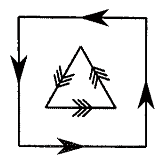
Geometry and Topology Graduate Exam
Spring 2016

Problem 1. Let Y be the space obtained by removing an open triangle from the interior of a compact square in $\mathbb{R}^2$. Let X be the quotient space of Y by the equivalence relation which identifies all four edges of the square and which identifies all three edges of the triangle according to the diagram below. Compute the fundamental group of X.



Problem 2. Let X be a path connected space with $\pi_1(X; x_0) = \mathbb{Z}/5$, and consider a covering space $\pi : \widetilde{X} \to X$ such that $p^{-1}(x_0)$ consists of 6 points. Show that $\widetilde{X}$ has either 2 or 6 connected components.

Problem 3. Compute the homology groups $H_k(S^1 \times S^n; \mathbb{Z})$ of the product of the circle S^1 and the sphere S^n, with $n \geq 1$.

Problem 4. Let M be a compact oriented manifold of dimension n, and consider a differentiable map $f: M \to \mathbb{R}^n$ whose image $f(M)$ has non-empty interior in $\mathbb{R}^n$.

(a) Show there there is at least one point $x \in M$ where f is a local diffeomorphism, namely such that there exists an open neighborhood $U \subset M$ of x such that restriction $f_{|U} : U \to f(U)$ is a diffeomorphism.

(b) Show that there exists at least two points x, $y \in M$ such that f is a local diffeomorphism at x and y, f is orientation-preserving at x, and f is orientation-reversing at y. Possible hint: What is the degree of f?

Problem 5. Consider the real projective space $\mathbb{RP}^n$, quotient of the sphere S^n by the equivalence relation that identifies each $x \in S^n$ to $-x$. Is there a degree n differential form such $\omega \in \Omega^n(\mathbb{RP}^n)$ such that $\omega(y) \neq 0$ at every $y \in \mathbb{RP}^n$? (The answer may depend on n.)

Problem 6. Let S^n denote the n–dimensional sphere, and remember that for $n \geq 1$ its de Rham cohomology groups are

$$H^k(S^n) \cong \begin{cases} 0 & \text{if } k \neq 0, n \\ \mathbb{R} & \text{if } k = 0, n. \end{cases}$$

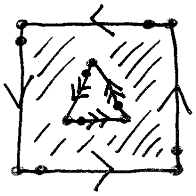
Consider a differentiable map $f : S^{2n-1} \to S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n on S^n such that $\int_{S^n} \alpha = 1$, let $f^*(\alpha) \in \Omega^n(S^{2n-1})$ be its pull-back under the map f.

(a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.

(b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of β and α.

Geometry Spring 2016

1. Y is the space obtained by removing an open triangle from the interior of a compact square in $\mathbb{R}^2$. Let X be the quotient of Y:



Compute the fundamental group of X.

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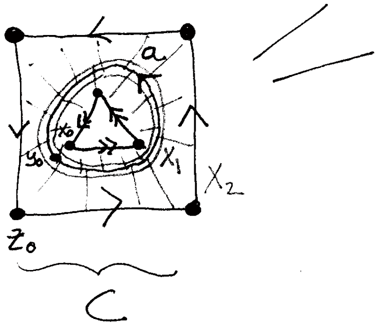
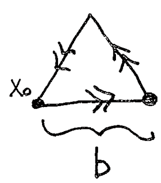
solution

By SvK I,

$$\pi_1(X;y_0) \cong \pi_1(X_1;y_0) \underset{\pi_1(X_1\cap X_2;y_0)}{*} \pi_1(X_2;y_0)$$





$$\overset{\text{deformation retract}}{\cong} \underbrace{\pi_1(X_1;x_0) \underset{\pi_1(X_1\cap X_2;y_0)}{*} \pi_1(X_2;z_0)}_{\text{What does this mean?}}$$

$$\begin{tikzcd}
\pi_1(X_1\cap X_2;y_0) \arrow[r] \arrow[dr] \arrow[rr, bend left, "f_1"] \arrow[drr, bend right, "f_2"'] & \pi_1(X_1;y_0) \arrow[r] & \pi_1(X_1;x_0) \\
 & \pi_1(X_2;y_0) \arrow[r] & \pi_1(X_2;z_0)
\end{tikzcd}$$

$$[a] \mapsto [a] \mapsto [b]^3 \qquad [a] \mapsto [a] \mapsto [c]^4$$

$$\pi_1(X;y_0) \cong \langle b,c \mid f_1(a)f_2^{-1}(a) = b^3c^{-4} = 1\rangle \cong \langle b,c \mid b^3 = c^4\rangle. \quad \square$$

(2) X path connected w/ $\pi_1(X;x_0) \cong \mathbb{Z}_5$.

Let $\pi : \tilde{X} \to X$ be a covering space s.t. $p^{-1}(x_0)$ consists of 6 points. Show $\tilde{X}$ has either 2 or 6 path connected components.

<u>solution</u> Let $F = p^{-1}(x_0)$ be the fiber of x_0. Recall the monodromy antihomomorphism

$\rho : \pi_1(X;x_0) \longrightarrow \mathrm{Bij}(F)$ defined by

$[\alpha] \overset{\rho}{\longmapsto} \left(\tilde{x} \overset{\rho([\alpha])}{\longmapsto} \tilde{\alpha}_{\tilde{x}}(1) \right)$.

where $\tilde{\alpha}_{\tilde{x}}$ is the unique lift of α to $\tilde{X}$ starting at $\tilde{x}$. (Note: $\rho([\alpha])$ is a bijection since it has the explicit inverse $\rho([\bar{\alpha}])$. And "antihomomorphism" means $\rho([\alpha]*[\beta]) = \rho([\beta]) * \rho([\alpha])$.)

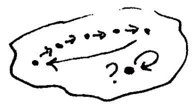
Observe since ρ is an antihomomorphism that $|\rho([\alpha])|$ divides 5. Indeed, $|\rho([\alpha])|$ is finite since $\mathrm{Bij}(F) \cong S_6$. And $\rho([\alpha])^5 = \rho([\alpha]^5) = \rho([c_{x_0}]) = id_F$. Here we used

$\pi_1(X;x_0) \cong \mathbb{Z}_5$. ✓ So $|\rho([\alpha])| = 1$ or 5.

If a component has two points $\tilde{x}$ and $\tilde{y}$ in the fiber F, then let $\tilde{\alpha}$ go from $\tilde{x}$ to $\tilde{y}$. Then $\rho([\pi \circ \tilde{\alpha}])(\tilde{x}) = \tilde{y}$. In particular, ρ is not the trivial monodromy. To summarize, if $|\rho([\alpha])| = 1$ for all $[\alpha] \in \pi_1(X;x_0)$, then every component intersects the fiber in at most one point.

Since X is connected, every component of $\tilde{X}$ contains an elt of the fiber F. Indeed, if $\tilde{y} \in \tilde{X}$, then let γ be a path in X from $\pi(\tilde{y})$ to x_0. Then the unique lift $\tilde{\gamma}_{\tilde{y}}$ starting at $\tilde{y}$ ends at a point of F.

From the last 2 paragraphs, we gather if ρ is the trivial monodromy, then every component has exactly one elt of F. Hence, there are 6 components. If ρ is not trivial, then there is some α s.t. $|\rho(\alpha)| = 5$. Therefore, some component has at least 5 elts of F. Q: Could there be 6? A: No. By the path connectedness of the component, $\pi_1(X;x_0)$ acts transitively via ρ on the component. So the size of



the orbit divides $|\pi_1(X;x_0)| = 5$. Therefore, to conclude, if p is not trivial (i.e. if there are not 6 p.c. components), then there is one component w/ 5 elts of F leaving a 2nd component w/ exactly one elt of F. $\square$.

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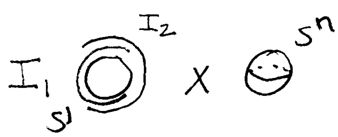
(3) Compute $H_k(S^1 \times S^n; \mathbb{Z})$, $n \geq 1$.

<u>solution</u>



MV: $\to H_k((I_1 \cap I_2)\times S^n) \to H_k(I_1 \times S^n) \oplus H_k(I_2 \times S^n) \to H_k(S^1 \times S^n) \to H_{k-1}((I_1 \cap I_2)\times S^n) \to$

$\to H_k(S^n) \oplus H_k(S^n) \to H_k(S^n) \oplus H_k(S^n) \to H_k(S^1 \times S^n) \to H_{k-1}(S^n) \oplus H_{k-1}(S^n) \to$

<u>$k \neq n, n+1, 1, 0$:</u>

$0 \to 0 \to H_k(S^1 \times S^n) \to 0 \to$ $\therefore \underline{\underline{H_k(S^1 \times S^n) \cong 0}}$.

<u>$k=0$:</u> $\underline{\underline{H_0(S^1 \times S^n) \cong \mathbb{Z}}}$.

$\underline{k=1,\ n=1}$:

$$\to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{f} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[g]{(x,y)\mapsto x+y} H_1(S^1\times S^1) \xrightarrow{h} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\ell} \mathbb{Z}\oplus\mathbb{Z} \to .$$

$f: (x,y) \mapsto (-x-y, x+y)$ $\qquad$ $\ell: (x,y) \mapsto (-x-y, x+y)$

$\cancel{0 \to \ker h \to H_1(S^1\times S^1) \to \operatorname{im} h \to 0}$ ~~exact.~~

$\cancel{\operatorname{im} f = \{(a,b) : a=-x-y,\ b=x+y \text{ for some } x,y\in\mathbb{Z}\}}$

$\cancel{= \{(a,-a) : a\in\mathbb{Z}\} \cong \mathbb{Z}.}$

$$0 \to \mathbb{Z}\oplus\mathbb{Z}/\ker g \xrightarrow{\bar{g}} H_1(S^1\times S^1) \xrightarrow{h} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\ell} \mathbb{Z}\oplus\mathbb{Z} \to$$

exact.

$$0 \to \mathbb{Z}\oplus\mathbb{Z}/\ker g \xrightarrow{\bar{g}} H_1(S^1\times S^1) \to \operatorname{im} h \equiv \ker \ell \to 0$$

exact.

$\ker g = \operatorname{im} f = \{(x,-x) : x\in\mathbb{Z}\} \Rightarrow \mathbb{Z}\oplus\mathbb{Z}/\ker g \cong \mathbb{Z}.$

$\ker \ell = \{(x,-x) : x\in\mathbb{Z}\} \cong \mathbb{Z}.$

$$0 \to \mathbb{Z} \to H_1(S^1\times S^1) \to \mathbb{Z} \to 0 \overset{\mathbb{Z}\text{ free}}{\Rightarrow} \underline{\underline{H_1(S^1\times S^1) \cong \mathbb{Z}\oplus\mathbb{Z}}}.$$

$\underline{k=1,\ n>1}$:

$$\to 0 \to 0 \to H_1(S^1\times S^n) \xrightarrow{f} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g} \mathbb{Z}\oplus\mathbb{Z} \to$$

$$(x,y) \mapsto (-x-y,\ x+y)$$

$$H_1(S^1\times S^n) \cong \operatorname{im} f \cong \ker g \cong \mathbb{Z} \Rightarrow \underline{H_1(S^1\times S^n) \cong \mathbb{Z}}.$$

$\underline{k=n,\ n>1}$:

$$\to \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{f} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g} H_n(S^1\times S^n) \xrightarrow{h} 0 \to$$

$$(x,y) \mapsto (-x-y,\ x+y)$$

$$H_n(S^1\times S^n) \cong \ker h \cong \operatorname{im} g \cong \mathbb{Z}\oplus\mathbb{Z}/\ker g = \mathbb{Z}\oplus\mathbb{Z}/\operatorname{im} f \cong \mathbb{Z}.$$

$$\Rightarrow \underline{H_n(S^1\times S^n) \cong \mathbb{Z}}.$$

$\underline{k=n+1}$:

$$\to 0 \to 0 \to H_{n+1}(S^1\times S^n) \xrightarrow{f} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow{g} \mathbb{Z}\oplus\mathbb{Z} \to$$

same as $(k=1, n>1)$ case : $\underline{H_{n+1}(S^1\times S^n) \cong \mathbb{Z}}$.

$\square$.

connected

(4) M compact, connected, oriented mfld, dim n.

Let $f : M \to \mathbb{R}^n$ smooth s.t. interior $(f(M)) \neq \emptyset$.

(a) Show there is at least one point $x \in M$ at which f is a *local* diffeomorphism.

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solution to (a)

By Sard's theorem, a.e. $y \in \mathbb{R}^n$ is a regular value of f. Since $f(M)$ has nonempty interior, $f(M)$ has nonzero Lebesgue measure, so $f(M)$ contains a critical value y. If $x \in f^{-1}(y)$, then $T_x f : \underbrace{T_x M}_{\cong \mathbb{R}^n} \to \mathbb{R}^n$ has full rank, hence is an iso. By the Inverse Function Theorem, f is a local diffeo at x. $\square$.

(b) Show there is $x, y \in M$ s.t. f is a local diffeo at x and y, f is orientation-preserving at x, and f is orientation-reversing at y.

For a proof, the e. f is then calculating the geometric degree at a point not in the image, we get that the degree is 0.

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solution to (b)

It is a corollary of the Degree Theorem equating cohomological degree w/ geometric degree that if M is compact and N is noncompact (both oriented, connected) that $\deg f = 0$. [Bonahon].

Therefore, since this degree is equal to the sum of the local intersection numbers, each of which is either $+1$ or -1, it follows there is a regular value z s.t. $x, y \in f^{-1}(z)$, f is a local diffeo at x & y, f is orientation reversing at x, f is orientation preserving at y. $\square$.

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(5) Consider real projective space $\mathbb{R}P^n$, the quotient of the sphere S^n by the equivalence relation $x \sim -x$. Determine the n s.t. there is a degree n differential form ω s.t. $\omega(y) \neq 0$ for every $y \in \mathbb{R}P^n$.

Solution

We know the answer: Such an n exists iff $\mathbb{R}P^n$ is orientable, which occurs iff n is odd.

Another Bonahon result: Let G be a group acting freely on M (i.e. if $g \in G$ is not the trivial elt, then $gx \neq x$ for any $x \in M$) and discontinuously (i.e. if for every compact subset $K \subset M$, then $\{g \in G : K \cap gK \neq \emptyset\}$ is finite), so that we may consider the quotient manifold M/G w/ quotient map $\pi : M \to M/G$. Let $\Omega^p_G(M)$ be the set of differential forms $\omega \in \Omega^p(M)$

satisfying the G-invariant property $\Omega^p(g)(\omega) = \omega$ for all $g \in G$, where we are viewing g as a smooth map $g : M \to M$ inducing a linear map $\Omega^p(g) : \Omega^p(M) \to \Omega^p(M)$. Then the map $\Omega^p(\pi) : \Omega^p(M/G) \to \Omega^p(M)$ induces a 1-1 correspondence between differential forms in $\Omega^p(M/G)$ and those differential forms in $\Omega^p_G(M)$.

Let $\varphi : S^n \to S^n$ be the antipodal map defined by $\varphi(x) = -x$ $(x \in S^n)$. We assume the fact that S^n is orientable and that the antipodal map induces a map $\Omega^n(\varphi) : \Omega^n(S^n) \to \Omega^n(S^n)$ ~~is multiplication by $(-1)^{n+1}$ where $\varphi : S^n \to S^n$, $\varphi(x) = -x$~~ such that for a specific volume form $\alpha \in \Omega^n(S^n)$ we have the identity $\Omega^n(\varphi)(\alpha) = (-1)^{n+1}\alpha$.

To show $\mathbb{RP}^n$ is orientable, we show $\mathbb{RP}^n$ has a volume form. Consider the volume form $\alpha \in \Omega^n(S^n)$.

Now, $\mathbb{RP}^n$ is $S^n/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts as the antipodal map $\varphi : S^n \to S^n$. By the result above, we wish to show α is mapped to by some $\tilde{\alpha} \in \Omega^n(S^n/\mathbb{Z}_2)$, for which it suffices to show $\alpha \in \Omega^n_{\mathbb{Z}_2}(S^n)$, namely, $\Omega^n(\varphi)(\alpha) = \alpha$. And indeed,

$\Omega^n(\varphi)(\alpha) = (-1)^{n+1}\alpha = \alpha$ we conclude, $\mathbb{RP}^n$ is orientable for n odd.

uses (*) $\tilde{\alpha} \in \Omega^n(S^n/\mathbb{Z}_2)$ is a volume form, hence

I suppose we should show the computation of $\Omega^n(\varphi)(\alpha) = (-1)^{n+1}(\alpha)$.

Recall generally if $f: M \to N$ then $\Omega^p(f): \Omega^p(N) \longrightarrow \Omega^p(M)$ is given by the formula $\Omega^p(f)(\omega)(x)(v_1, \ldots, v_p) =$

$= \omega(f(x))(T_x f(v_1), \ldots, T_x f(v_p))$ for $\omega \in \Omega^p(N)$, $x \in M$, $v_1, \ldots, v_p \in T_x M$.

(*) Hence if $\omega \in \Omega^p(N)$ and $\Omega^p(f)(\omega)(x) \neq 0$. So $\Omega^p(f)(\omega)$ volume form $\Rightarrow \omega$ volume form.

So here, $\varphi: S^n \to S^n$, $\varphi(x) = -x$, satisfies $T_x\varphi(v_i) = -v_i$.

Fact

We need a fact:

We may take the volume form $\alpha \in \Omega^n(S^n)$ to be defined by

$\alpha(x)(v_1, v_2, \ldots, v_n) = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_{n+1}(v_1, v_2, \ldots, v_n, x)$

for $x \in S^n$, $v_1, v_2, \ldots, v_n \in T_x S^n \subset \mathbb{R}^{n+1}$.

$\mathbb{Z}$

~~Since $\Omega^n(\varphi)$ is linear, we only need to check $\Omega^n(\varphi)(\alpha) = (-1)^{n+1}\alpha$ because of the theorem. Indeed,~~

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we calculate

$$\Omega^n(f)(\alpha)(x)(v_1, \ldots, v_n) = \alpha(\varphi(x))(T_x\varphi(v_1), \ldots, T_x\varphi(v_n))$$

$$= \alpha(-x)(-v_1, \ldots, -v_n) = dx_1 \wedge \cdots \wedge dx_n \wedge dx_{n+1}(-v_1, \ldots, -v_n, -x)$$

$$= (-1)^{n+1} dx_1 \wedge \cdots \wedge dx_n \wedge dx_{n+1}(v_1, \ldots, v_n, x) =$$

$$= (-1)^{n+1} \alpha(x)(v_1, \ldots, v_n) \text{ for all } x \in S^n$$

and $v_1, \ldots, v_n \in T_x S^n$, as desired. So we conclude $\Omega^n(\varphi)(\alpha) = (-1)^{n+1}\alpha$.

$\mathbb{Z}$

Next, we want to show that when n is even $\mathbb{R}P^n$ is not orientable.

Suppose $\mathbb{R}P^n$ is orientable w/ volume form $\beta \in \Omega^n(S^n/\mathbb{Z}_2)$.

Let $\alpha \in \Omega^n(S^n)$ be the volume form from above. Consider $\Omega^n(\pi): \Omega^n(S^n/\mathbb{Z}_2) \to \Omega^n(S^n)$. This must then define a smooth function $f: S^n \to \mathbb{R}$ s.t. $\Omega^n(\pi)(\beta)(x) = f(x)\alpha(\underset{\mathrm{Alt}^n(T_xS^n)}{x})$ $(x \in S^n)$. Indeed, this is because $\mathrm{Alt}^n(T_xS^n) \cong \mathbb{R}$. Since β is a volume form, $\Omega^n(\pi)(\beta)(x) \neq 0$ for all x:

$$\Omega^n(\pi)(\beta)(x)(v_1, \dots, v_n) = \beta(\pi(x))(T_x\pi(v_1), \dots, T_x\pi(v_n));$$

chose $w_1, \dots, w_n \in T_{\pi(x)}\mathbb{R}P^n$ s.t. $\beta(\pi(x))(w_1, \dots, w_n) \neq 0$;

recall the smooth structure on $\mathbb{R}P^n$ is specially designed s.t. π restricts to a local diffeomorphism around every $x \in S^n$, hence $T_x\pi$ is an isomorphism and we may choose $v_1, \dots, v_n$ s.t. $\Omega^n(\pi)(\beta)(x)(v_1, \dots, v_n) = \beta(\pi(x))(w_1, \dots, w_n) \neq 0$; hence, $\Omega^n(\pi)(\beta)(x) \neq 0$.

From this we gather $f(x) \neq 0$ for all $x \in S^n$.

Since S^n is connected, either $f > 0$ or $f < 0$. Since β is a volume form, $\Omega^n(\pi)(\beta) \in \Omega^n_{\mathbb{Z}_2}(S^n)$, i.e. $f\alpha \in \Omega^n_{\mathbb{Z}_2}(S^n)$. So $f\alpha = \Omega^n(\varphi)(f\alpha) = \Omega^0(\varphi)(f)\Omega^n(\varphi)(\alpha) = (f\circ\varphi)(-1)^{n+1}\alpha =$
$= -(f\circ\varphi)\alpha$ since n is even. But $\mathrm{sign}(f) = \mathrm{sign}(f\circ\varphi)$, so α must vanish everywhere, an obvious contradiction. $\Rightarrow\Leftarrow$. So $\mathbb{R}P^n$ is not orientable. $\square$

First assume $\alpha \in \Omega^n(S^n)$ is fixed and $\beta, \beta' \in \Omega^{n-1}(S^{2n-1})$ satisfy $d\beta = d\beta' = \Omega^n(f)(\alpha) \in \Omega^n(S^{2n-1})$.

So $d(\beta - \beta') = 0$ hence $[\beta - \beta'] \in H^{n-1}(S^{2n-1})$ is w.d.

So since $H^{n-1}(S^{2n-1}) = 0$ $(n \geq 2)$, there is $\gamma \in \Omega^{n-2}(S^{2n-1})$ s.t. $d\gamma = \beta - \beta'$.

We want

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} \beta \wedge d\beta' \iff \int_{S^{2n-1}} d\gamma \wedge d\beta = 0.$$

Since $d(\gamma \wedge d\beta) = d\gamma \wedge d\beta \pm \gamma \wedge \cancel{d d\beta}^{\,0}$

by the corollary to Stokes's theorem $\int_{S^{2n-1}} d\gamma \wedge d\beta = \int_{S^{2n-1}} d(\gamma \wedge d\beta) = 0$.

Next we need to show if $\alpha, \alpha' \in \Omega^n(S^n)$ and $\beta, \beta' \in \Omega^{n-1}(S^{2n-1})$ satisfy $d\beta = \Omega^n(f)(\alpha)$ and $d\beta' = \Omega^n(f)(\alpha')$ then $\int_{S^{2n-1}} \beta \wedge d\beta - \beta' \wedge d\beta' = 0$. (So the previous paragraph is a special case of this, when $\alpha = \alpha'$.)

(6) Recall $H^k(S^n) = \begin{cases} 0, & k \neq 0, n \\ \mathbb{R}, & k = 0, n \end{cases}$ (de Rahm cohomology).

Consider $f : S^{2n-1} \to S^n$, $n \geq 2$.

Let $\alpha \in \Omega^n(S^n)$ s.t. $\int_{S^n} \alpha = 1$, and consider the pullback $\Omega^n(f)(\alpha) \in \Omega^n(S^{2n-1})$.

(a) Show there is $\beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $\Omega^n(f)(\alpha) = d\beta$.

(b) Show the integral $\int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of β and α.

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solution

(a)

$$d\,\Omega^n(f)(\alpha) = \Omega^{n+1}(f)(d\alpha) = \Omega^n(f)(0) = 0$$

where $d\alpha = 0$ since $\Omega^{n+1}(S^n) = 0$.

So $[\ \Omega^n(f)(\alpha)\ ] \in H^n(S^{2n-1})$ makes sense.

Since $H^n(S^{2n-1}) = 0$ as $n \geq 2$, we have that $\Omega^n(f)(\alpha)$ is a boundary, as desired.

(b) Recall the corollary of Stokes's Theorem:

$\int_M d\omega = 0$ if M is an oriented mfld of dim m w/out boundary and $\omega \in \Omega_c^{m-1}(M)$ is a compactly supported $(m-1)$-form.

Rewrite the integral

$$\int_{S^{2n-1}} \beta \wedge d\beta - \beta' \wedge d\beta + \beta' \wedge d\beta - \beta' \wedge d\beta'$$

$$= \int_{S^{2n-1}} (\beta - \beta') \wedge d\beta \quad + \quad \int_{S^{2n-1}} \beta' \wedge (d\beta - d\beta').$$

Our argument from the special case gives the first integral to be 0.

Observe $\beta' \wedge (d\beta - d\beta') =$

$$= \beta' \wedge \left(\Omega^n(f)(\alpha) - \Omega^n(f)(\alpha')\right) =$$

$$= \beta' \wedge \Omega^n(f)(\alpha - \alpha').$$

Now, $d(\alpha - \alpha') = 0$ and $\int_{S^n} \alpha - \alpha' = 1 - 1 = 0$, hence $\alpha - \alpha'$ is a boundary as $H^n(S^n) \cong \mathbb{R}$ via integration. Write $\alpha - \alpha' = d\gamma$, $\gamma \in \Omega^{n-1}(S^n)$.

So we are interested in the integral

$\int_{S^{2n-1}} \beta' \wedge \Omega^n(f)(d\gamma) = \int_{S^{2n-1}} \beta' \wedge d\,\Omega^{n-1}(f)(\gamma)$ which we want to vanish. Note

$$\int_{S^{2n-1}} d(\beta' \wedge \Omega^{n-1}(f)(\gamma)) = \int_{S^{2n-1}} d\beta' \wedge \Omega^{n-1}(f)(\gamma) \pm \int_{S^{2n-1}} \beta' \wedge d\Omega^{n-1}(f)(\gamma).$$

The first integral vanishes by the corollary to Stokes. So we have reduced to showing the top right integral vanishes. And indeed

$$\int_{S^{2n-1}} d\beta' \wedge \Omega^{n-1}(f)(\gamma) = \int_{S^{2n-1}} \Omega^{n}(f)(\alpha) \wedge \Omega^{n-1}(f)(\gamma)$$

$$= \int_{S^{2n-1}} \Omega^{2n-1}(f)(\alpha \wedge \gamma) \underset{2n-1>n}{=} \int_{S^{2n-1}} \Omega^{2n-1}(f)(0)$$

$$= 0.$$

$\square$