**Geometry and Topology Graduate Exam**
Fall 2015

**Problem 1.** (15 points)

(a) Define the two notions "homotopy between two maps" and "homotopy equivalences between two spaces".
(b) Give an example of two topological spaces $X$ and $Y$ that are homotopy equivalent but are not homeomorphic.
(c) Give an example of path-connected topological spaces $X$ and $Y$ that have isomorphic fundamental groups but are not homotopy equivalent.
(d) Give an example of path-connected topological spaces $X$ and $Y$ that have isomorphic first homology groups $H_1(X;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$ but whose fundamental groups are not isomorphic.

**Problem 2.** (15 points) Let $T$ be the 2–dimensional torus, and let $K$ be the Klein bottle.

(a) Describe a twofold covering map $p\colon T \to K$. ("Twofold" means that the preimage of each point of $K$ consists of two points of $T$.)
(b) Pick base points $x_0 \in T$ and $y_0 \in K$ such that $y_0 = p(x_0)$. Give generators for the fundamental groups $\pi_1(T;x_0)$ and $\pi_1(K;y_0)$ and, for each generator of $\pi_1(T;x_0)$, express its image under the induced homomorphism $p_*\colon \pi_1(T;x_0) \to \pi_1(K;y_0)$ in terms of the generators of $\pi_1(K;y_0)$.

**Problem 3.** (25 points) Let $\Sigma_g$ and $\Sigma_{g'}$ be closed orientable surfaces of genus $g$ and $g' > 0$, respectively. Let $f\colon B^2 \to \Sigma_g$ be an embedding of the 2–dimensional disk $B^2$, and consider the simple closed curve $\gamma = f(S^1) \subset \Sigma_g$. Similarly, let $\gamma' = f'(S^1) \subset \Sigma_{g'}$ be associated to an embedding $f'\colon B^2 \to \Sigma_{g'}$. Finally, let $X$ be the topological space obtained by gluing $\Sigma_g$ and $\Sigma_{g'}$ along $\gamma$ and $\gamma'$; namely, $X$ is obtained from the disjoint union $\Sigma_g \sqcup \Sigma_{g'}$ by gluing $f(x)$ to $f'(x)$ for every $x \in S^1$.

(a) Compute the fundamental group of $X$.
(b) Compute all homology groups of $X$.
(c) Is $X$ homotopy equivalent to the product $\Sigma_g \times \Sigma_{g'}$?

**Problem 4.** (15 points) Let $M$ be a manifold of dimension $n$, and let $\omega$ be a differential form of degree $n-1$ on $M$. Suppose that $\int_N \omega = 0$ for every $(n-1)$-dimensional oriented closed submanifold $N$ of $M$. Show that $dw = 0$. (Possible hint: look at small spheres.)

**Problem 5.** (15 points) Consider the vector fields $\mathbf{v} = \partial_x + xz\partial_z$ and $\mathbf{w} = \partial_y + yz\partial_z$ in $\mathbb{R}^3$. If $P$ is a point of $\mathbb{R}^3$, does there exist a local coordinate system in a neighborhood of $P$ in which $\mathbf{v}$ and $\mathbf{w}$? Namely, is there a diffeomorphism $\phi\colon U \to V$ from a neighborhood $U$ of $P$ to an open subset $V \subset \mathbb{R}^3$ that sends $\mathbf{v}$ to $\partial_x$ and $\mathbf{w}$ to $\partial_y$?

**Problem 6.** (15 points) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial of one complex variable. Recall that the one-point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ is homeomorphic to the sphere $S^2$.

(a) Show that $f$ extends to a continuous map $\bar{f}\colon S^2 \to S^2$.
(b) Show that the degree of $\bar{f}$ (in the sense of topology or geometry) is equal to the degree of the polynomial $f$ (in the algebraic sense).

# Geometry Fall 2015

① (a) Define "homotopy between two maps" and "homotopy equivalences between two spaces".

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<u>solution to (a)</u>

Two maps $f, g : X \to Y$ between top spaces $X, Y$ are homotopic if there is a map $F : X \times [0,1] \to Y$ s.t. $F(x,0) = f(x)\ \forall x \in X$, and $F(x,1) = g(x)\ \forall x \in X$ ; $F$ is called a homotopy from $f$ to $g$; we write $f \simeq g$.

To top spaces $X$ and $Y$ are homotopy equivalent if there are maps $f : X \to Y$ and $g : Y \to X$ s.t. $fg \simeq id_Y$ and $gf \simeq id_X$; we write $X \simeq Y$. $\square$.

(b) Give $X, Y$ homotopy equivalent but not homeomorphic.

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<u>solution to (b)</u>

$$\mathbb{R}^n \simeq \mathbb{R}^0 \quad (n \geqslant 0)$$

but $\mathbb{R}^n \not\cong \mathbb{R}^0 \quad (n > 0)$. $\square$.

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(c) Give $X, Y$ path connected w/ isomorphic fundamental groups that are not homotopy equivalent.

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<u>solution to (c)</u>

$$X = \mathbb{R}^0, \quad Y = S^n \quad (n \geqslant 2).$$

$\square$.

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(d) Give $X, Y$ path connected s.t.

$H_1(X;\mathbb{Z}) \cong H_1(Y;\mathbb{Z})$ but

$\pi_1(X) \not\cong \pi_1(Y)$.

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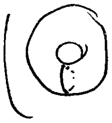
<u>solution to (d)</u>

$\pi_1(\infty) \cong F_2$.

$\pi_1($  $) \cong \mathbb{Z} \oplus \mathbb{Z}$

$H_1(\infty;\mathbb{Z}) \cong H_1($  $;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

$\square$

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(2) $T$ is the 2-dim torus.
$K$ is the Klein bottle.

(a) Describe a two-fold covering map $p: T \to K$.

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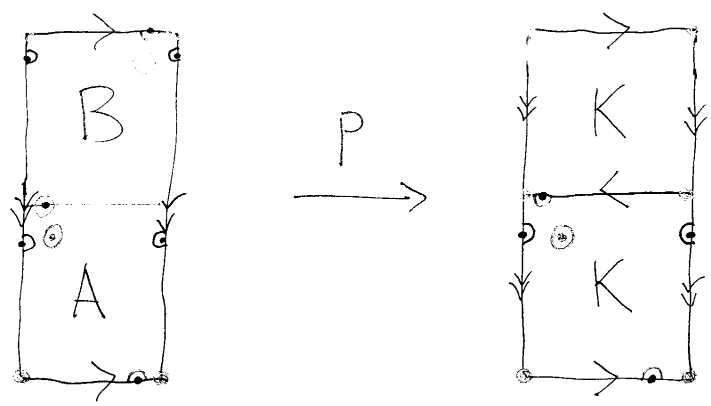
solution to (a)

$T$ : $K$ :




$B \to K$ is well-defined. ✓. It is continuous because it is the composition of a continuous map followed by a quotient map. ✓. Similarly, $A \to K$ is w.d. & continuous. ✓. By the pasting lemma this induces a continuous map $T \to K$ that is two-fold. ✓. By inspection, $p$ is a covering map. $\square$.

(b) Pick base points $x_0 \in T$ and $y_0 \in K$ s.t. $y_0 = p(x_0)$. Give generators for the fundamental groups $\pi_1(T; x_0)$ and $\pi_1(K; y_0)$. For each generator of $\pi_1(T; x_0)$ express its image under the induced homomorphism $p_*: \pi_1(T; x_0) \to \pi_1(K; y_0)$ in terms of the generators of $\pi_1(K; y_0)$.

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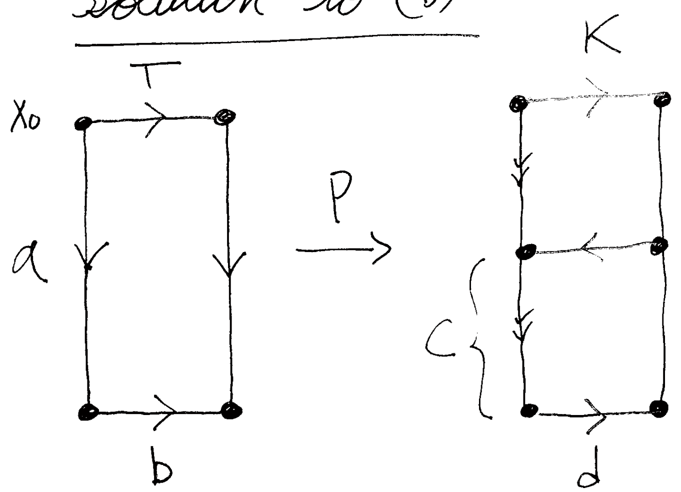
<u>solution to (b)</u>




$p_*([a]) = 2[c]$

$p_*([b]) = [d]$

$\pi_1(T; x_0) = \langle a, b \mid aba^{-1}b^{-1} = 1 \rangle$ $\quad \pi_1(K; y_0) = \langle c, d \mid dcdc^{-1} = 1 \rangle$

$\square$.

(3) $\Sigma_g$, $\Sigma_{g'}$ closed orientable surfaces of genus $g, g' > 0$, respectively. Let $f: B^2 \to \Sigma_g$ be an embedding of the 2-dim disk $B^2$, and consider the simple closed curve $\gamma = f(S^1) \subset \Sigma_g$. Similarly, let $\gamma' = f'(S^1) \subset \Sigma_{g'}$ be associated to an embedding $f': B^2 \to \Sigma_{g'}$. Finally, let $X$ be the topological space obtained by gluing $\Sigma_g$ and $\Sigma_{g'}$ along $\gamma$ and $\gamma'$; namely, $X$ is obtained from the disjoint union $\Sigma_g \amalg \Sigma_{g'}$ by gluing $f(x)$ to $f'(x)$ for all $x \in S^1$.

(a) Compute the fundamental group of $X$.

(b) Compute all homology groups of $X$.

(c) Is $X$ homotopy equivalent to the product $\Sigma_g \times \Sigma_{g'}$?

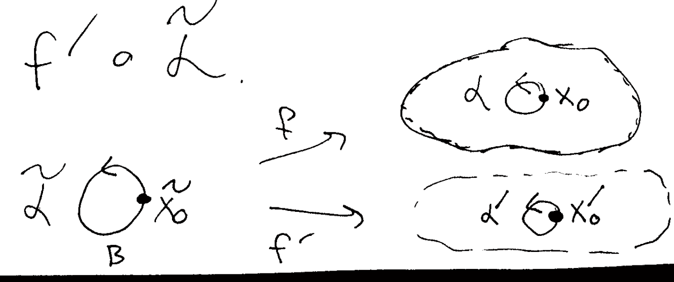
## solution to (a)

$$\pi_1(\Sigma_g ; x_0) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1} = \ldots = a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle$$

$$\pi_1(\Sigma_{g'} ; x_0') = \langle c_1, d_1, c_2, d_2, \ldots, c_{g'}, d_{g'} \mid c_1 d_1 c_1^{-1} d_1^{-1} = c_2 d_2 c_2^{-1} d_2^{-1} = \ldots = c_{g'} d_{g'} c_{g'}^{-1} d_{g'}^{-1} = 1 \rangle.$$

$$H_k(\Sigma_g) = \begin{cases} \mathbb{Z}, & k = 0,2 \\ \mathbb{Z}^{2g}, & k = 1 \\ 0, & k > 2 \end{cases} \qquad H_k(\Sigma_{g'}) = \begin{cases} \mathbb{Z}, & k = 0,2 \\ \mathbb{Z}^{2g'}, & k = 1 \\ 0, & k > 2 \end{cases}$$

we may assume $x_0 \in \gamma$, $x_0' \in \gamma'$ and $p(x_0) = p(x_0')$ where $p$ is the quotient map $\Sigma_g \amalg \Sigma_{g'} \longrightarrow X$. Let $\tilde{\alpha}$ be a generator for $S^1$ s.t. $f(\tilde{\alpha}(0)) = f(\tilde{x}_0) = x_0$ and $f'(\tilde{\alpha}(0)) = f'(\tilde{x}_0) = x_0'$; let $\alpha := f \circ \tilde{\alpha}$ and $\alpha' := f' \circ \tilde{\alpha}$.



We claim $p \circ \alpha = p \circ \alpha'$ as a path in $X$. Indeed,

$$p \circ \alpha(t) = p \circ \alpha'(t) \in X \iff$$

$$\begin{array}{ccc} \alpha'(t) & = & f' \circ f^{-1} \circ \alpha(t) \\ \| & & \| \\ f' \circ \tilde{\alpha}(t) & & f' \circ f^{-1} \circ f \circ \tilde{\alpha}(t). \end{array} \quad \checkmark.$$

Let $X_1 = p(\Sigma_g) \subset X$, $X_2 = p(\Sigma_{g'}) \subset X$.

Then $[\xi] := [p \circ \alpha] = [p \circ \alpha']$ is a generator for $\pi_1(X; z_0)$ where $z_0 := p(x_0) = p(x_0')$.

By the SvK closed version (It is obvious $X$ is Hausdorff, so $p$ is a closed map, as $\Sigma_1 \amalg \Sigma_2$ is compact, so $X_1$ & $X_2$ are closed; $X_1 \cap X_2 = \operatorname{im} \xi \subset X$ is p.c. & clearly a nbhd def retract in $X_1$ & $X_2$.)

$$\pi_1(X;z_0) \cong \pi_1(X_1;z_0) *_{\pi_1(X_1 \cap X_2; z_0)} \pi_1(X_2;z_0)$$

where the amalgamation occurs along

$$(i_j)_* : \pi_1(X_1 \cap X_2; z_0) \to \pi_1(X_j;z_0) \qquad (j=1,2).$$

Therefore,

$$\pi_1(X;z_0) \cong \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c_1, d_1, c_2, d_2, \ldots, c_{g'}, d_{g'} \mid$$

$$a_1 b_1 a_1^{-1} b_1^{-1} = a_2 b_2 a_2^{-1} b_2^{-1} = \cdots = a_g b_g a_g^{-1} b_g^{-1} = c_1 d_1 c_1^{-1} d_1^{-1} = c_2 d_2 c_2^{-1} d_2^{-1} = \cdots = c_{g'} d_{g'} c_{g'}^{-1} d_{g'}^{-1}$$

$$= (i_1)_*([\xi])(i_2)_*^{-1}([\xi]) = 1 \rangle$$

where the last term goes to 1 since $(i_1)_*([\xi]) = [\alpha] = 1$ as $\alpha$ is the boundary of an embedded disk, and similarly for $(i_2)_*([\xi])$. $\square$.

## solution to (b)

Use also $H_k(S^1) = \begin{cases} \mathbb{Z}, & k=0,1 \\ 0, & k>1. \end{cases}$

Since the hypotheses of Z-vK were satisfied, so are those for MV (recall for MV you don't need $X_1 \cap X_2$ p.c., in fact). So we get LES

$$\cdots \xrightarrow{\delta_{k+1}} H_k(X_1 \cap X_2) \xrightarrow{(-H_k(i_1),\, H_k(i_2))} H_k(X_1) \oplus H_k(X_2) \xrightarrow{H_k(j_1) + H_k(j_2)} H_k(X_1 \cup X_2) \xrightarrow{\delta_k} H_{k-1}(X_1 \cap X_2) \to \cdots$$

$k > 2$ / $\cdots \xrightarrow{\delta_{k+1}} 0 \to 0 \oplus 0 \to H_k(X) \to 0 \xrightarrow{\delta_k} \cdots$

$\therefore \boxed{H_k(X) = 0.}$

the third hole is that 2 sphere

$k = 2$ / $\cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} H_2(X) \xrightarrow{g = \delta_2} \mathbb{Z} \xrightarrow{h \equiv 0} \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g'} \to \cdots$

$H_2(X) \cong \operatorname{im} f$ $\mathbb{Z} \times \mathbb{Z}$

$\mathbb{Z}$

" since the image of a 1-simplex on the boundary of a disk is contractible.

$\therefore \boxed{H_2(X) \cong \mathbb{Z}^3}$

$k=1$ |

$$\cdots \xrightarrow{\delta_2} \mathbb{Z} \xrightarrow{f \equiv 0} \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g'} \xrightarrow{g} H_1(X) \xrightarrow{\delta_1} \mathbb{Z} \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z} \to \cdots$$

$$x \overset{h}{\mapsto} (-x, x)$$

$$\underbrace{H_1(X) / \operatorname{img}}_{\cong \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g'}} \cong \frac{H_1(X)}{\ker \delta_1} \cong \operatorname{im} \delta_1 \cong \ker h \cong 0$$

$$\therefore \boxed{H_1(X) \cong \mathbb{Z}^{2g} \oplus \mathbb{Z}^{2g'}}$$

$k=0$ | $\boxed{H_0(X) \overset{\text{p.c.}}{\cong} \mathbb{Z}}$. $\square$.

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solution to (c)

no. $H_2(\Sigma_g \times \Sigma_{g'}) \cong \mathbb{Z} \times \mathbb{Z}$

$\not\cong \mathbb{Z}^3$, and homology is preserved under homotopy equivalence. $\square$.

(4) $M$ mfld dim $n$,

$\omega \in \Omega^{n-1}(M)$. Suppose $\int_N \omega = 0$ for all $(n-1)$-dimensional oriented closed submflds $N$ of $M$. Show $d\omega = 0$.

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<u>*solution*</u> <u>Special case, $M = \mathbb{R}^n$</u>

Lemma/ If $\alpha \in \Omega^n(\mathbb{R}^n)$ and $\int_{\overline{B(X,\varepsilon)}} \alpha = 0$ for all

$B(X,\varepsilon) = \{ y \in \mathbb{R}^n \mid \|X - y\| < \varepsilon \}$,

then <u>$\alpha = 0 \in \Omega^n(\mathbb{R}^n)$</u>.

Pf (lemma)/ $\alpha = f(X_1, \dots, X_n)\, dX_1 \wedge \dots \wedge dX_n$

for some $\underline{\text{smooth}}$ $f$. Since

$$\int_{\overline{B(X,\varepsilon)}} \alpha := \int_{\overline{B(X,\varepsilon)}} f(X_1,\dots,X_n)\,dX_1\cdots dX_n$$

$$\left( \underset{\uparrow}{=} \int_{B(X,\varepsilon)} f(X_1,\dots,X_n)\,dX_1\cdots dX_n \right)$$

Lebesgue measure of $\partial\overline{B(X,\varepsilon)}$ is zero

$$= 0 \quad \text{for all } X, \varepsilon \;)$$

continuity of $f$ alone shows $f \equiv 0$. ✓

By $\overline{\text{Stokes' theorem}}$,

$$\int_{\overline{B(X,\varepsilon)}} d\omega = \int_{\partial\overline{B(X,\varepsilon)}} \omega \overset{\text{hyp.}}{=} 0.$$

By the lemma, $d\omega \equiv 0$.

This completes the proof of the special case. ✓

## General case, $M$ arbitrary

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$d\omega = 0$ is a local property.

By restricting $\omega$ to a chart we are done by the special case. $\square$.

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5) Consider the vector fields

$$X = \frac{\partial}{\partial x} + xz\frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y} + yz\frac{\partial}{\partial z}$$

in $\mathbb{R}^3$. If $p$ is a point of $\mathbb{R}^3$, is there a local coordinate system $(\tilde{x}, \tilde{y})$ around $p$ s.t. $\frac{\partial}{\partial \tilde{x}} = X$ and $\frac{\partial}{\partial \tilde{y}} = Y$?

## solution to (5)

A sufficient condition is

$[X, Y] = 0$.

I'm pretty sure, like 90%, that it is necessary.

↑

Yes. Since $[\partial x, \partial y] = 0$ always.

Recall $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$.

Here, $Xg(p) := X_p(g) \in \mathbb{R}$.

$$[X, Y] = [\partial_x + xz\partial_z, \partial_y + yz\partial_z]$$

$$= \cancelto{0}{[\partial_x, \partial_y]} + [\partial_x, yz\partial_z] + [xz\partial_z, \partial_y] + [xz\partial_z, yz\partial_z]$$

$$= yz\cancelto{0}{[\partial_x, \partial_z]} + 1 \cdot \cancelto{0}{\partial_x(yz)}\partial_z - yz\cancelto{0}{\partial_z(1)}\partial_x + xz\cancelto{0}{[\partial_z, \partial_y]} + xz\cancelto{0}{\partial_z(1)} - 1 \cdot \cancelto{0}{\partial_y(xz)}\partial_z$$

$$+ \; XZyz[\partial z, \partial z] + XZ\partial_z(yz)\,dz$$

$$- \; yz\,\partial_z(XZ)\,\partial z$$

$$= \; XZy - yZX = 0.$$

$\therefore$ such a coordinate system exists. $\square$.

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Let $f : \mathbb{C} \to \mathbb{C}$ be a *polynomial* of one complex variable. Recall the one-point compactification $\mathbb{C} \cup \{\infty\}$ is homeomorphic to $S^2$.

(a) Show $f$ extends to a continuous map $\bar{f} : S^2 \to S^2$.

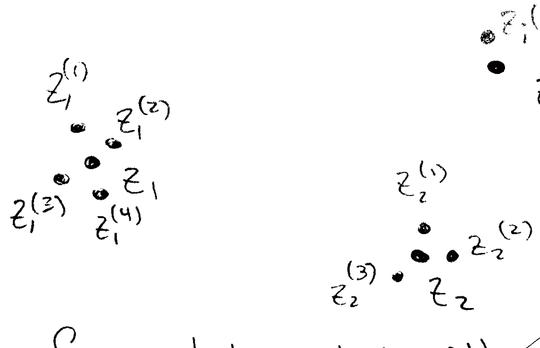
(b) Show the degree of $\bar{f}$, in either the sense of topology or geometry, is equal to the degree of the polynomial $f$, in the sense of algebra.

## Solution to (a) & (b) (sketch)

For (a), setting $\bar{f}(\infty) = \infty$

is clearly the desired extension.

For (b), there are regular values $w \in S^2$ very close to $0$. The preimages of this regular value form little groups $z_i^{(j)}$ around the various roots $z_i$, where each group has $m_i$ elts if $z_i$ has multiplicity $m_i$.




because the tangent map is orientation-preserving at every point

Since $f$ is holomorphic, all the local degrees of the preimages are $+1$. We conclude the geometric degree is $\sum_i m_i = n$. $\square$.