ An m-dim mfld M is <u>parallelizable</u> if its tangent bundle $TM = \{(x,v) : x \in M, v \in T_xM\}$ admits a smooth map $f : TM \to M \times \mathbb{R}^m$ s.t.
(i) $f(x,v) \in \{x\} \times \mathbb{R}^m$ for every $x \in M$ and $v \in T_xM$,
and (ii) the map $v \mapsto f(x,v)$ induces a linear isomorphism $T_xM \to \{x\} \times \mathbb{R}^m \cong \mathbb{R}^m$.

Show M is parallelizable iff M admits m vector fields $X_1, X_2, \dots, X_m$ s.t. the vectors $X_1(x), X_2(x), \dots, X_m(x)$ are l.i. in T_xM for every $x \in M$.

<u>solution</u>

$(\Longrightarrow)$

Define $X_i(x) = \overset{\overset{T_xM}{\rotatebox{90}{$\in$}}}{f^{-1}(x, -)(e_i)}$ for $x \in M$ and $i = 1, 2, \dots, m$. Each X_i is smooth since f is, and satisfies the desired l.i. property since $f^{-1}(x, -)$ is an isomorphism. ✓

($\Leftarrow$)

Define $f(x, -) : T_x M \to \mathbb{R}^m$ by the reverse map $\mathbb{R}^m \xrightarrow{g_x} T_x M$, $e_i \mapsto X_i(x)$, which is an isomorphism by the freeness of vector spaces & since $X_i(x)$ is l.i. in $T_x M$ by assumption; so $f(x,-) := g_x^{-1}$.

Thus we have defined $f : TM \to M \times \mathbb{R}^m$ which is clearly smooth, by the smoothness of the X_i, and satisfies (i) and (ii). ✓. $\square$.

---

⑥ Let $p \geqslant 2$ and S^{p-1} be the unit sphere in $\mathbb{R}^p$.

(a) Consider the inclusion $j : S^{p-1} \to (\mathbb{R}^p - \{0\}) \times \mathbb{R}^q$ defined by $j(x) = (x, 0)$. Show the induced map $H^{p-1}(j) : H^{p-1}((\mathbb{R}^p - \{0\}) \times \mathbb{R}^q) \longrightarrow H^{p-1}(S^{p-1})$ is an isomorphism.

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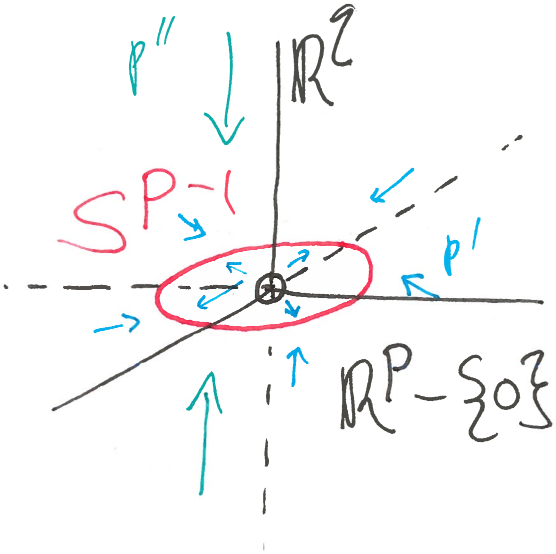
solution to (a)



j can be decomposed as $S^{p-1} \xrightarrow{j'} \mathbb{R}^p - \{0\} \xrightarrow{j''} (\mathbb{R}^p - \{0\}) \times \mathbb{R}^q$ where j'' is the inclusion into the 0-section. This decomposes $H^{p-1}(j)$ into

$$H^{p-1}((\mathbb{R}^p - \{0\}) \times \mathbb{R}^q) \xrightarrow{H^{p-1}(j'')} H^{p-1}(\mathbb{R}^p - \{0\}) \xrightarrow{H^{p-1}(j')} H^{p-1}(S^{p-1}).$$

And these maps are each isomorphisms coming from the deformation retractions

$$(\mathbb{R}^p - \{0\}) \times \mathbb{R}^q \underset{p''}{\twoheadrightarrow} \mathbb{R}^p - \{0\} \underset{p'}{\twoheadrightarrow} S^{p-1}.$$

So $H^{p-1}(j)$ is an iso. $\square$.

(b) Show that, for each smooth map $f : \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^q$ s.t. $f(X) = (X, 0)$ for all $X \in S$, the image $f(\mathbb{R}^p)$ has nonempty intersection w/ $0 \times \mathbb{R}^q$.

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solution to (b)

Suppose otherwise.

Consider $\mathbb{R}^p \xrightarrow{f} (\mathbb{R}^p - \{0\}) \times \mathbb{R}^q$. More specifically,

$$\begin{tikzcd}
S^{p-1} \arrow[r, "i"] \arrow[rr, bend right, "j"'] & \mathbb{R}^p \arrow[r, "f"] & (\mathbb{R}^p - \{0\}) \times \mathbb{R}^q
\end{tikzcd}$$

is a commutative diagram, and induces

$$\begin{tikzcd}
H^{p-1}((\mathbb{R}^p - \{0\}) \times \mathbb{R}^q) \arrow[r, "H_{p-1}(f)"] \arrow[rr, bend right, "\cong", "H_{p-1}(j)"'] & H^{p-1}(\mathbb{R}^p) \arrow[r, "H_{p-1}(i)"] & H^{p-1}(S^{p-1}) \\
 & 0 \ (p \geq 2) & \mathbb{Z} \ (p \geq 2)
\end{tikzcd}$$

Which says the 0 map surjects onto $\mathbb{Z}$, which is a contradiction. $\square$.
