Geometry and Topology Graduate Exam
Fall 2016

Problem 1. Give an example of a connected topological space whose fundamental group is the free product $\mathbb{Z} * \mathbb{Z}_3$ (where $\mathbb{Z}_3$ is the cyclic group of order 3).

Problem 2. Recall that two covering spaces $p\colon \widetilde{X} \to X$ and $p'\colon \widetilde{X}' \to X$ are isomorphic if there exists a homeomorphism $\varphi\colon \widetilde{X} \to \widetilde{X}'$ such that $p' \circ \varphi = p$. Up to isomorphism, how many covering spaces $p\colon \widetilde{X} \to S^1 \times S^1$ are there with $p^{-1}(x_0)$ consisting of 3 points?

Problem 3. For p, $q > 0$, compute all homology groups $H_k(S^p \times S^q; \mathbb{Z})$ where $S^n = \{x \in \mathbb{R}^{n+1}; \|x\| = 1\}$ denotes the n–dimensional sphere.

Problem 4. Show that the special linear group

$$\mathrm{SL}_n(\mathbb{R}) = \{A \in \mathrm{M}_n(\mathbb{R}); \det(A) = 1\}$$

is a submanifold of the vector space $\mathrm{M}_n(\mathbb{R})$ of all $n \times n$ real matrices. Explicitly describe its tangent space $T_{\mathrm{I}_n}\mathrm{SL}_n(\mathbb{R})$ at the identity matrix I_n, as a subspace of $\mathrm{M}_n(\mathbb{R})$.

Problem 5. Let S be a (nonempty) compact 2-dimensional submanifold of $\mathbb{R}^3$. Show that there exists infinitely many vertical lines $\{x\} \times \{y\} \times \mathbb{R}$ whose intersection with S is finite and nonempty.

Problem 6. The group SO_3, consisting of all linear rotations of $\mathbb{R}^3$, respects the 2-dimensional phere S^2. Let $\omega \in \Omega^1(S^2)$ be a 1–form on S^2 that is invariant under action of SO_3, namely such that $\Omega^1(r)(\omega) = \omega$ for every $r \in \mathrm{SO}_3$. Show that $\omega = 0$.

Problem 7. Let M be a submanifold of the manifold N, and let $i\colon M \to N$ be the inclusion map. Assume $\dim M < \dim N$. Else M is open and could have a pole or something.

a. Show that, for every differential form $\alpha \in \Omega^p(M)$, there exists a differential form $\beta \in \Omega^p(N)$ such that $\Omega^p(i)(\beta) = \alpha$. Possible hint: First consider the case where $M = \mathbb{R}^m \times 0$ in $N = \mathbb{R}^n$, with $m \leq n$.

b. If $d\alpha = 0$, can you always arrange that $d\beta = 0$?

GEOMETRY FALL 2016.

(1) Find X s.t. $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}_3$.

Let $\tilde{X}$ be the closed unit disk $\overline{D}$ in $\mathbb{C}$ quotiented by the relation $z \sim w$ iff $z, w \in S^1$ and $\exists k = 0,1,2$ s.t. $z = e^{\frac{2\pi i k}{3}} w$.

Then $\pi_1(\tilde{X}) \cong \mathbb{Z}_3$.

Then $\pi_1(S^1 \vee \tilde{X}) \cong \mathbb{Z} * \mathbb{Z}_3$. $\square$

(2) Up to isomorphism, how many 3-sheeted coverings of $S^1 \times S^1$ are there?

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solution

Generally, n-sheeted coverings are in 1-to-1 correspondence w/ the collection $\{\rho : \pi_1(X, x_0) \to S_n\}/\sim$

(see Hatcher's section on "Representing Covering Spaces by Permutations").

Here S_n is the symmetric group on n letters, and $\rho_1 \sim \rho_2 \iff \exists\, \sigma \in S_n$ s.t. $\rho_1([\alpha]) = \sigma \rho_2([\alpha]) \sigma^{-1}$ for all $[\alpha] \in \pi_1(X, x_0)$.

In our case, we are first interested in $\{\rho : \mathbb{Z} \oplus \mathbb{Z} \to S_3\}$. Viewing $\mathbb{Z} \oplus \mathbb{Z} = \langle x, y \mid xy = yx \rangle$ and $S_3 = \langle s, t \mid s^2 = t^3 = 1, st = t^2 s \rangle$), a necessary and sufficient condition for ρ to be a homomorphism, determined by the images of x&y, is that $\rho(x)\rho(y) = \rho(y)\rho(x)$. So either $\{x, y\} \xrightarrow{\rho} \{1, s\}$ or $\{1, t, t^2\}$.

Possibilities for where ρ takes x & y

$t = (123)$

$s = (12)$

	x	y	
	1	1	
	1	s	
	s	1	
	s	s	
	1	1	
	1	t	(12)
	1	t^2	
(12)	t	1	
	t	t	
	t	t^2	(12)
(12)	t^2	1	
	t^2	t	
	t^2	t^2	

Which homomorphisms are conjugate?

So it looks like there are 8 iso classes of covering spaces.
(I would check w/ someone...) $\square$.

③

For $p,q>0$, compute the homology groups of $S^p \times S^q$ over $\mathbb{Z}$.

<u>Answer</u> (i) $p \neq q$: $H_k = \begin{cases} \mathbb{Z}, & k=0,p,q,p+q \\ 0, & \text{else} \end{cases}$.

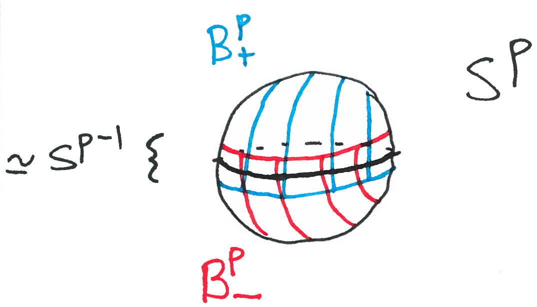
(ii) $p=q$: $H_k = \begin{cases} \mathbb{Z}, & k=0,\ p+q \\ \mathbb{Z}\oplus\mathbb{Z}, & k=p=q \\ 0, & \text{else} \end{cases}$.

<u>*solution*</u>

Prove by induction on p.

Use MV, where $X_1 = \underbrace{B_+^p \times S^q}_{\simeq S^q}$,

$X_2 = \underbrace{B_-^p \times S^q}_{\simeq S^q}$, $X_1 \cap X_2 \underset{\substack{\uparrow \\ \text{homotopy equivalent}}}{\simeq} S^{p-1} \times S^q$.



The details for the base case $p=1$ are given in the solutions to Bonahon's Fall 2015 final exam (in topology).
The arguments are completely analogous for $p>1$.

$\square$.

④ This is a standard problem.

The key calculation is that if $A \in SL_n(\mathbb{R})$ and $B \in M_n(\mathbb{R})$,

then $$\frac{d}{dt}\Big|_{t=0} \operatorname{Det}(A + tB)$$

$$= \frac{d}{dt}\Big|_{t=0} \overset{1}{\cancel{\operatorname{Det} A}} \cdot \operatorname{Det}(I + tA^{-1}B)$$

$$= \operatorname{Tr}(A^{-1}B),$$ since generally

$$\operatorname{Det}(I + tC) = \begin{vmatrix} 1+tC_{11} & tC_{21} & \cdots \\ tC_{12} & 1+tC_{22} & \\ \vdots & & \ddots \end{vmatrix}$$

$$= 1 + t\operatorname{Tr}(C) + \mathcal{O}(t^2).$$

$\square$

(5) S is a (nonempty) compact 2-dim submfld of $\mathbb{R}^3$. Show there are infinitely many vertical lines $\{x\} \times \{y\} \times \mathbb{R}$ whose intersection w/ S is finite & nonempty.

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solution

Lemma/ If M is a compact submfld of dim $n-1$ in $\mathbb{R}^n$, then there is a point $p \in M$ s.t. the projection of $T_pM \subset \mathbb{R}^n$ to $\mathbb{R}^{n-1} \times 0$ is an isomorphism.

Pf/ Since M is compact, choose p w/ highest n^{th} coordinate. This point p satisfies the desired property. ✓

Define $\Pi : \mathbb{R}^3 \to \mathbb{R}^2 \times 0 \subset \mathbb{R}^3$

by $\Pi(x, y, z) := (x, y, 0)$.

Define $f: S \to \mathbb{R}^2 \times 0$ by $f := \Pi|_S$.

If $(x, y, 0)$ is a regular value of f, then since $\dim S = 2 = \dim(\mathbb{R}^2 \times 0)$, $f^{-1}(x, y, 0)$ is a finite set, as M is compact.

If in addition $(x, y, 0)$ is in the image of f, then the line $\{x\} \times \{y\} \times \mathbb{R}$ intersects S in a nonempty finite set.

So we have reduced to showing there are infinitely many regular values in the image of f. By Sard's thm, the regular values of f are dense in $\mathbb{R}^2 \times 0$; and by the lemma, the image of f has positive measure in $\mathbb{R}^2 \times 0$. So we are done. $\square$

(6) Consider $SO(3)$ acting on $S^2 \subset \mathbb{R}^3$.

Let $\omega \in \Omega^1(S^2)$ s.t. $\Omega^1(r)(\omega) = \omega$ for every $r \in SO(3)$. Show $\omega = 0$.

Solution

Let $p \in S^2$. We show $\omega(p) \in T_p^* S^2$ is equal to the zero map, which suffices.

Identify $T_p S^2$ w/ the linear subspace of $\mathbb{R}^3$ perpendicular to the normal vector to S^2 at p. So $\omega(p)$ is a linear map $T_p S^2 \to \mathbb{R}$. Let $r \in SO(3)$, $v \in T_p S^2$. Assume $r(p) = p$.

$$\Omega^1(r)(\omega)(p)(v) := \omega(p)(T_p r(v)) = \omega(p)(rv).$$

$\|$ hypothesis

$\omega(p)(v)$

<u>Lemma</u> If $T : \mathbb{R}^2 \to \mathbb{R}$ is a linear map s.t. $T(e^{i\theta} v) = T(v)$ for all $v \in \mathbb{R}^2$, $\theta \in [0, 2\pi]$, then $T \equiv 0$.

Since $e_2 = \overline{i e_1}$, $T(e_2) \overset{\text{hyp.}}{=} T(e_1) =: a \in \mathbb{R}$.

$$T(e^{i\theta}e_1) = T(\cos\theta\, e_1 + \sin\theta\, e_2) = \cos\theta\, T(e_1) + \sin\theta\, T(e_2)$$

$$= a(\cos\theta + \sin\theta) \overset{\text{hyp.}}{=} a \quad \forall\, \theta \in [0, 2\pi].$$

$\therefore$ $a = 0$, hence $T \equiv 0$. $\checkmark$

Applying the lemma, we see $\omega(p) \equiv 0$. $\square$

⑦ M submfld of mfld N.

$i : M \to N$ the inclusion map.

(a) Show, for every $\alpha \in \Omega^p(M)$,

there exists $\beta \in \Omega^p(N)$ s.t. $\Omega^p(i)(\beta) = \alpha$.

Hint: First consider $M = \mathbb{R}^m \times 0$ in $N = \mathbb{R}^n$ w/ $m \leq n$.

(b) If $d\alpha = 0$, can we always arrange that $d\beta = 0$?

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solution to (a)

Special case: $M = \mathbb{R}^m \times 0$, $N = \mathbb{R}^n$, $m \leq n$.

Recall generally if $f : \mathbb{R}^m \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$,

then $\Omega^p(f) : \Omega^p(\mathbb{R}^n) \to \Omega^p(\mathbb{R}^m)$.

And if $\beta \in \Omega^p(\mathbb{R}^n)$ is given by

$$\beta = \sum_{1 \leq j_1 < \cdots < j_p \leq n} \beta_{j_1 \cdots j_p}(y_1, \ldots, y_n)\, dy_{j_1} \wedge \cdots \wedge dy_{j_p}\,, \quad \text{then}$$

$\Omega^p(f)(\beta)$ is given by

$$\Omega^p(f)(\beta) = \sum_{1 \leq j_1 < \cdots < j_p \leq n} \sum_{i_1=1}^{m} \cdots \sum_{i_p=1}^{m} (\beta_{j_1 \cdots j_p} \circ f) \frac{\partial f_{j_1}}{\partial X_{i_1}} \cdots \frac{\partial f_{j_p}}{\partial X_{i_p}} dX_{i_1} \wedge \cdots \wedge dX_{i_p}.$$

Taking $\mathbb{R}^m \equiv \mathbb{R}^m \times 0 \subset \mathbb{R}^n$ and taking $f \equiv i : \mathbb{R}^m \times 0 \to \mathbb{R}^n$, the above becomes: for $\beta = \sum_{1 \leq j_1 < \cdots < j_p \leq n} \beta_{j_1 \cdots j_p} dX_{j_1} \wedge \cdots \wedge dX_{j_p}$,

since $\frac{\partial i_a}{\partial X_b} = \delta_{ab}$, i.e. 0 unless $a = b$ in which case 1,

$$\Omega^p(i)(\beta) = \sum_{1 \leq j_1 < \cdots < j_p \leq m} \beta_{j_1 \cdots j_p}(X_1, \ldots, X_m, 0, \ldots, 0)\, dX_{j_1} \wedge \cdots \wedge dX_{j_p}.$$

Clearly then, if $\alpha \in \Omega^p(\mathbb{R}^m \times 0)$ is given to start, as $\alpha = \sum_{1 \leq i_1 < \cdots < i_p \leq m} \alpha_{i_1 \cdots i_p}(X_1, \ldots, X_m)\, dX_{i_1} \wedge \cdots \wedge dX_{i_p}$,

then we simply need define

$\beta_{j_1 \cdots j_p}(X_1, \ldots, X_m, X_{m+1}, \ldots, X_n) := \alpha_{j_1 \cdots j_p}(X_1, \ldots, X_m)$ if $1 \leq j_1 < \cdots < j_p \leq m$;

and 0 else, and we will have satisfied $\Omega^p(i)(\beta) = \alpha$. ✓

This completes the special case. ✓

General Case: $M \subset N$ arbitrary.

we want to show for each $\alpha \in \Omega^p(M)$, there is $\beta \in \Omega^p(N)$ s.t. $\Omega^p(i)(\beta) = \alpha$.

Since M is a submfld of N, for every $p \in M$, there is a chart $(U_p; \varphi_p : U_p \to \varphi_p(U_p) \subset \mathbb{R}^n)$ s.t. $\varphi_p(M \cap U_p) \subset \mathbb{R}^m \times 0$.

WOLG

we may assume $\varphi_p(M \cap U_p) = \mathbb{R}^m \times 0$.

Then we obtain a p-form $\tilde{\alpha}_p \in \Omega^p(\mathbb{R}^m \times 0)$ (abusing letter p) defined by

$$\tilde{\alpha}_p := \Omega^p(\varphi_p^{-1}|_{\mathbb{R}^m \times 0})(\alpha|_{M \cap U_p}).$$

By the special case, there is $\tilde{\beta}_p \in \Omega^p(\mathbb{R}^n)$ s.t. $\Omega^p(\tilde{i})(\tilde{\beta}_p) = \tilde{\alpha}_p$, where $\tilde{i} : \mathbb{R}^m \times 0 \to \mathbb{R}^n$ is the natural inclusion.

Therefore, we may locally define $\beta_p \in \Omega^p(U_p)$ by

$$\beta_p := \Omega^p(\varphi_p)(\tilde{\beta}_p|_{\varphi_p(U_p)}). \quad (p \in M)$$

Then, on $U_p \cap M$,

$$\beta_p = \Omega^p(\varphi_p)(\tilde{\alpha}_p)$$

$$= \Omega^p(\varphi_p) \circ \Omega^p(\varphi_p^{-1})(\alpha)$$

$$= \Omega^p(\varphi_p^{-1} \circ \varphi_p)(\alpha)$$

$$= \alpha. \Longrightarrow \boxed{\beta_p(x) = \alpha(x) \quad (p \in M, x \in U_{p \cap M})}$$

Now, let, for each $p \in N - M$, $(U_p ; \varphi_p : U_p \to \varphi_p(U_p))$ be any chart, and define $\beta_p \in \Omega^p(U_p)$ by

$$\boxed{\beta_p(x) := 0 \quad \left(\begin{matrix} x \in U_p \\ p \in N - M \end{matrix} \right).}$$

solution to (a)

Obviously $\infty \to \infty$ gives the desired extension. $\square$.

solution to (b)

~~Let $z_0 \in \mathbb{C}$ be a root of f w/ multiplicity n. We can write $f(z) = (z - z_0)^n g(z)$~~

We show $f(z)$ is homotopic to z^n as a function $\mathbb{R}^2 \to \mathbb{R}^2$; the result follows.

write $f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0$.

Let $\{\phi_p : U_p \to [0,1]\}_{p\in N}$ be a smooth partition of unity subordinate to the open cover $\{U_p : p \in N\}$ of N. So

(i) $\operatorname{supp}\phi_p \subset U_p \quad (\forall p \in N)$.

(ii) $\{p \in N \mid \operatorname{supp}\phi_p \cap K \neq \emptyset\}$ is finite for every compact set $K \subset N$.

(iii) $\sum_{p\in N} \phi_p(x) = 1 \quad (x \in N)$.

~~Define a~~ $\cancel{p}$~~-form~~ $\cancel{\beta \in \Omega^p(N)}$

~~by~~ $\cancel{\beta(x) =}$

~~Up until now now, we have not required any <u>finiteness</u> assumptions. However, we need the property, as will become apparant, that the U_p, for $p \in N - M$, does not intersect~~

We have to make a minor adjustment to what we've done. In particular, we must assume, for $p \in N - M$, that $U_p \cap M = \phi$; this is possible because, IF $m < n$, then M is <u>closed</u> in N. ($\dim M = n \Leftrightarrow$ M is open in N)

WE MAKE THE ASSUMPTION $m < n$, ELSE THE RESULT MIGHT NOT BE TRUE. Indeed, you can have a pole or something.

So M is closed, and we may choose U_p, for $p \in N - M$, s.t. $U_p \cap M = \phi$.

Proceeding as we were, given the partition of unity $\{\phi_p\}_{p \in N}$ dominated by $\{U_p\}_{p \in N}$, define a p-form $\beta \in \Omega^p(N)$ by

$$\beta(x) = \sum_{p \in N} \phi_p(x) \beta_p(x).$$

Then β is well-defined, since for each $x \in N$, there are only finitely many $p \in N$ for which $\phi_p(x) \neq 0$. ✓

And β is smooth. ✓

If $x \in M$, and if $\phi_p(x) \neq 0$, then $\operatorname{supp}\phi_p \cap M \neq \emptyset$; since $\operatorname{supp}\phi_p \subset U_p$, and since $U_p \cap M = \emptyset$ when $p \in N - M$, it must be that

$$\left[\phi_p(x) \neq 0 \implies p \in M \right].$$

Therefore, for $x \in M$, every nonzero-coefficient term in the def of $\beta(x)$ satisfies $\beta_p(x) = \alpha(x)$, as $p \in M$; therefore,

$$\beta(x) = \alpha(x) \sum_{p \in N} \phi_p(x) = \alpha(x). \qquad \square$$

(b) No!

Any $i : M \to N$ induces a map

$H^p(i) : H^p(N) \to H^p(M)$ defined

by $H^p(i)([\beta]) \longrightarrow [\Omega^p(i)(\beta)]$.

Part (a) shows this map is surjective.

If $[d\alpha = 0 \Rightarrow d\beta = 0]$

then this map would be injective,

hence an isomorphism.

Let N be any compact p-dim

mfld, and let M be a point in N.

(Note $p > 0$ since $m \neq n$).

Then $H^p(N) = H^p_c(N) \cong \mathbb{R}$ and

$H^p(M) = H^p_c(M) \cong \{0\}$. So $H^p(i) : H^p_c(N) \to H^p_c(M)$ cannot be an iso. $\square$